PROGRAM ROZWOJOWY
POLITECHNIKI WARSZAWSKIEJ

FLUID MECHANICS 3 - LECTURE 4

INCOMPRESSIBLE POTENTIAL FLOWS
PART 1

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Prologue – the Crocco equation

The Euler Equation in Lamb-Gromeko form

$$\nabla\left(\tfrac{1}{2}\upsilon^2\right)+\boldsymbol{\omega}\times\boldsymbol{v}=-\tfrac{1}{\rho}\nabla p+\nabla\Phi_f$$

First Principle of Thermodynamics

$$Tds=de+pd\left(\tfrac{1}{\rho}\right)=d\left(e+\tfrac{p}{\rho}\right)-\tfrac{1}{\rho}dp=di-\tfrac{1}{\rho}dp \quad\Rightarrow\quad T\nabla s-\nabla i=-\tfrac{1}{\rho}\nabla p$$

After insertion to EE we obtain Crocco Equation

$$\nabla\left(\tfrac{1}{2}\upsilon^2+i-\Phi_f\right)+\boldsymbol{\omega}\times\boldsymbol{v}=T\nabla s$$

Assume:

- $\tfrac{1}{2}\upsilon^2+i-\Phi=const$ - homoenergetic flow
- $s=const$ - homoentropic flow

Hence $\boldsymbol{\omega}\times\boldsymbol{v}=\boldsymbol{0}$! In the 2D case, it implies that $\boldsymbol{\omega}\equiv\boldsymbol{0}$, i.e., the velocity is a potential vector field. There exists the velocity potential function φ such that

$$\boldsymbol{v}=\nabla\varphi$$

Stationary incompressible potential flows

Assume flow incompressibility. The velocity field satisfies simultaneously the following conditions

$$\nabla \cdot \boldsymbol{v} = 0 \quad , \quad \nabla \times \boldsymbol{v} = 0$$

One has $\nabla \cdot \boldsymbol{v} = 0 \;\; \Rightarrow \;\; \nabla \cdot \nabla \varphi \equiv \nabla^2 \varphi = 0$, i.e., the velocity potential φ **is the harmonic function.**

On the other hand, the divergence-free velocity field can be expressed as

$$\boldsymbol{v} = \nabla \times \boldsymbol{\psi}$$

where $\boldsymbol{\psi}$ is the vector streamfunction. One can assume that $\nabla \cdot \boldsymbol{\psi} = 0$. For the potential flow we have

$$\boldsymbol{0} = \nabla \times \boldsymbol{v} = \nabla \times (\nabla \times \boldsymbol{\psi}) = \nabla (\underbrace{\nabla \cdot \boldsymbol{\psi}}_{0}) - \boldsymbol{\Delta \psi} = -\boldsymbol{\Delta \psi}$$

Hence, $\boldsymbol{\Delta \psi} = \boldsymbol{0}$, i.e., the vector streamfunction is a harmonic vector field.

In the 2D case $$\boldsymbol{\upsilon} = u\boldsymbol{e}_x + \upsilon\boldsymbol{e}_y$$

The vorticity field is

$$\nabla \times \boldsymbol{\upsilon} = \underbrace{\left(\tfrac{\partial}{\partial x}\upsilon - \tfrac{\partial}{\partial y}u\right)}_{\omega}\boldsymbol{e}_z \equiv \omega\boldsymbol{e}_z$$

In the 2D case, the vector streamfunction can be expressed as $\boldsymbol{\psi} = \psi\boldsymbol{e}_z$, where the scalar field ψ is called (just) a streamfunction. One can write

$$\boldsymbol{\upsilon} = \nabla \times \psi\boldsymbol{e}_z = \begin{vmatrix} \boldsymbol{e}_x & \boldsymbol{e}_y & \boldsymbol{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & \psi \end{vmatrix} = \underbrace{\tfrac{\partial}{\partial y}\psi}_{u}\,\boldsymbol{e}_x + \underbrace{\left(-\tfrac{\partial}{\partial x}\psi\right)}_{\upsilon}\boldsymbol{e}_y$$

Hence $$u = \tfrac{\partial}{\partial y}\psi \quad , \quad \upsilon = -\tfrac{\partial}{\partial x}\psi$$

For the potential velocity field we obtain

$$\tfrac{\partial}{\partial x}\upsilon - \tfrac{\partial}{\partial y}u = 0 \;\Rightarrow\; -\tfrac{\partial^2}{\partial x^2}\psi - \tfrac{\partial^2}{\partial y^2}\psi = 0 \;\Rightarrow\; \nabla^2\psi = 0$$

Isolines of φ - equipotential lines

Isolines of ψ - streamlines (for a stationary flows they are identical to fluid element trajectories)

Note: **the equipotential lines and streamlines are mutually orthogonal.**

It is sufficient to show that the vectors $\nabla\varphi$ and $\nabla\psi$ are perpendicular at each point of the flow domain. One can write

$$\nabla\varphi\cdot\nabla\psi = \frac{\partial}{\partial x}\varphi\frac{\partial}{\partial x}\psi + \frac{\partial}{\partial y}\varphi\frac{\partial}{\partial y}\psi = -u\upsilon + \upsilon u = 0$$

The equipotential lines and the streamlines form an orthogonal grid covering the flow domain.

Complex potential function and velocity

The functions φ and ψ form the Riemann pair, meaning that

$$\partial_x \varphi = \partial_y \psi = u \quad , \; \partial_y \varphi = -\partial_x \psi = \upsilon$$

Thus, the complex-valued function of the complex variable $z = x + iy$ can be defined

$$\Phi(z) = \varphi(x, y) + i\psi(x, y)$$

The function Φ is called the complex velocity potential. Its derivative exists and can be computed as follows

$$\Phi'(z) = \partial_x \varphi + i\,\partial_x \psi = \partial_y \psi - i\partial_y \phi = u - i\upsilon$$

One can define the complex velocity $V(z) = \Phi'(z)$. It is the complex-valued function such that

$$u(x, y) = \mathfrak{Re}\{V(x+iy)\} \quad , \; \upsilon(x, y) = -\mathfrak{Im}\{V(x+iy)\}$$

Potential flows in polar coordinates

Polar coordinates in the plane $x = r\cos\theta \quad , \quad y = r\sin\theta$

$$r = \sqrt{x^2 + y^2} \quad , \quad \theta = \mathrm{atan}(y/x)$$

Transformation of the velocity field components

$$\upsilon_r = u\cos\theta + \upsilon\sin\theta \quad , \quad \upsilon_\theta = -u\sin\theta + \upsilon\cos\theta$$
$$u = \upsilon_r\cos\theta - \upsilon_\theta\sin\theta \quad , \quad \upsilon = \upsilon_r\sin\theta + \upsilon_\theta\cos\theta$$

Gradient in polar coordinates $\nabla\varphi(r,\theta) = \frac{\partial}{\partial r}\varphi\,\boldsymbol{e}_r + \frac{1}{r}\frac{\partial}{\partial\theta}\varphi\,\boldsymbol{e}_\theta$

Hence $\upsilon_r = \frac{\partial}{\partial r}\varphi \quad , \quad \upsilon_\theta = \frac{1}{r}\frac{\partial}{\partial\theta}\varphi$

Polar components of the velocity field from the streamfunction

$$\upsilon_r = \frac{1}{r}\frac{\partial}{\partial\theta}\psi \quad , \quad \upsilon_\theta = -\frac{\partial}{\partial r}\psi$$

Scalar Laplace operator in polar coordinates

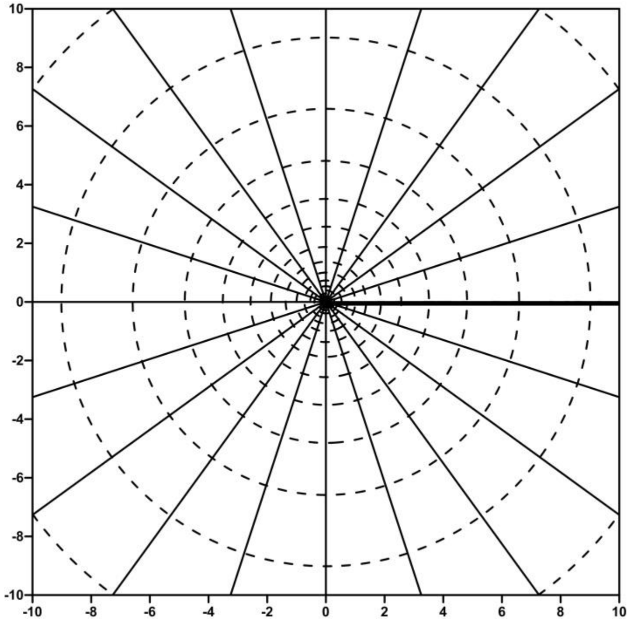
$$\nabla^2 f = \frac{\partial^2}{\partial r^2} f + \frac{1}{r}\frac{\partial}{\partial r} f + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} f \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r} f\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} f$$

Elementary potential flows in 2D

1. Uniform stream

$$\varphi(x,y) = U_\infty x + V_\infty y \quad , \quad \psi(x,y) = -V_\infty x + U_\infty y$$
$$\varphi(r,\theta) = U_\infty r \cos\varsigma + V_\infty r \sin\theta \quad , \quad \psi(r,\theta) = -V_\infty r\cos\theta + U_\infty r \sin\theta$$

2. Source/sink



$$\upsilon_r = \frac{Q}{2\pi r} \quad , \quad \upsilon_\theta \equiv 0$$

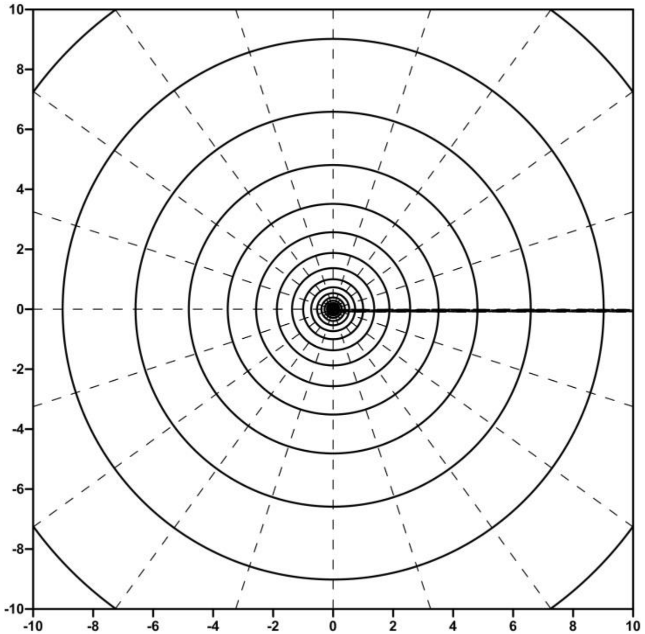
$Q > 0$ - source, $Q < 0$ - sink.

$$\varphi(r,\theta) = \frac{Q}{2\pi}\ln r \quad , \quad \psi(r,\theta) = \frac{Q}{2\pi}\theta$$

Q - source/sink efficiency (flow rate).

$$\oint_{K_a} \boldsymbol{\upsilon}\cdot\boldsymbol{n}\, ds = a\int_0^{2\pi} \upsilon_r(a,\theta)d\theta = a\frac{Q}{2\pi a}2\pi = Q$$

3. Potential vortex



$$\upsilon_r \equiv 0 \quad , \quad \upsilon_\theta \equiv \frac{\Gamma}{2\pi r} \ .$$

Γ measure of the vortex intensity (actual sense – later)

$$\varphi(r,\theta) = \frac{\Gamma}{2\pi}\theta \quad , \quad \psi(r,\theta) = -\frac{\Gamma}{2\pi}\ln r$$

Circulation along the circular contour K_a (the center at the origin, radius a)

$$\oint_{K_a} \boldsymbol{\upsilon} \cdot d\boldsymbol{s} = \oint_{K_a} \nabla\varphi \cdot \boldsymbol{\tau} ds = [\varphi]_{K_a}$$

Note: the potential φ is the multivalued! In the above formula, the symbol $[\,f\,]$ denotes the increment of the function f during a single passage (in the anticlockwise direction) along the integration path K_a.

For the potential function, this increment is equal to the vortex “charge” of circulation

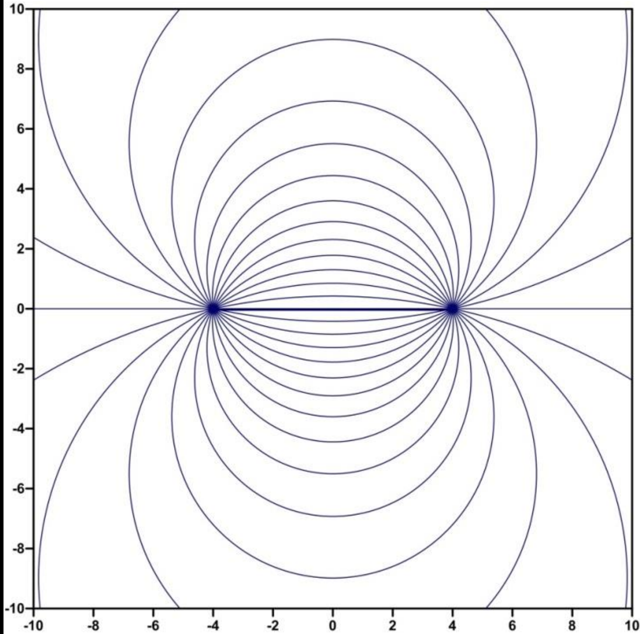
$$[\varphi]_{K_a} = \frac{\Gamma}{2\pi}[\theta]_{K_a} = \frac{\Gamma}{2\pi}2\pi = \Gamma$$

Note that the flow induced by the vortex is potential on the whole plane except the vortex center (here – the origin).

The curvilinear integral of the induced velocity field along the path which does not circumvent the vortex center is zero. More generally, the circulation of the velocity field along arbitrarily chosen path is equal to $(n_1 - n_2)\Gamma$, where n_1 (n_2) is a number of anticlockwise (clockwise) turns around the vortex center.

4. Doublet (with the axis parallel to *0x*)

The flow obtained by shifting sink and source with opposite flow rates to the same point (the origin). The flow rate rises without bounds in the process …



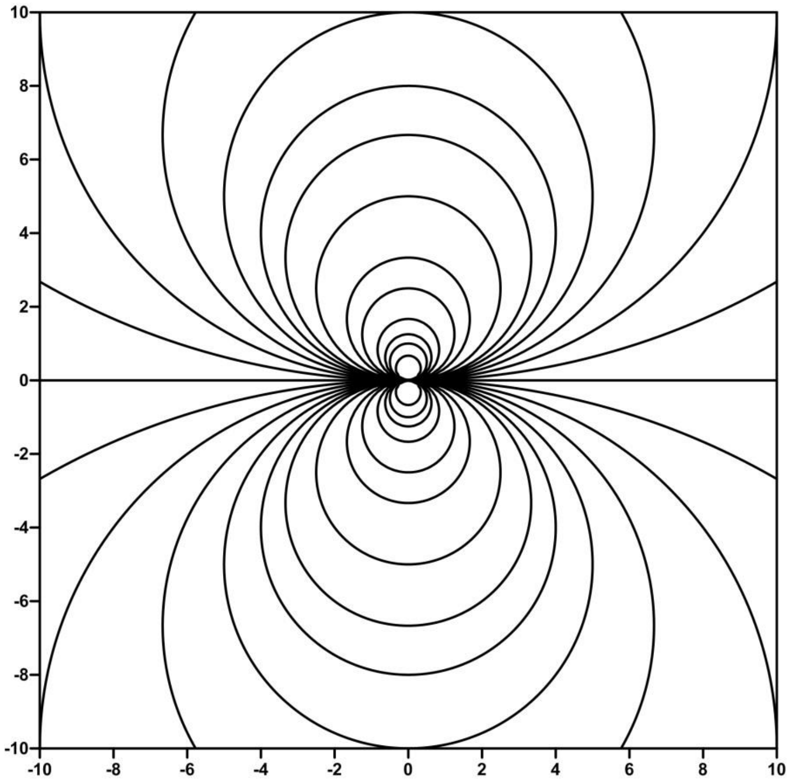
$$\varphi_\varepsilon(x,y) = \tfrac{D}{\varepsilon}\ln\sqrt{(x+\tfrac{1}{2}\varepsilon)^2+y^2} - \tfrac{D}{\varepsilon}\ln\sqrt{(x-\tfrac{1}{2}\varepsilon)^2+y^2} =$$

$$= \frac{D}{\varepsilon}\left[\ln\sqrt{(x+\tfrac{1}{2}\varepsilon)^2+y^2} - \ln\sqrt{(x-\tfrac{1}{2}\varepsilon)^2+y^2}\right]$$

Passage to the limit $\varepsilon \to 0$ (D - moment of the doublet)

$$\varphi(x,y) = \lim_{\varepsilon\to 0}\varphi_\varepsilon(x,y) = D\lim_{\varepsilon\to 0}\frac{\ln\sqrt{(x+\frac{1}{2}\varepsilon)^2+y^2} - \ln\sqrt{(x-\frac{1}{2}\varepsilon)^2+y^2}}{\varepsilon} =$$

$$= \underset{de\ l'Hospital}{\ldots} = \frac{Dx}{x^2+y^2}$$



Streamlines of the doublet flow

Exercise:

- Show that $\psi(x,y) = -\dfrac{Dy}{x^2+y^2}$
- Derive formulae for the Cartesian and polar components of the velocity field

Construction of more complex flows by superposition principle

Since the problem at hand is linear, more complex potential flows can be obtained by superposition of the elementary flows.

Example 1: uniform stream plus a source/sink

$$\varphi(r,\theta) = \varphi_\infty(r,\theta) + \varphi_{src}(r,\theta) = V_\infty \underbrace{r\cos\theta}_{x} + \tfrac{Q}{2\pi}\ln r$$

$$\psi(r,\theta) = \psi_\infty(r,\theta) + \psi_{src}(r,\theta) = V_\infty \underbrace{r\sin\theta}_{y} + \tfrac{Q}{2\pi}\theta$$

Exercise:

- Find polar components of the velocity field
- Find a such that $u(-a,0)=0$ (stagnation point)
- Show that $\psi(-a,0) = \pm\tfrac{1}{2}Q$
- Find the shape of the line $\psi(r,\theta) = \pm\tfrac{1}{2}Q$

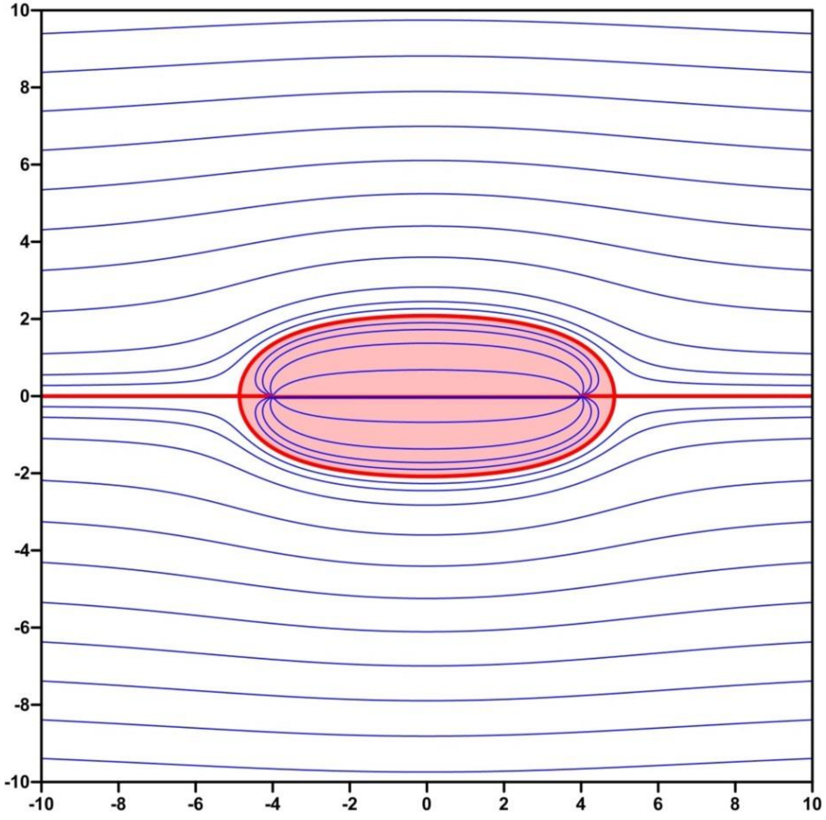
Example 2: uniform stream plus sink plus source (flow past the Rankin oval)

$$\psi(r,\theta) = V_\infty r\sin\theta + \tfrac{Q}{2\pi}\theta_2(r,\theta) - \tfrac{Q}{2\pi}\theta_1(r,\theta)$$

where

$$\theta_1 = \operatorname{atan}\left(\frac{y}{x-a}\right) = \operatorname{atan}\left(\frac{r\sin\theta}{r\cos\theta - a}\right)$$

$$\theta_2 = \operatorname{atan}\left(\frac{y}{x+a}\right) = \operatorname{atan}\left(\frac{r\sin\theta}{r\cos\theta + a}\right)$$



Exercise:

1) show that $u(x,y) = V_\infty + \frac{Q}{2\pi}\left[\frac{x+a}{(x+a)^2+y^2} - \frac{x-a}{(x-a)^2+y^2}\right]$

2) show that the stagnation points are $(x,y) = (\pm b, 0)$,

where $b^2 = a^2 + \dfrac{aQ}{\pi V_\infty}$

3) show that the line $\psi = 0$ is described as

$$x^2 = a^2 - Y^2(x) + \frac{2aY(x)}{\tan[2\pi V_\infty Y(x)/Q]}$$

Example 3: Symmetric flow past a circular contour (a cylinder)

Consider the superposition of the uniform stream and the doublet

$$\varphi(x,y) = U_\infty x + \frac{U_\infty a^2 x}{x^2 + y^2}$$

In polar coordinates …

$$\varphi(r,\theta) = U_\infty r\cos\theta + \frac{U_\infty a^2 \cos\theta}{r} = U_\infty r\left(1 + \frac{a^2}{r^2}\right)\cos\theta$$

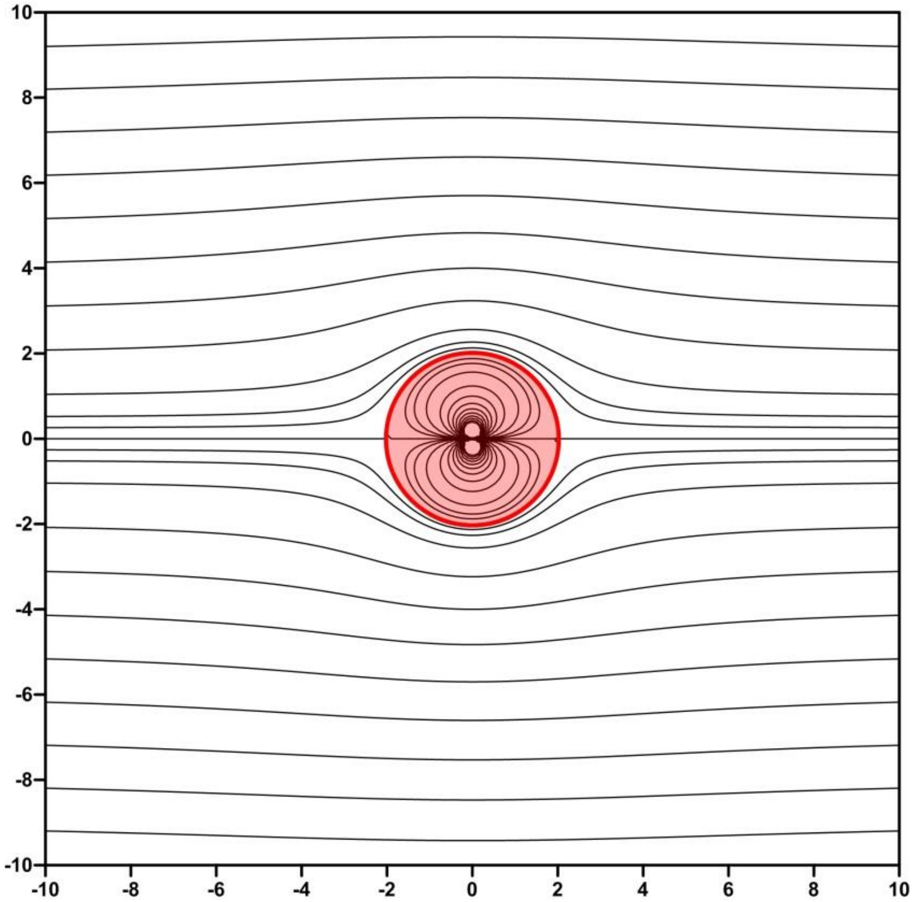
Velocity field in polar coordinates …

$$\begin{cases} \upsilon_r = \frac{\partial}{\partial r}\varphi = U_\infty \cos\theta - \dfrac{U_\infty a^2}{r^2}\cos\theta = U_\infty\left(1 - \dfrac{a^2}{r^2}\right)\cos\theta \\ \upsilon_\theta = \frac{1}{r}\frac{\partial}{\partial\theta}\varphi = -U_\infty \sin\theta - \dfrac{U_\infty a^2}{r^2}\sin\theta = -U_\infty\left(1 + \dfrac{a^2}{r^2}\right)\sin\theta \end{cases}$$

For $r = a$ we obtain

$$\begin{cases} \upsilon_r(a,\theta) = 0 \\ \upsilon_\theta(a,\theta) = -2U_\infty \sin\theta \end{cases}$$

Thus, the contour $r = a$ is one of the streamlines – we have obtained the flow past a circular contour with the center at the origin and the radius a.



Pressure at the contour can be computed from the Bernoulli Equation

$$p_\infty + \tfrac{1}{2}\rho U_\infty^2 = p(a,\theta) + \tfrac{1}{2}\rho V^2(a,\theta)$$

Since $V^2(a,\theta) = 4U_\infty^2 \sin^2\theta$, we get

$$p(\theta) = p_\infty + \tfrac{1}{2}\rho U_\infty^2 (1 - 4\sin^2\theta)$$

In Aerodynamics, we often use the pressure coefficient

$$c_p(\theta) = \frac{p(\theta) - p_\infty}{\tfrac{1}{2}\rho U_\infty^2} = 1 - 4\sin^2\theta$$

Note that:

$$p_{\max} = p(a,0) = p(a,\pi) = p_\infty + \tfrac{1}{2}\rho U_\infty^2 = p_\infty + q$$ - stagnation pressure ($c_p = 1$)

$$p_{\min} = p(a,\tfrac{1}{2}\pi) = p(a,\tfrac{3}{2}\pi) = p_\infty - \tfrac{3}{2}\rho U_\infty^2 = p_\infty - 3q$$ - minimal pressure ($c_p = -3$)

Note also that the pressure distribution is symmetric with respect to both $0x$ and $0y$ axis. Hence, total aerodynamic force is equal zero! In particular, there is no aerodynamic drag. This results is in clear contradiction to the properties of real fluid flows.

However, it is possible to modify the flow in order to obtain the lift force. To this aim, an additional component must be included – the potential vortex located at the circle center.

$$\varphi(r,\theta) = \underbrace{U_\infty r\cos\theta}_{\textit{uniform stream}} + \underbrace{\frac{U_\infty a^2 \cos\theta}{r}}_{\textit{doublet}} + \underset{\textit{vortex}}{\frac{\Gamma}{2\pi}\theta}$$

Note that the presence of the vortex does not spoil the circular streamline!

Upon this modification, the velocity field reads

$$\begin{cases} \upsilon_r = \frac{\partial}{\partial r}\varphi = U_\infty\left(1-\dfrac{a^2}{r^2}\right)\cos\theta \\ \upsilon_\theta = \frac{1}{r}\frac{\partial}{\partial\theta}\varphi = -U_\infty\left(1+\dfrac{a^2}{r^2}\right)\sin\theta + \dfrac{\Gamma}{2\pi r} \end{cases}$$

The velocity distribution at the circular contour is

$$\begin{cases} \upsilon_r(a,\theta) = 0 \\ \upsilon_\theta(a,\theta) = -2U_\infty\sin\theta + \dfrac{\Gamma}{2\pi a} \end{cases}$$

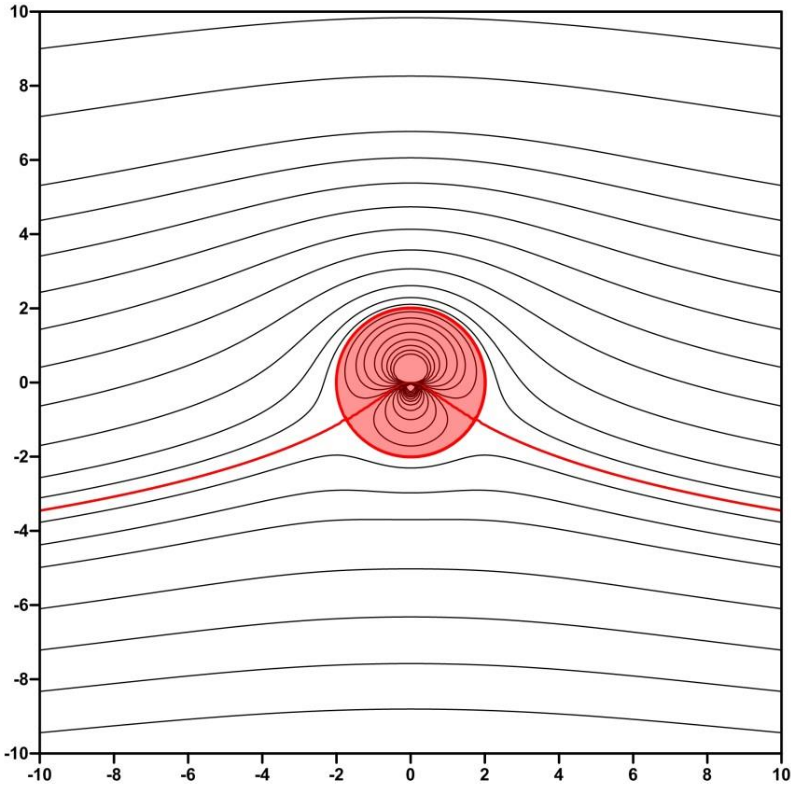
The stagnation points can be determined (if they exist) …

$$-2U_\infty\sin\theta + \frac{\Gamma}{2\pi a} = 0 \;\Rightarrow\; \sin\theta_* = \frac{\Gamma}{4\pi U_\infty a}$$

The solutions

$$\theta_{*,1} = \text{asin}\frac{\Gamma}{4\pi U_\infty a} \quad , \quad \theta_{*,1} = \text{asin}\frac{\Gamma}{4\pi U_\infty a} + \pi$$

Exist as long as $|\Gamma| \le 4\pi U_\infty a$. If $|\Gamma| = 4\pi U_\infty a$ then only one stagnation point appears on the contour (dependently on the sign of the circulation Γ the angular location of this point is $\theta = \frac{1}{2}\pi$ or $\theta = \frac{3}{2}\pi$). If $|\Gamma| > 4\pi U_\infty a$ then the stagnation point appears inside the flow, not on the contour.



Again, the pressure distribution follows from the BE …

$$p(a,\theta) = p_\infty + \tfrac{1}{2}\rho[U_\infty^2 - V^2(a,\theta)]$$

This time

$$V^2(a,\theta) = (-2U_\infty \sin\theta + \frac{\Gamma}{2\pi a})^2 =$$

$$= 4U_\infty^2 \sin^2\theta - 2\frac{\Gamma U_\infty}{\pi a}\sin\theta + \frac{\Gamma^2}{4\pi^2 a^2}$$

Hence

$$p(a,\theta) = p_\infty + \tfrac{1}{2}\rho\left[U_\infty^2(1-4\sin^2\theta) + 2\frac{\Gamma U_\infty}{\pi a}\sin\theta - \frac{\Gamma^2}{4\pi^2 a^2}\right]$$

Note that this distribution is still symmetric with respect to the axis $0y$ - the drag force is again equal zero! However, the presence of the vortex breaks the symmetry with respect to the axis $0x$. The lift force can be computed from the formula

$$\boldsymbol{L} = -\left[a\int_0^{2\pi} p(a,\theta)\sin\theta d\theta\right]\boldsymbol{e}_y$$

as follows …

$$\int_0^{2\pi} p(a,\theta)\sin\theta d\theta =$$

$$= \int_0^{2\pi}[p_\infty \sin\theta + \tfrac{1}{2}\rho U_\infty^2(\sin\theta - 4\sin^3\theta) + \frac{\rho\Gamma U_\infty}{\pi a}\sin^2\theta - \frac{\rho\Gamma^2}{8\pi^2 a^2}\sin\theta]d\theta =$$

$$= \frac{\rho\Gamma U_\infty}{\pi a}\underbrace{\int_0^{2\pi}\sin^2\theta\, d\theta}_{\pi} = \frac{\rho\Gamma U_\infty}{a}$$

We have arrived to a very simple result – known as the Kutta-Joukovski formula.

$$\boldsymbol{L} = -\rho \Gamma U_\infty \boldsymbol{e}_y$$

We will show later that this formula is valid also for contours of general shape.

Again, due to symmetry the drag force defines as

$$\boldsymbol{D} = -\left[a \int_0^{2\pi} p(a,\theta) \cos\theta d\theta \right] \boldsymbol{e}_x \equiv \boldsymbol{0}$$

vanishes identically. This (nonphysical) effect is known as the d'Alembert Paradox.

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Milne-Thomson Theorem

Let the potential flow is given with $\hat{\varphi}(x,y)$ and $\hat{\psi}(x,y)$. Milne-Thomson Theorem explains how to modify this flow in order to achieve two goals:

- The circular contour $x^2 + y^2 = a^2$ is one of the streamlines of the modified flow
- Total charge of the circulation remains unchanged.

The appropriate formulae for the modified streamfunction and the velocity potential are following

$$\psi(x,y) = \hat{\psi}(x,y) - \hat{\psi}\left(\frac{a^2 x}{x^2+y^2}, \frac{a^2 y}{x^2+y^2}\right)$$

$$\varphi(x,y) = \hat{\varphi}(x,y) + \hat{\varphi}\left(\frac{a^2 x}{x^2+y^2}, \frac{a^2 y}{x^2+y^2}\right)$$

Proof – exercise.

Analogical formulae in the polar coordinate are even simpler

$$\psi(r,\theta) = \hat{\psi}(r,\theta) - \hat{\psi}\left(\frac{a^2}{r},\theta\right) \quad , \quad \varphi(r,\theta) = \hat{\varphi}(r,\theta) + \hat{\varphi}\left(\frac{a^2}{r},\theta\right)$$

Indeed, the radial component of the velocity can be computed as follows

$$\upsilon_r(r,\theta) = \tfrac{\partial}{\partial r}\varphi(r,\theta) = \tfrac{\partial}{\partial r}\hat{\varphi}(r,\theta) + \tfrac{\partial}{\partial r}\hat{\varphi}(\tfrac{a^2}{r},\theta) = \hat{\upsilon}_r(r,\theta) - \tfrac{a^2}{r^2}\hat{\upsilon}_r(\tfrac{a^2}{r},\theta)$$

At the circular contour one gets $\upsilon_r(a,\theta) = \hat{\upsilon}_r(a,\theta) - \tfrac{a^2}{a^2}\hat{\upsilon}_r(\tfrac{a^2}{a},\theta) = 0.$

Let us check what happens to the tangent component. To this aim, we calculate azimuthal component

$$\upsilon_\theta(r,\theta) = \tfrac{1}{r}\tfrac{\partial}{\partial\theta}\varphi(r,\theta) = \tfrac{1}{r}\tfrac{\partial}{\partial\theta}\hat{\varphi}(r,\theta) + \tfrac{1}{r}\tfrac{\partial}{\partial\theta}\hat{\varphi}(\tfrac{a^2}{r},\theta) = \hat{\upsilon}_\theta(r,\theta) + \hat{\upsilon}_\theta(\tfrac{a^2}{r},\theta)$$

The, on the contour $r = a$ we obtain

$$\upsilon_\theta(a,\theta) = \hat{\upsilon}_\theta(a,\theta) + \hat{\upsilon}_\theta(\tfrac{a^2}{a},\theta) = 2\hat{\upsilon}_\theta(a,\theta)$$

We conclude that **the flow modification proposed by Milne-Thomson cancels the normal velocity component and doubles the tangent component**.

<u>Examples:</u>

1. Cylinder immersed in the uniform flow

We have $\hat{\varphi}(x,y) = U_\infty x$. Accordingly to MT Theorem we have

$$\varphi(x,y) = \hat{\varphi}(x,y) + \hat{\varphi}\left(\tfrac{a^2 x}{x^2+y^2}, \tfrac{a^2 y}{x^2+y^2}\right) = U_\infty x + U_\infty a^2 \frac{x}{x^2+y^2}$$

which is exactly the right formula. Starting from the polar form, we obtain

$$\varphi(r,\theta) = \hat{\varphi}(r,\theta) + \hat{\varphi}\left(\tfrac{a^2}{r}, \theta\right) = U_\infty r\cos\theta + U_\infty \frac{a^2}{r}\cos\theta$$

which is also correct.

2. Cylinder immersed in the flow induced by a point vortex

Assume that the original flow is induced by the potential vortex located at the point *(c,0)*. The streamfunction is

$$\hat{\psi}(x,y) = -\frac{\Gamma}{2\pi}\ln\sqrt{(x-c)^2 + y^2}$$

Then, the modified flow is

$$\psi(x,y) = \hat{\psi}(x,y) - \hat{\psi}\left(\frac{a^2 x}{x^2+y^2}, \frac{a^2 y}{x^2+y^2}\right)$$

In explicit form

$$\psi(x,y) = -\frac{\Gamma}{2\pi}\ln\frac{\sqrt{(x-c)^2 + y^2}}{\sqrt{\left(\frac{a^2 x}{x^2+y^2} - c\right)^2 + \frac{a^4 x^2}{(x^2+y^2)^2}}} = -\frac{\Gamma}{4\pi}\ln\frac{(x-c)^2 + y^2}{\left(\frac{a^2 x}{x^2+y^2} - c\right)^2 + \frac{a^4 x^2}{(x^2+y^2)^2}}$$

We will show that the modified flow is actually induced by three potential vortices.

To see this, we transform the expression under the logarithm as follows

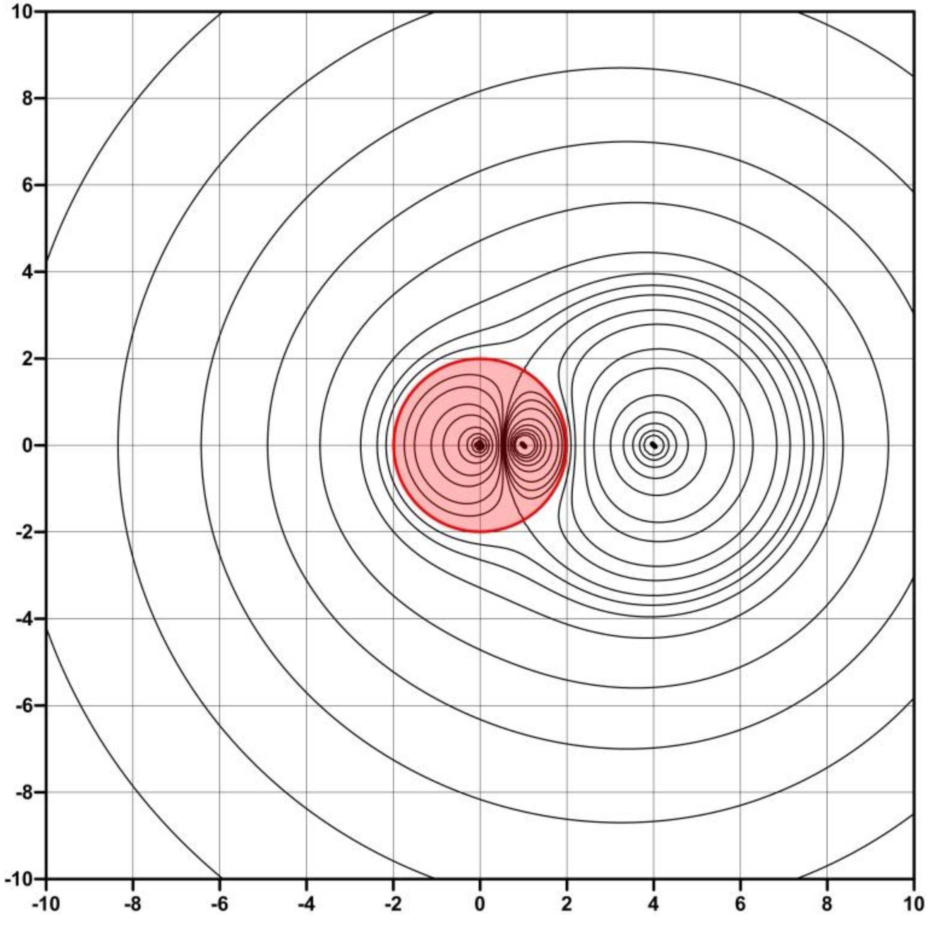
$$\frac{(x-c)^2+y^2}{(\frac{a^2x}{x^2+y^2}-c)^2+\frac{a^4x^2}{(x^2+y^2)^2}} = \frac{[(x-c)^2+y^2](x^2+y^2)}{(x^2+y^2)[\frac{a^4x^2}{x^2+y^2}-2a^2xc+c^2(x^2+y^2)+\frac{a^4y^2}{x^2+y^2}]} =$$

$$= \frac{[(x-c)^2+y^2](x^2+y^2)}{a^4-2a^2xc+c^2(x^2+y^2)} = \frac{[(x-c)^2+y^2](x^2+y^2)}{c^2[(\frac{a^2}{c})^2-2\frac{a^2}{c}x+x^2+y^2]} = \frac{[(x-c)^2+y^2](x^2+y^2)}{c^2[(x-\frac{a^2}{c})^2+y^2]}$$

Thus, the streamfunction can be written as follows

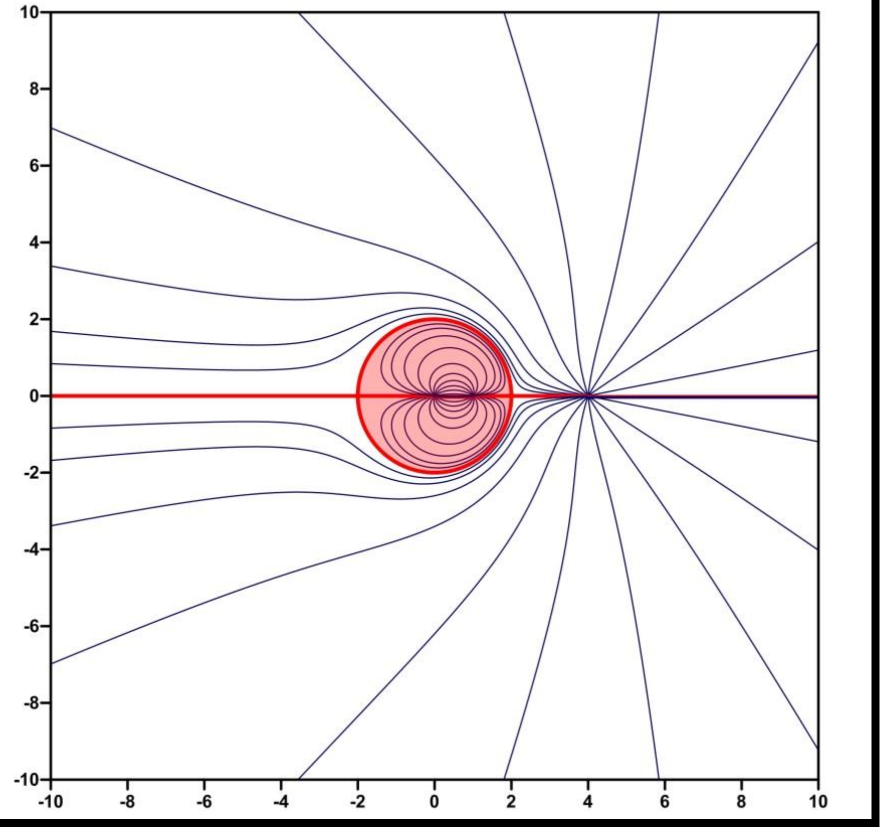
$$\psi(x,y) = -\frac{\Gamma}{4\pi}\ln\frac{(x-c)^2+y^2}{(\frac{a^2x}{x^2+y^2}-c)^2+\frac{a^4x^2}{(x^2+y^2)^2}} = -\frac{\Gamma}{4\pi}\ln\frac{[(x-c)^2+y^2](x^2+y^2)}{c^2[(x-\frac{a^2}{c})^2+y^2]} =$$

$$= \underbrace{-\frac{\Gamma}{2\pi}\ln\sqrt{(x-c)^2+y^2}}_{\textit{original vortex } (\Gamma)} + \left(\underbrace{-\frac{\Gamma}{2\pi}\ln\sqrt{x^2+y^2}}_{\textit{vortex at } (0,0)\ (\Gamma)}\right) + \underbrace{\frac{\Gamma}{2\pi}\ln\sqrt{(x-\frac{a^2}{c})^2+y^2}}_{\textit{vortex at the inversion point } (-\Gamma)} + \underbrace{\frac{\Gamma}{2\pi}\ln c}_{\textit{insignificant constant}}$$



The corresponding pattern of streamlines ….

One can also put the cylinder into the flow induced by a source/sink. The resulting flow is shown in the right …



Even more complex flow is presented below …