Turbulence Modelling

(CFD course)

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Contents

1. Reynolds-averaged Navier-Stokes equations

The discussion of the Reynolds or time-averaged Navier-Stokes (RANS) equations and turbulence transport equations is limited to the constant-density (incompressible) fluids. A linear relationship is assumed between the components of the stress and deformation tensors. An extension to compressible fluids is straightforward and can be found in many textbooks (see for instance Wilcox, 2006, 2008). In case of compressible fluid one has to apply the Favre-averaging (density-based averaging) instead of mentioned Reynolds-averaging to allow for some simplifications in the mass and momentum conservation equations.

For incompressible fluid, the instantaneous velocity component $u_i(\mathbf{x},t)$ can be written as the sum of a mean $\bar{u}_i(\mathbf{x})$ and a fluctuating part $u_i'(\mathbf{x},t)$ (fig. 1):

$$u_i(\mathbf{x},t) = \bar{u}_i(\mathbf{x}) + u_i'(\mathbf{x},t) \tag{1.1}$$

where the mean velocity is defined as the time-averaged value

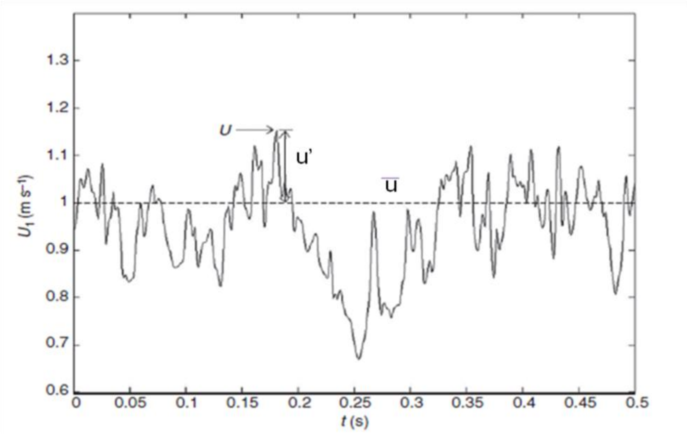

Fig. 1 Decomposition of the signal in mean and fluctuating components.

$$\bar{u}_i(\mathbf{x}) = \lim_{T\to\infty} \frac{1}{T} \int_t^{t+T} u_i(\mathbf{x},t)dt \tag{1.2}$$

T is the averaging time interval. We assume that T→∞. This corresponds to the steady RANS model. For time-accurate RANS (called URANS) this time interval has to be sufficiently large with respect to the time scale T_1 of the turbulent fluctuations (fig. 2) and small with respect to the time scale T_2 of large scale unsteadiness. So for time-accurate problems Eq. (1.2) takes the form:

$$\bar{u}_i(\mathbf{x},t)=\frac{1}{T}\int_t^{t+T}u_i(\mathbf{x},t)dt$$

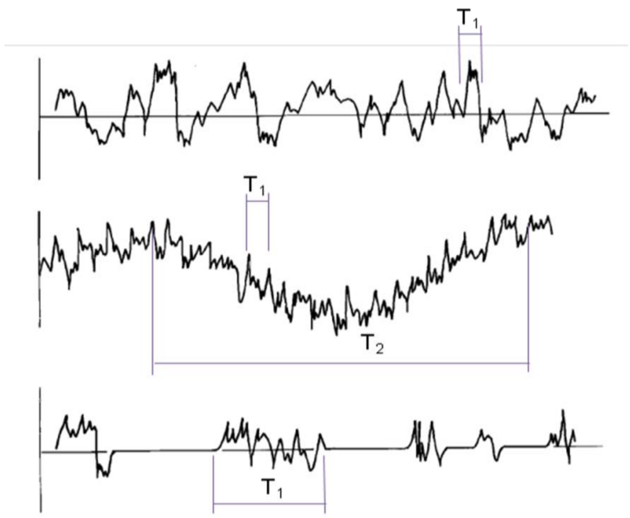

Fig. 2 Time-averaging windows. T_1 – time scale of turbulent fluctuations, T_2 – time scale of unsteady motion. The time-averaging window T should be: $T_1 < T < T_2$

Two useful properties are (steady RANS):

- the time-average of a time-averaged value is again the same value (the averaged value
$\bar{u}_i(\mathbf{x})$ is not a function of time)

$$\bar{\bar{u}}_i(\mathbf{x})=\lim_{T\to\infty}\frac{1}{T}\int_t^{t+T}\bar{u}_i(\mathbf{x})dt=\lim_{T\to\infty}\frac{\bar{u}_i(\mathbf{x})}{T}\int_t^{t+T}dt=\bar{u}_i(\mathbf{x}) \tag{1.3}$$

- and that the time-averaged value of the fluctuating part is zero

$$\overline{u'}_i(\mathbf{x})=\lim_{T\to\infty}\frac{1}{T}\int_t^{t+T}[u_i(\mathbf{x},t)-\bar{u}_i(\mathbf{x})]dt=\bar{u}_i(\mathbf{x})-\bar{\bar{u}}_i(\mathbf{x})=0 \tag{1.4}$$

The other averaging properties are listed below for arbitrary quantities ξ, ψ and ζ:

$$\overline{\xi+\psi}=\bar{\xi}+\bar{\psi} \qquad \text{(time averaging is linear)} \tag{1.5}$$

$$\overline{\frac{\partial \xi}{\partial x_i}} = \frac{\partial \overline{\xi}}{\partial x_i} \tag{1.6}$$

$$\overline{\overline{\xi}\,\overline{\psi}} = \overline{\xi}\,\overline{\psi} \tag{1.7}$$

$$\overline{\overline{\xi}\psi'\zeta'} = \overline{\xi}\,\overline{\psi'\zeta'} \tag{1.8}$$

$$\overline{\overline{\xi}\psi'} = 0 \tag{1.9}$$

$$\overline{\overline{\xi}\,\overline{\psi}\zeta'} = 0 \tag{1.10}$$

$$\overline{\xi'\psi'} \neq 0 \tag{1.11}$$

The Reynolds-averaging is linear (1.5) and it commutes with the space derivatives (1.6). Relations (1.7) and (1.8) come from the property (1.3). With (1.9) and (1.10) we take advantage of the observation that the product of mean and fluctuating quantity is zero because the mean of the latter is zero. The quantities ξ' and ψ' are correlated. It means the average of their product is not zero (Eq. 1.11).

For incompressible fluid the conservation of mass and momentum equations (with the constitutive relation introduced already, see first lecture) are

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{1.12}$$

$$\frac{\partial u_i}{\partial t} + \frac{\partial (u_i u_j)}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left(2\nu S_{ji}\right) \tag{1.13}$$

where ρ is the fluid density, μ is the dynamic molecular viscosity, p is the pressure and S_{ij} is the strain-rate tensor $S_{ij}=1/2(\partial u_i/\partial x_j+\partial u_j/\partial x_i)$.

First, we average the continuity equation (1.12). Taking the property (1.6) we obtain:

$$\overline{\frac{\partial u_i}{\partial x_i}} = \frac{\partial \overline{u_i}}{\partial x_i}. \tag{1.14}$$

Next, we average the momentum equations (1.13).

$$\overline{\frac{\partial u_i}{\partial t} + \frac{\partial (u_i u_j)}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left(2\nu S_{ji}\right)}$$

Using the property (1.5) we obtain

$$\overline{\frac{\partial u_i}{\partial t}} + \overline{\frac{\partial (u_i u_j)}{\partial x_j}} = -\frac{1}{\rho}\overline{\frac{\partial p}{\partial x_i}} + \overline{\frac{\partial}{\partial x_j}\left(2\nu S_{ji}\right)} \tag{1.15}$$

Here we assume that the time-averaging commutes with the local time derivative. We can see from figure 1 that for $T \to \infty$ the time-derivative of the mean velocity is zero

$$\overline{\frac{\partial u_i}{\partial t}} \cong \frac{\partial \overline{u_i}}{\partial t} \to 0 \tag{1.16}$$

The convective derivative deserves attention. We average the product of u_i and u_j:

$$\overline{u_i u_j} = \overline{\left(\overline{u_i} + u_i'\right)\left(\overline{u_j} + u_j'\right)} = \overline{\overline{u_i}\,\overline{u_j}} + \overline{\overline{u_i} u_j'} + \overline{u_i' \overline{u_j}} + \overline{u_i' u_j'} \tag{1.17}$$

The second and third term on r.h.s. of eq. (1.17) cancels out (property 1.9). The last term is nonzero because there is some correlation between u_i' and u_j' fluctuating velocity components. So using property (1.7) we obtain:

$$\overline{u_i u_j} \cong \overline{u_i}\,\overline{u_j} + \overline{u_i' u_j'} \tag{1.18}$$

Figure 3 shows the momentum exchange process in the shear layers of the plane jet at Re=20000. This example is used to illustrate a correlation between u' and v' fluctuating velocity components. Let us assume that the red fluid element (located on line R) is shifted to the left. This shift is visible by negative u' fluctuating velocity component (see the coordinate system). Since the red fluid element is moving from the high mean velocity zone V_R to the small mean velocity zone V_L this transfer generates the positive v' fluctuating velocity ($v'=V_R-V_L$, $v'>0$). The product of these two fluctuations is negative $u'v'<0$. The time-averaged product of u' and v' is shown in fig. 3 (bottom). This is the turbulent shear stress profile $\tau_{ii} = -\overline{u_i' u_j'}$. We clearly see that the negative u' fluctuation generates the positive v' fluctuation. The momentum exchange process can also be realized the other way around, so from left to right (see blue fluid element located on line L). This is due to the fact that formation of the shear layer is typically related to evolution (in space and time) of the coherent vortex structures which subsequently breakdown into smaller forms. It means that at one time instant there is a momentum exchange from right to left. But at the other time instance there might be the momentum exchange from left to right. Let us, therefore, assume that the fluid element is shifted from left to right (blue contour). This shift results in positive u' fluctuating velocity. Since the blue fluid element is moving from the low mean velocity zone V_L to the high mean velocity zone V_R this transfer generates the negative v' fluctuating velocity ($v' = V_L-V_R$, $v'<0$). The product of these two fluctuating velocities is, again, negative $u'v'<0$. It means that it leads to generation of the negative shear stress after time averaging (fig. 3 bottom).

The Reader can perform a similar analysis for the right part of fig. 3. The result of this analysis will be a formation of the positive shear stress profile (fig. 3, bottom). The sign depends on the coordinate system. In both cases the net effect is the momentum transfer from the mean flow to the fluctuating flow (resulting in increased turbulent shear stress). But the most important remark is that in turbulent flow the momentum exchange in one flow direction causes the momentum exchange in the other flow directions (flow is three dimensional). So there is some correlation between both u_i' and u_j' fluctuating components. This results in the shear stress tensor (rightmost term in Eq. 1.18).

Going back to the time-averaging. The terms on r.h.s. of equation (1.15) can be easily averaged:

$$-\overline{\frac{1}{\rho}\frac{\partial p}{\partial x_i}} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_i} \qquad \overline{\frac{\partial}{\partial x_j}\left(2\nu S_{ji}\right)} = \frac{\partial}{\partial x_j}\left(2\nu \overline{S_{ji}}\right) \tag{1.19}$$

Finally, we obtain the Reynolds-Averaged Navier-Stokes (RANS) equations (eq. 1.14, 1.18, 1.19):

$$\frac{\partial \overline{u}_i}{\partial x_i} = 0 \tag{1.20}$$

$$\frac{\partial(\overline{u}_j \overline{u}_i)}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(2\nu \overline{S}_{ji} - \overline{u_j' u_i'}\right) \tag{1.21}$$

where

$$\overline{S}_{ij} = \frac{1}{2}\left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}\right) \tag{1.22}$$

The last term on r.h.s of Eq. (1.21) is the divergence of the Reynolds-stress tensor. The components of the Reynolds –stress tensor are denoted by $\tau_{ij} = -\overline{u_i' u_j'}$ .This term requires closure model (see below).

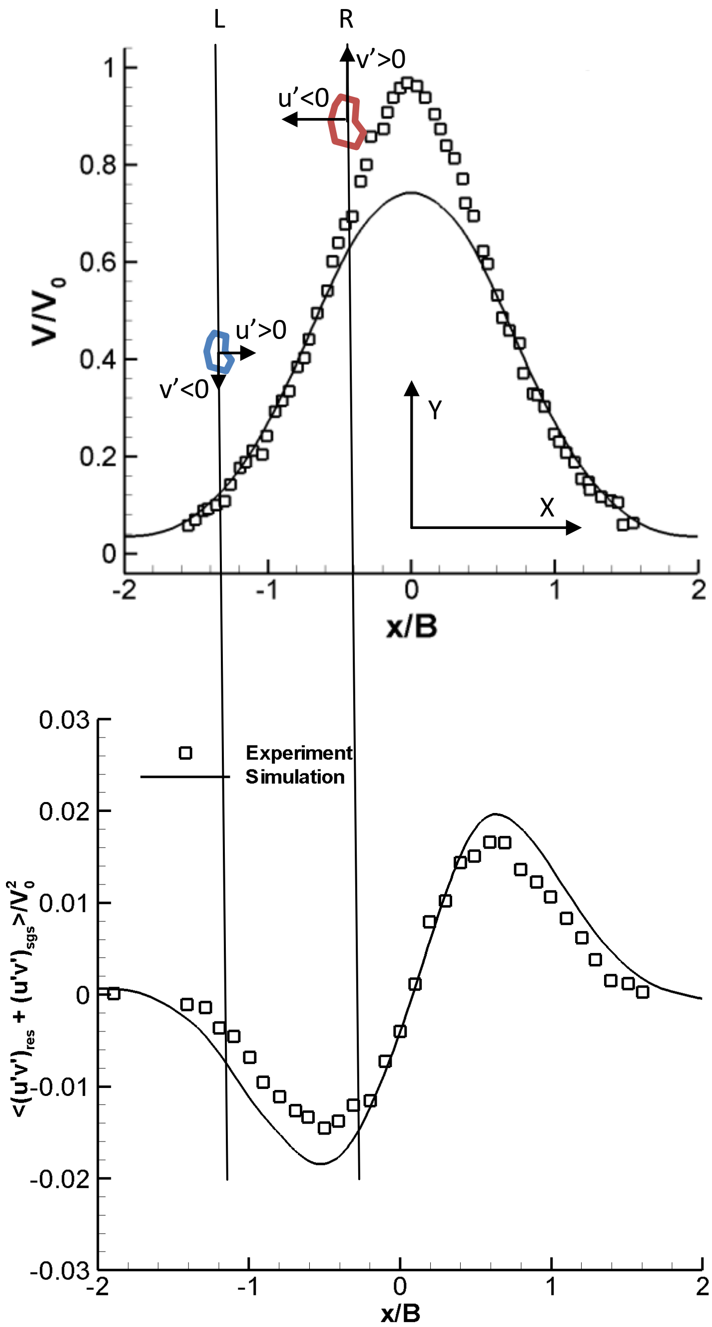

Fig. 3. Momentum exchange in the shear layers of the plane jet. Mean V velocity component (top) and the shear stress profile (bottom).

2. Closure of the modelled terms

2.1. Exact Reynolds-stress transport equations

If we denote by N(u_i) the Navier-Stokes operator:

$$N(u_i)=\frac{\partial u_i}{\partial t}+u_k\frac{\partial u_i}{\partial x_j}+\frac{1}{\rho}\frac{\partial p}{\partial x_i}-\nu\frac{\partial^2 u_i}{\partial x_k \partial x_k}=0 \tag{2.1}$$

We can obtain the exact Reynolds-stress transport equation, by multiplying the momentum equation (2.1) by fluctuating quantity u'_j and adding to this term a similar term with the indexes interchanged and averaging. The following transport equation is obtained (Wilcox, 2006):

$$\begin{aligned}\frac{\partial \tau_{ij}}{\partial t}+\bar{u}_k\frac{\partial \tau_{ij}}{\partial x_k}&=-\tau_{ik}\frac{\partial \bar{u}_j}{\partial x_k}-\tau_{jk}\frac{\partial \bar{u}_i}{\partial x_k}+2\nu\overline{\frac{\partial u'_i}{\partial x_k}\frac{\partial u'_j}{\partial x_k}}-\frac{1}{\rho}\overline{p'\left(\frac{\partial u'_i}{\partial x_j}+\frac{\partial u'_j}{\partial x_i}\right)}+\\&+\frac{\partial}{\partial x_k}\left[\nu\frac{\partial \tau_{ij}}{\partial x_k}+\overline{u'_i u'_j u'_k}+\frac{1}{\rho}\overline{p'u'_i}\delta_{jk}+\frac{1}{\rho}\overline{p'u'_j}\delta_{ik}\right].\end{aligned} \tag{2.2}$$

Derivation of Eq. (2.2) is show in Appendix A.

The physical meaning of the terms on r.h.s. of Eq. (2.2) can be described as follows.

- Production (first and second term). Turbulent stresses are generated at the expense of mean flow energy by mean flow deformation. This term does not need any closure.

- The third term represents the stress dissipation which mainly occurs at the smallest scales (it should be modelled).

- Pressure fluctuations (fourth term) redistribute the turbulent stress among components to make turbulence more isotropic.

- Transport term (last term in Eq. 2.2). This term consists of several parts (in square bracket):

First term is the transport term and it is called the molecular diffusion.

Second, there is the turbulent diffusion term (transport through velocity fluctuations). A closure model is necessary for this term.

A third and fourth part of the transport term is the pressure transport. This term also needs closure model.

2.2 Transport equation for turbulent kinetic energy

An exact equation for the turbulent kinetic energy follows from the equations for the Reynolds stress components (2.2) by contraction trough putting j = i and making the sum over i = 1,2,3:

$$\begin{aligned}\frac{\partial}{\partial t}(\tau_{11}+\tau_{22}+\tau_{33})+\overline{u}_k\frac{\partial}{\partial x_k}(\tau_{11}+\tau_{22}+\tau_{33}) &= -2\Big(\tau_{1k}\frac{\partial \overline{u}_1}{\partial x_k}+\tau_{2k}\frac{\partial \overline{u}_2}{\partial x_k}+\tau_{3k}\frac{\partial \overline{u}_3}{\partial x_k}\Big)+\\ &+2\nu\left(\overline{\frac{\partial u_1'}{\partial x_k}\frac{\partial u_1'}{\partial x_k}}+\overline{\frac{\partial u_2'}{\partial x_k}\frac{\partial u_2'}{\partial x_k}}+\overline{\frac{\partial u_3'}{\partial x_k}\frac{\partial u_3'}{\partial x_k}}\right)+0+\\ &+\frac{\partial}{\partial x_k}\Big[\nu\frac{\partial}{\partial x_k}(\tau_{11}+\tau_{22}+\tau_{33})+\overline{u_1'u_1'u_k'}+\overline{u_2'u_2'u_k'}+\overline{u_3'u_3'u_k'}+\\ &+\frac{2}{\rho}(\overline{p'u_1'}\delta_{1k}+\overline{p'u_2'}\delta_{2k}+\overline{p'u_3'}\delta_{3k})\Big].\end{aligned} \tag{2.3}$$

We take advantage of the continuity equation for the fluctuating velocity components.

$$\partial u_1'/\partial x_1+\partial u_2'/\partial x_2+\partial u_3'/\partial x_3=0 \tag{2.4}$$

which means that for an incompressible fluid studied here the pressure strain-rate term cancels out in Eq. (2.3). We define the turbulent kinetic energy as:

$$k=\frac{1}{2}(u_1'u_1'+u_2'u_2'+u_3'u_3')=-\frac{1}{2}(\tau_{11}+\tau_{22}+\tau_{33})\Rightarrow(\tau_{11}+\tau_{22}+\tau_{33})=-2k \tag{2.5}$$

Next, by inserting Eq. (2.5) to Eq. (2.2) we obtain an exact equation for the turbulent kinetic energy

$$\frac{\partial k}{\partial t}+\overline{u}_k\frac{\partial k}{\partial x_k}=\tau_{ik}\frac{\partial \overline{u}_i}{\partial x_k}-\nu\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_i'}{\partial x_k}}+\frac{\partial}{\partial x_k}\Big[\nu\frac{\partial k}{\partial x_k}-\frac{1}{2}\overline{u_i'u_i'u_k'}-\frac{1}{\rho}\overline{p'u_k'}\Big] \tag{2.6}$$

In Eq. (2.6) the first term on r.h.s describes production of the turbulent kinetic energy. This term is modeled using the Boussinesq hypothesis:

$$\tau_{ik}=2\nu_t\overline{S}_{ik}-\frac{2}{3}k\delta_{ik} \tag{2.7}$$

We later write the production term as $P_k\equiv 2\nu_t\overline{S}_{ik}\partial\overline{u}_i/\partial x_k=2\nu_t\overline{S}_{ik}\overline{S}_{ik}$. ν_t in Eq. (2.7) denotes the turbulent viscosity (see discussion below).

The second term on r.h.s. of eq. (2.6) represents dissipation of the turbulent kinetic energy

$$\epsilon = \nu \overline{\frac{\partial u_i'}{\partial x_k} \frac{\partial u_i'}{\partial x_k}} \tag{2.8}$$

In the frame of the two-equation models discussed here the term (2.8) is obtained by solving an additional transport equation (discussion below).

The second and third term in brackets describe the turbulent transport process by velocity fluctuations and the fluctuating pressure–velocity induced diffusion. Both terms are modeled using the gradient hypothesis:

$$-\frac{1}{2}\overline{u_i' u_i' u_k'} - \frac{1}{\rho}\overline{p' u_k'} = \frac{\nu_t}{\sigma_k}\frac{\partial k}{\partial x_k} \tag{2.9}$$

where σ_k is a certain constant. Finally, the transport equation for the turbulent kinetic energy reads:

$$\frac{\partial k}{\partial t} + \overline{u}_k \frac{\partial k}{\partial x_k} = P_k - \epsilon + \frac{\partial}{\partial x_k}\left[\left(\nu + \frac{\nu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_k}\right] \tag{2.10}$$

2.3. The standard k-ε model

The model which is called the standard k-ε turbulence model was developed by Jones and Launder (1972). In this model, the RANS-equations are used together with the k-equation (2.10), the ε-equation (introduced below) and the eddy-viscosity expression $\nu_t = C_\mu \frac{k^2}{\varepsilon}$. The eddy viscosity expression $\nu_t = C_\mu \frac{k^2}{\varepsilon}$ requires the knowledge of the turbulent kinetic energy, which we can derive from equation (2.10). The second ingredient is the dissipation rate ε. Remark that we are not restricted to an equation of ε. Any quantity which is a combination of k and ε may be used. The most obvious is to use an equation for ε. The expression for ε is known, Eq. (2.11). So, we could use an approach similar to the one as used for the turbulent kinetic energy to derive a transport equation. Technically, this is possible, but the equation is much more complicated than the equation for k and contains many more terms that need modelling. So, looking at the k-equation (2.10) and taking into account that the parameter that we can form with k and ε is the time scale τ=k/ε, the k-equation can be transformed into an ε-equation by

$$\frac{\partial \varepsilon}{\partial t} + \bar{u}_k \frac{\partial \varepsilon}{\partial x_k} = \frac{c_{\varepsilon 1} P - c_{\varepsilon 2}\varepsilon}{\tau} + \frac{\partial}{\partial x_k}\left(\left(\nu + \frac{\nu_t}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial x_k}\right) \tag{2.11}$$

where we introduce the constants $c_{\varepsilon 1}$ and $c_{\varepsilon 2}$ in the transformation of the production term and the dissipation term. We further introduce the diffusion coefficient σ_ε (see discussion latter).

The standard values for the model parameters are:

$$c_\mu = 0.09, \quad \sigma_k = 1.0, \quad \sigma_\varepsilon = 1.3, \quad c_{\varepsilon 1} = 1.44, \quad c_{\varepsilon 2} = 1.92$$

The value for c_μ comes from the observation of thin shear flows with approximate balance between production and dissipation: 2D free jet mixing layers and the so-called inertial region or logarithmic layer in a boundary layer flow. Figure 4 shows the terms in the turbulent kinetic energy equation (Eq. 2.6) along normal to the wall direction in the turbulent boundary layer at Re_θ=1410. The results have been obtained using DNS by Laurent et al. (2012). The molecular diffusion is important at the wall. Both production and dissipation become dominant at $y^+>5$. Note a sign change on the profiles of the turbulent and molecular diffusion inside the boundary layer (diffusion partly acts as a sink and partly as a source of the turbulent kinetic energy). This makes the production and dissipation far more important than the diffusion processes at sufficiently large distance from the wall. Moreover, we can assume that production almost equals dissipation for $y^+>10$: $P_k=\varepsilon$. With y as cross-stream coordinate this results in:

$$c_\mu = \nu_t \frac{\varepsilon}{k^2} = \nu_t \frac{\nu_t (\frac{\partial u}{\partial y})^2}{k^2} = (\frac{-\overline{u'v'}}{k})^2$$

For thin shear flows, the experimental observation is $(\frac{-\overline{u'v'}}{k}) \approx 0.3$, resulting in c_μ = 0.09.

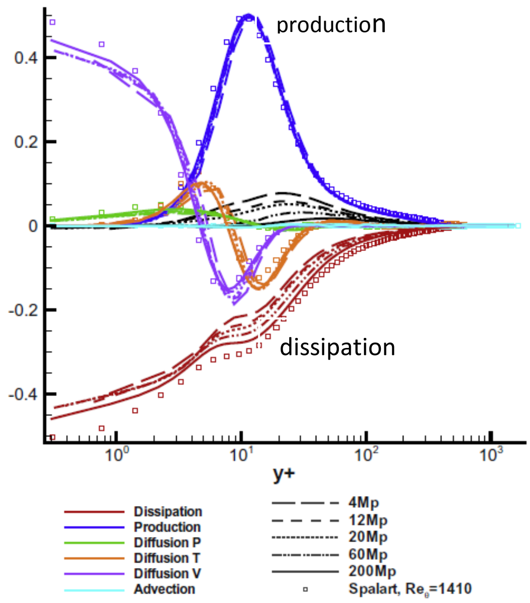

Figure 4. Terms in the turbulent kinetic energy equation inside the turbulent boundary layer at Re_θ=1410. DNS results by Laurent et al., (2012).

The constant $c_{\varepsilon 2}$ is determined by considering decaying homogeneous turbulence. Figure 5 shows the vortex structures obtained for DNS simulation of decaying isotropic and homogeneous turbulence in the periodic box (Dubief i Delcayre, 2000). In this interesting flow the statistics of all quantities in brackets on r.h.s. of Eq. (2.6) are constant. This means that there is no diffusion. Moreover, the statistics of mean velocity derivatives are zero. It means that there is no production either. The k-equation reduces to

$$\frac{Dk}{Dt} = -\varepsilon, \qquad \frac{D\varepsilon}{Dt} = -c_{\varepsilon 2}\frac{\varepsilon^2}{k} \tag{2.12}$$

These equations are satisfied for an observer moving with the flow by

$$k = t^{-n}, \quad \varepsilon = t^{-m}, \quad \text{dla} \quad n = \frac{1}{c_{\varepsilon 2} - 1}, \quad m = n + 1 \tag{2.13}$$

Experiments of Comte-Bellot and Corrsin (1966) indicate values n = 1.2 to 1.3 which implies $c_{\varepsilon 2}$ **=** 1.83 to 1.77. The value $c_{\varepsilon 2}$ **=** 1.92, selected by Jones and Launder differs somewhat because they did some numerical optimisation of this constant over a range of flows.

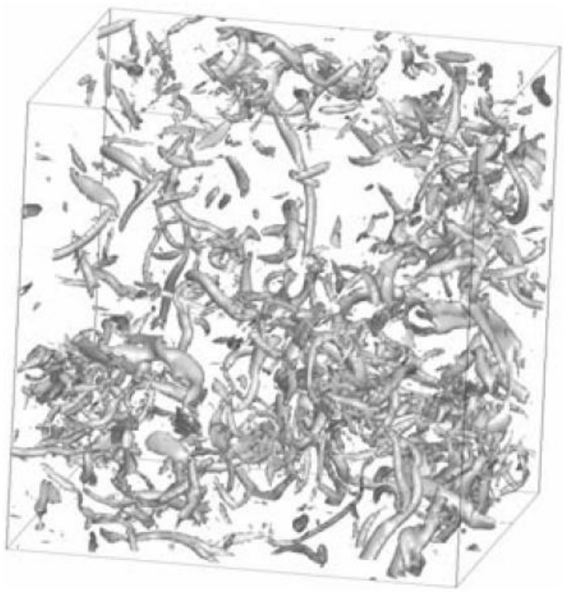

Figure 5. Homogeneous and isotropic turbulence in a periodic box. (Dubief i Delcayre, 2000)

The constant $c_{\varepsilon 1}$ is determined from a homogeneous shear flow experiment. Figure 6 shows a sketch of such flow. Homogeneous shear flow appears to reach an equilibrium state with k and ε growing in such a manner that the turbulent time-scale k/ε approaches an approximately constant value. The governing equations are

$$\frac{Dk}{Dt} = P_k - \varepsilon, \qquad \frac{D\varepsilon}{Dt} = \frac{c_{\varepsilon 1} P_k - c_{\varepsilon 2}\varepsilon}{k/\varepsilon} \tag{2.14}$$

which can be combined with the assumption of a constant turbulent time-scale to yield the following relations. Assuming Dτ/Dt=0 we arrive at:

$$\begin{aligned} \frac{D(k/\varepsilon)}{Dt} &= \frac{Dk}{Dt}\frac{1}{\varepsilon} - \frac{k}{\varepsilon^2}\frac{D\varepsilon}{Dt} \\ &= \frac{P_k}{\varepsilon} - 1 - \frac{k}{\varepsilon^2}\left(\frac{c_{\varepsilon 1} P_k \varepsilon}{k} - \frac{c_{\varepsilon 2}\varepsilon^2}{k}\right) \\ &= c_{\varepsilon 2} - 1 - (c_{\varepsilon 1} - 1)\frac{P_k}{\varepsilon} = 0 \end{aligned} \tag{2.15}$$

which leads to

$$c_{\varepsilon 1} = \frac{c_{\varepsilon 2} - 1}{P_k/\varepsilon} + 1$$

Using $c_{\varepsilon 2}$ =1.83 and the shear flow data of Tavoularis and Corrsin (1981), where $P_k/\varepsilon \approx 1.8$ and $k/\varepsilon \approx$ constant, the value $c_{\varepsilon 1} = 1.46$ results. Again, Jones and Launder take the somewhat different value $c_{\varepsilon 1} = 1.44$ by numerical optimisation.

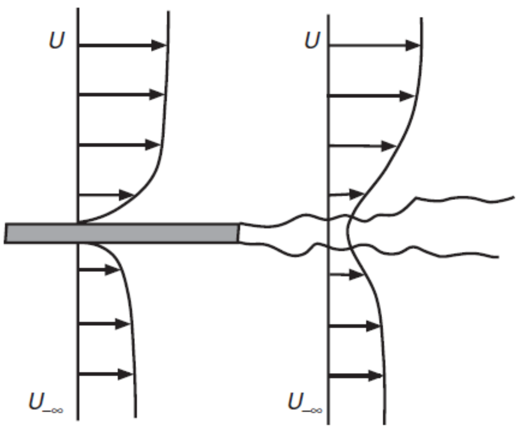

Figure 6. Flow in 2D mixing shear layer

The diffusion coefficient σ_k is a priori taken to unity, since it governs the turbulent diffusion of the turbulent kinetic energy by the turbulent motion itself.

The second diffusion coefficient σ_ε comes from consideration of the diffusion of ε in the logarithmic zone of the boundary layer. The following equations can be written in a zero pressure gradient boundary layer flow:

$$0 = \frac{\partial}{\partial y}(\nu_t \frac{\partial u}{\partial y}) \Rightarrow \nu_t \frac{\partial u}{\partial y} = \mathrm{cons\,tan\,t} = \rho\tau_w = u_\tau^2 \tag{2.16}$$

$$0 = \nu_t (\frac{\partial u}{\partial y})^2 - \varepsilon + \frac{\partial}{\partial y}(\frac{\nu_t}{\sigma_k}\frac{\partial k}{\partial y}) \tag{2.17}$$

$$0 = c_{\varepsilon 1}\frac{\varepsilon}{k}\nu_t (\frac{\partial u}{\partial y})^2 - c_{\varepsilon 2}\frac{\varepsilon^2}{k} + \frac{\partial}{\partial y}(\frac{\nu_t}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial y}) \tag{2.18}$$

In the log-layer,

$$\frac{du}{dy} = \frac{u_\tau}{\kappa y} \tag{2.19}$$

From Eq. (2.16) and (2.19) we obtain

$$\nu_t = \kappa u_\tau y \tag{2.20}$$

If we assume $P_k=\varepsilon$ and the constant *k*-profile in the logarithmic part of the turbulent boundary layer (no diffusion of k) Eq. (2.10) reduces to:

$$P_k = \varepsilon = -\overline{u'v'}\frac{\partial u}{\partial y} = \nu_t (\frac{\partial u}{\partial y})^2$$

$$\Rightarrow \varepsilon = \nu_t (\frac{\partial u}{\partial y})^2 = \kappa u_\tau y (\frac{u_\tau}{\kappa y})^2$$

$$\Rightarrow \varepsilon = \frac{u_\tau^3}{\kappa y} \tag{2.21}$$

Taking the definition of the dynamic turbulent viscosity $\mu_t = \rho C_\mu \frac{k^2}{\varepsilon}$ and introducing it into Eq. (2.16) we obtain:

$$\tau_w = \rho u_\tau^2 = -\rho\overline{u'v'} = \mu_t \frac{\partial u}{\partial y} = \rho c_\mu \frac{k^2}{\varepsilon} \frac{u_\tau}{\kappa y}$$

$$\Rightarrow u_\tau^2 = c_\mu \frac{k^2}{\varepsilon} \frac{u_\tau}{\kappa y} = c_\mu \frac{k^2}{u_\tau^3} \kappa y \frac{u_\tau}{\kappa y} = \frac{c_\mu k^2}{u_\tau^2}$$

$$\Rightarrow k = \frac{u_\tau^2}{\sqrt{c_\mu}} \tag{2.22}$$

Introducing Eq. (2.19), (2.21), (2.22) and $\nu_t = C_\mu \frac{k^2}{\varepsilon}$ into Eq. (2.18) we obtain:

$$\begin{aligned} 0 &= c_{\varepsilon 1} \frac{u_\tau^3}{\kappa y} \frac{\sqrt{c_\mu}}{u_\tau^2} \kappa u_\tau y \frac{u_\tau^2}{(\kappa y)^2} - c_{\varepsilon 2} \frac{u_\tau^6}{(\kappa y)^2} \frac{\sqrt{c_\mu}}{u_\tau^2} + \frac{\partial}{\partial y}\left(\frac{\kappa u_\tau y}{\sigma_\varepsilon} (-\frac{u_\tau^3}{\kappa y^2}) \right) \\ &= c_{\varepsilon 1} \frac{u_\tau^4 \sqrt{c_\mu}}{(\kappa y)^2} - c_{\varepsilon 2} \frac{u_\tau^4 \sqrt{c_\mu}}{(\kappa y)^2} + \frac{\partial}{\partial y} (-\frac{u_\tau^4}{\sigma_\varepsilon y}) \\ &= \left((c_{\varepsilon 1} - c_{\varepsilon 2}) \frac{\sqrt{c_\mu}}{(\kappa y)^2} + \frac{1}{\sigma_\varepsilon} \frac{1}{y^2} \right) u_\tau^4 \end{aligned}$$

This results in

$$\sigma_\varepsilon = \frac{\kappa^2}{(c_{\varepsilon 2} - c_{\varepsilon 1})\sqrt{c_\mu}} \tag{2.23}$$

In standard k-ε, the values $c_{\varepsilon 1}$ = 1.44, $c_{\varepsilon 2}$ = 1.92, c_μ = 0.09 and σ_ε = 1.3 are used, which following (2.23) implies κ = 0.43, which is a somewhat larger value than what is usually assumed for the Von Karman constant (κ = 0.40-0.41). Again, this difference comes from the numerical optimisation by Jones and Launder.

One of the shortcoming of the standard k-ε model is its inability to reproduce a correct level of the turbulent shear stress in the near-wall region. Figure 7 shows the predicted by the k-ε model (dashed line) and computed using DNS (solid line) profiles of the turbulent viscosity inside the boundary layer for simulation of the channel flow. As shown the k-ε model overestimates the turbulent viscosity in the near-wall region. In order to limit this shortcoming some damping terms are employed in front of the eddy-viscosity formula. The other approach consists in using separate turbulence model, which shows better performance than the classic k-ε model in the near-wall region, and its blending with standard k-ε further away from walls. This obviously complicates the modeling approach using the standard k-ε model.

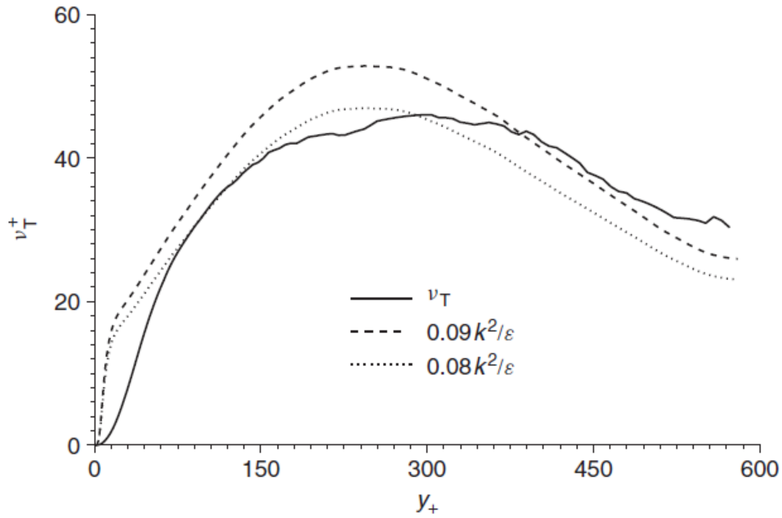

Figure 7. Predicted using the k-ε model (dashed line) and computed using DNS (solid line) profiles of the turbulent viscosity inside the boundary layer for simulation of the channel flow at Re_τ=590. Durbin and Pettersson Reif, (2003).

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Appendix A

Multiplying Eq. (2.1) by the fluctuating velocity u_j' and adding this term to a similar term with the indexes interchanged and averaging we obtain:

$$\overline{u_i'\mathcal{N}(u_j) + u_j'\mathcal{N}(u_i)} = 0 \tag{A.1}$$

After some algebra we obtain:

$$\begin{aligned}
&\overline{u_i'\left(\rho\frac{\partial u_j}{\partial t} + \rho u_k\frac{\partial u_j}{\partial x_k} + \frac{\partial p}{\partial x_j} - \mu\frac{\partial^2 u_j}{\partial x_k \partial x_k}\right) + u_j'\left(\rho\frac{\partial u_i}{\partial t} + \rho u_k\frac{\partial u_i}{\partial x_k} + \frac{\partial p}{\partial x_i} - \mu\frac{\partial^2 u_i}{\partial x_k \partial x_k}\right)} = \\
&= \overline{u_i'\rho\frac{\partial u_j}{\partial t} + u_j'\rho\frac{\partial u_i}{\partial t}} + \\
&+ \overline{u_i'\rho u_k\frac{\partial u_j}{\partial x_k} + u_j'\rho u_k\frac{\partial u_i}{\partial x_k}} + \\
&+ \overline{u_i'\frac{\partial p}{\partial x_j} + u_j'\frac{\partial p}{\partial x_i}} + \\
&- \mu\overline{\left(u_i'\frac{\partial^2 u_j}{\partial x_k \partial x_k} + u_j'\frac{\partial^2 u_i}{\partial x_k \partial x_k}\right)} = 0
\end{aligned} \tag{A.2}$$

Assuming the Reynolds decomposition (mean plus fluctuation):

$$u_i = \overline{u}_i + u_i' \tag{A.3}$$

and putting Eq. (A.3) to (A.2) we obtain several terms which are discussed below:

1. Local time derivative

$$\begin{aligned}
\overline{u_i'\rho\frac{\partial u_j}{\partial t} + u_j'\rho\frac{\partial u_i}{\partial t}} &= \rho\overline{u_i'\frac{\partial(\overline{u}_j + u_j')}{\partial t}} + \rho\overline{u_j'\frac{\partial(\overline{u}_i + u_i')}{\partial t}} \\
&= \rho\overline{u_i'\frac{\partial \overline{u}_j}{\partial t}} + \rho\overline{u_i'\frac{\partial u_j'}{\partial t}} + \rho\overline{u_j'\frac{\partial \overline{u}_i}{\partial t}} + \rho\overline{u_j'\frac{\partial u_i'}{\partial t}} \\
&= \rho\overline{u_i'\frac{\partial u_j'}{\partial t}} + \rho\overline{u_j'\frac{\partial u_i'}{\partial t}} \\
&= \rho\frac{\partial(\overline{u_i'u_j'})}{\partial t} \\
&= -\rho\frac{\partial \tau_{ij}}{\partial t}
\end{aligned} \tag{A.4}$$

The property (1.10) was used in order to simplify Eq. (A.4) (second line). The following relation was used to derive the last term in Eq. (A.4)

$$\frac{\partial \overline{(u_i'u_j')}}{\partial t} = \overline{\frac{\partial u_i'}{\partial t}u_j'} + \overline{u_i'\frac{\partial u_j'}{\partial t}} \tag{A.5}$$

2. Convective term

$$\begin{aligned}
\overline{u_i'\rho u_k\frac{\partial u_j}{\partial x_k} + u_j'\rho u_k\frac{\partial u_i}{\partial x_k}} &= \overline{\rho u_i'(\overline{u}_k+u_k')\frac{\partial(\overline{u}_j+u_j')}{\partial x_k}} + \overline{\rho u_j'(\overline{u}_k+u_k')\frac{\partial(\overline{u}_i+u_i')}{\partial x_k}} \\
&= \overline{\rho u_i'\overline{u}_k\frac{\partial \overline{u}_j}{\partial x_k}} + \overline{\rho u_i'\overline{u}_k\frac{\partial u_j'}{\partial x_k}} + \overline{\rho u_i'u_k'\frac{\partial \overline{u}_j}{\partial x_k}} + \overline{\rho u_i'u_k'\frac{\partial u_j'}{\partial x_k}} + \\
&+ \overline{\rho u_j'\overline{u}_k\frac{\partial \overline{u}_i}{\partial x_k}} + \overline{\rho u_j'\overline{u}_k\frac{\partial u_i'}{\partial x_k}} + \overline{\rho u_j'u_k'\frac{\partial \overline{u}_i}{\partial x_k}} + \overline{\rho u_j'u_k'\frac{\partial u_i'}{\partial x_k}} \\
&= \rho\overline{u}_k\frac{\partial\overline{(u_i'u_j')}}{\partial x_k} + \rho\overline{u_i'u_k'}\frac{\partial \overline{u}_j}{\partial x_k} + \rho\overline{u_k'\frac{\partial(u_i'u_j')}{\partial x_k}} + \rho\overline{u_j'u_k'}\frac{\partial \overline{u}_i}{\partial x_k} \\
&= -\rho\overline{u}_k\frac{\partial \tau_{ij}}{\partial x_k} - \rho\tau_{ik}\frac{\partial \overline{u}_j}{\partial x_k} + \rho\frac{\partial\overline{(u_i'u_j'u_k')}}{\partial x_k} - \rho\tau_{jk}\frac{\partial \overline{u}_i}{\partial x_k}
\end{aligned} \tag{A.6}$$

3. Pressure gradient term

$$\begin{aligned}
\overline{u_i'\frac{\partial p}{\partial x_j} + u_j'\frac{\partial p}{\partial x_i}} &= \overline{u_i'\frac{\partial(\overline{p}+p')}{\partial x_j}} + \overline{u_j'\frac{\partial(\overline{p}+p')}{\partial x_i}} \\
&= \overline{u_i'\frac{\partial p'}{\partial x_j}} + \overline{u_j'\frac{\partial p'}{\partial x_i}}
\end{aligned} \tag{A.7}$$

4. Viscous term

$$\begin{aligned}
\overline{\mu\left(u_i'\frac{\partial^2 u_j}{\partial x_k\partial x_k} + u_j'\frac{\partial^2 u_i}{\partial x_k\partial x_k}\right)} &= \overline{\mu u_i'\frac{\partial^2(\overline{u}_j+u_j')}{\partial x_k\partial x_k}} + \overline{\mu u_j'\frac{\partial^2(\overline{u}_i+u_i')}{\partial x_k\partial x_k}} \\
&= \overline{\mu u_i'\frac{\partial^2\overline{u}_j}{\partial x_k\partial x_k}} + \overline{\mu u_i'\frac{\partial^2 u_j'}{\partial x_k\partial x_k}} + \\
&+ \overline{\mu u_j'\frac{\partial^2\overline{u}_i}{\partial x_k\partial x_k}} + \overline{\mu u_j'\frac{\partial^2 u_i'}{\partial x_k\partial x_k}} \\
&= \mu\frac{\partial^2\overline{(u_i'u_j')}}{\partial x_k\partial x_k} - 2\mu\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}} \\
&= -\mu\frac{\partial^2\tau_{ij}}{\partial x_k\partial x_k} - 2\mu\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}}.
\end{aligned} \tag{A.8}$$

Finally, putting Eq. (A.4), (A.6), (A.7) and (A.8) to Eq. (A.2) we obtain the exact form of the Reynolds-stress equation:

$$
\begin{aligned}
\frac{\partial \tau_{ij}}{\partial t} + \overline{u}_k \frac{\partial \tau_{ij}}{\partial x_k} &= -\tau_{ik}\frac{\partial \overline{u}_j}{\partial x_k} - \tau_{jk}\frac{\partial \overline{u}_i}{\partial x_k} + 2\nu \overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}} + \overline{\frac{u_i'}{\rho}\frac{\partial p'}{\partial x_j}} + \overline{\frac{u_j'}{\rho}\frac{\partial p'}{\partial x_i}} + \\
&+ \frac{\partial}{\partial x_k}\left[\nu \frac{\partial \tau_{ij}}{\partial x_k} + \overline{u_i' u_j' u_k'}\right].
\end{aligned}
\tag{A.9}
$$

Note that Eq. (A.9) can be rewritten in the form given by Eq. (19) using the following relations:

$$
\begin{aligned}
\overline{\frac{u_i'}{\rho}\frac{\partial p'}{\partial x_j}} &= \frac{1}{\rho}\overline{\frac{\partial}{\partial x_j}(p' u_i')} - \overline{\frac{p'}{\rho}\frac{\partial u_i'}{\partial x_j}} \\
\overline{\frac{u_j'}{\rho}\frac{\partial p'}{\partial x_i}} &= \frac{1}{\rho}\overline{\frac{\partial}{\partial x_i}(p' u_j')} - \overline{\frac{p'}{\rho}\frac{\partial u_j'}{\partial x_i}}
\end{aligned}
\tag{A.10}
$$