PROGRAM ROZWOJOWY
POLITECHNIKI WARSZAWSKIEJ

LECTURE 13

ISENTROPIC MOTION OF THE CLAPEYRON GAS

KAPITAŁ LUDZKI
NARODOWA STRATEGIA SPÓJNOŚCI

UNIA EUROPEJSKA
EUROPEJSKI
FUNDUSZ SPOŁECZNY

DYNAMICS OF SMALL (ACOUSTIC) DISTURBANCES

Consider nonstationary motion in 1D. We have

- mass conservation equation $\partial_t \rho + \rho \partial_x u + u \partial_x \rho = 0$
- Euler equation $\rho \left(\partial_t u + u \partial_x u \right) = -\partial_x p$

Assume that the flow is smooth. Then, the energy equation can be replaced by the **isentropic condition** $S = const$.

Consider the **First Principle of Thermodynamics** written in the following form

$$TdS = \underbrace{c_v dT}_{dU - \text{int.} energ.} + p \underbrace{d(1/\rho)}_{d\vartheta - spec.vol.}$$

The differential of (mass-specific) entropy can be expressed as follows

$$dS = \frac{c_v}{T} dT - \frac{p}{T\rho^2} d\rho \underset{\substack{Clapeyron \\ equation}}{=} \frac{c_v}{T} dT - \frac{R}{\rho} d\rho = \frac{c_v}{T} dT - \frac{(\kappa - 1) c_v}{\rho} d\rho$$

Using the **Clapeyron equation** can write

$$\frac{1}{T}dT = \frac{1}{T}d\left(\frac{p}{R\rho}\right) = \frac{1}{TR}\left(\frac{dp}{\rho} - \frac{p}{\rho^2}d\rho\right) = \frac{dp}{p} - \frac{d\rho}{\rho}$$

Thus
$$dS = \frac{c_v}{p}dp - \frac{\kappa c_v}{\rho}d\rho = \frac{c_v}{p}dp - \frac{c_p}{\rho}d\rho$$

Since the flow is isentropic, we have $dS = 0$, hence

$$\frac{dp}{p} = \kappa\frac{d\rho}{\rho} \Rightarrow \left.\frac{dp}{d\rho}\right|_{S=const} = \kappa\frac{p}{\rho} = \kappa RT \geq 0$$

We see that the **flow is barotropic** and the derivative of the pressure as the function of density is always **nonnegative function**. We can introduce the quantity a defined as $a = \sqrt{\kappa RT}$

Then
$$\left.\frac{dp}{d\rho}\right|_{S=const} = a^2.$$

Note that the **physical unit of a is [m/s]**, so a seems to be the velocity of „something". But what is this "something"?. To figure it out, consider the **motion of weak (or small) disturbances** in the motionless gas.

The fields of density, pressure and velocity can be written as the sums of undisturbed (background) values and disturbances denoted by the "primed" symbols.

$$\rho = \rho_0 + \rho' \ , \ p = p_0 + p' \ , \ u = u_0 + u' = u'$$

Since disturbances are assumed small, the nonlinear (product) terms can be neglected. In effect, we get linearized equations as follows

$$\begin{cases} \partial_t \rho' + \rho_0 \partial_x u' = 0 \\ \rho_0 \partial_t u' + \partial_x p' = 0 \end{cases}$$

The key point is to express the pressure disturbances by means of the density disturbances (it is possible as the flow is barotropic). To this end we write

$$p = p_0 + p' = p(\rho_0 + \rho') \cong \underbrace{p(\rho_0)}_{p_0} + \left.\frac{dp}{d\rho}\right|_{\rho=\rho_0} \rho'$$

Thus

$$p' = \left.\frac{dp}{d\rho}\right|_{\rho=\rho_0} \rho' = \kappa R T_0 \rho' = a_0^2 \rho'$$

The next step is to differentiate the linearized mass conservation equation with respect to time and the equation of motion with respect to the spatial variable x. Then we subtract the second equation from the first one. The results reads

$$\partial_{tt}\rho' - a_0^2 \partial_{xx}\rho' = 0$$

Similar procedure (provide details!) leads to the formally identical PDE for the velocity disturbances.

$$\partial_{tt}u' - a_0^2 \partial_{xx}u' = 0$$

We see that the **spatio-temporal dynamics of the density and velocity disturbances (also pressure - show!) is governed by the linear wave equation**.

We know from the analysis that the **general solution the wave equation** (for the unbounded domain, i.e. the whole line) can be written in the following form

$$\rho'(t,x) = F_1(x - a_0 t) + F_2(x + a_0 t)$$

The functions F_1 and F_2 are arbitrary. The **physical interpretation** is straightforward: the solution (in 1D) is the superposition of two (arbitrary shaped) wave forms, moving with the constant velocity a_0 in the positive and negative directions of the *x*-axis, respectively.

In general 3D case, the linear wave equation takes the form of

$$\partial_{tt}\rho' - a_0^2 \nabla^2 \rho' = 0 \quad \text{where} \quad \nabla^2 = \partial_{xx} + \partial_{yy} + \partial_{zz} \quad \text{(scalar Laplace operator)}$$

We see that **small disturbances travel through the gas in the form of linear waves** (of small amplitudes). The speed of the wave is equal

$$a_0 = \sqrt{\kappa R T_0}$$

where T_0 is the background temperature. Such small (linear) disturbances are called **acoustic** ones and they represent the **sound waves**. The velocity of this waves is called the **speed of sound**.

Note:

- in general situation, the speed of sound is different at different points in space (and – possibly – at different time instants),
- the local speed of the gas can be either smaller (subsonic conditions), or equal (sonic or critical conditions) or larger (supersonic conditions) that the local speed of sound.
- The motion of ideal gas **need not to be spatially continuous**! Large disturbances have nonlinear dynamics and they can beveloped into strong discontinuities called the **shock waves**. **The flow across the shock wave is not isentropic!**

ENERGY INTEGRAL. ISENTROPIC RELATIONS

The energy integral can be written as

$$i + \tfrac{1}{2}\upsilon^2 = const$$

The mass-specific enthalpy can be expressed in several forms

$$i = c_p T = \tfrac{\kappa}{\kappa - 1} RT = \tfrac{\kappa}{\kappa - 1} p/\rho = \tfrac{1}{\kappa - 1} a^2$$

Mach number: $M = \dfrac{V}{a}$

Stagnation parameter: the parameter's value at such point where $\upsilon = 0$; e.g. T_0

Critical parameter: the parameter's value at such point where $\upsilon = a$ $(M = 1)$; e.g. T_*

Equivalent forms of the energy equation

$$c_p T + \tfrac{1}{2}\upsilon^2 = c_p T_0$$

$$\frac{\kappa p}{(\kappa-1)\rho} + \tfrac{1}{2}\upsilon^2 = \frac{\kappa p_0}{(\kappa-1)\rho_0}$$

$$\frac{a^2}{\kappa-1} + \tfrac{1}{2}\upsilon^2 = \frac{a_0^2}{\kappa-1} = \frac{\kappa+1}{2(\kappa-1)} a_*^2$$

Maximal velocity $(T \to 0)$

$$c_p T + \tfrac{1}{2}\upsilon^2 = c_p T_0 = \tfrac{1}{2}\upsilon_{\max}^2 \;\Rightarrow\; \upsilon_{\max} = \sqrt{2 c_p T_0}$$

$$1 + \frac{\upsilon^2}{2 c_p T} = \frac{T_0}{T} \;\Rightarrow\; 1 + \frac{\upsilon^2}{\frac{2}{\kappa-1} a^2} = \frac{T_0}{T} \;\Rightarrow\; 1 + \frac{\kappa-1}{2} M^2 = \frac{T_0}{T}$$

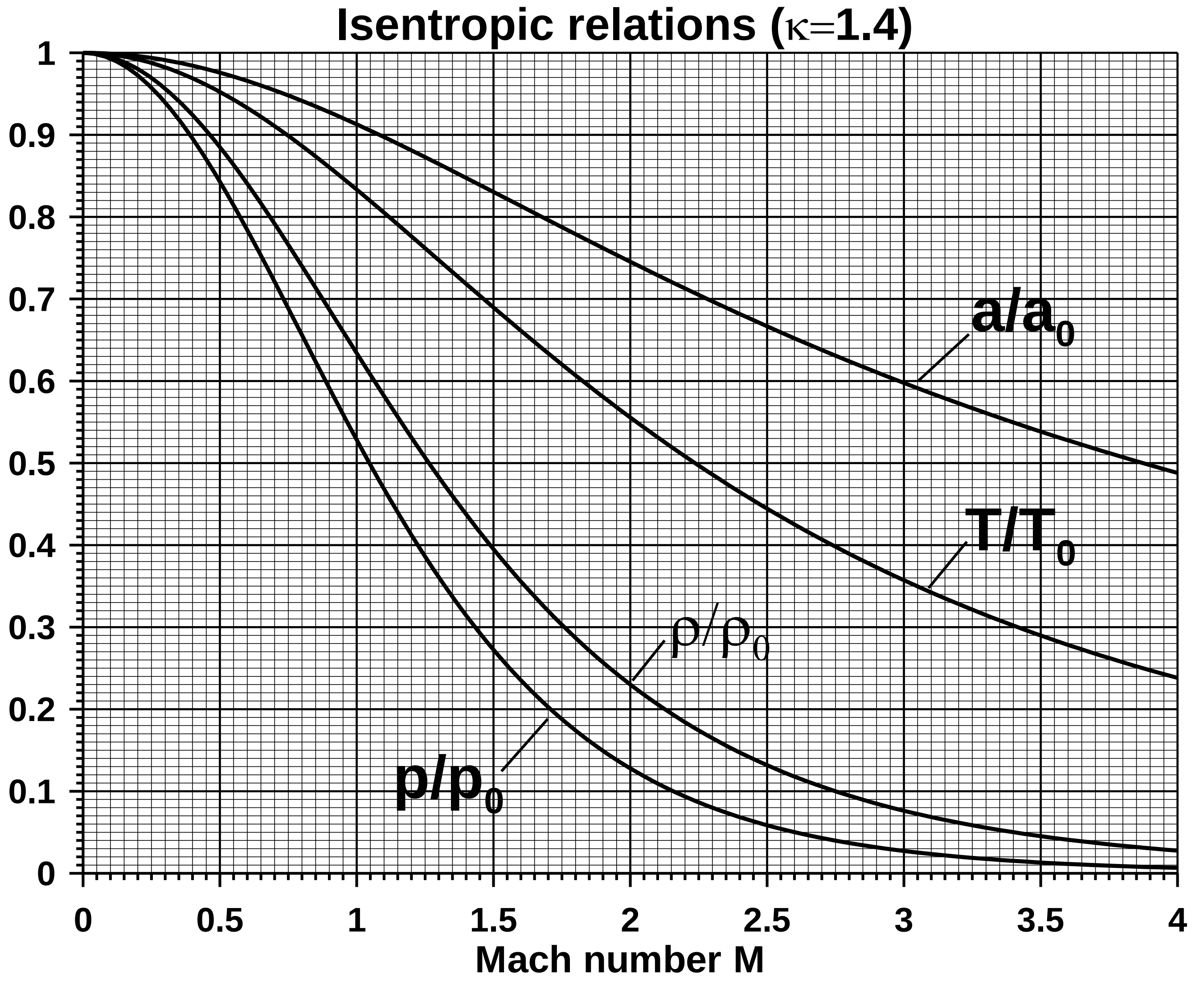
We have $\dfrac{T}{T_0}(M) = \left(1 + \dfrac{\kappa-1}{2} M^2\right)^{-1}$ (**true whenever total energy is conserved!**)

If the **flow is also isentropic**, we have $p/\rho^{\kappa} = const$ and $p = \rho RT$. Then

$$\frac{\rho}{\rho_0}(M) = \left(1 + \frac{\kappa - 1}{2} M^2\right)^{\frac{1}{1-\kappa}}$$

$$\frac{p}{p_0}(M) = \left(1 + \frac{\kappa - 1}{2} M^2\right)^{\frac{\kappa}{1-\kappa}}$$

$$\frac{a}{a_0}(M) = \left(1 + \frac{\kappa - 1}{2} M^2\right)^{-\frac{1}{2}}$$



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