Basic flow equations – based on general conservation principles:

- **mass conservation principle**
- **momentum conservation principle**
- **energy conservation principle (first thermodynamics law)**

and

- **second thermodynamics law (entropy rise)**

and can be formulated in various, mathematically equivalent manners. In the case of continuum we can consider balance of any (scalar) quantity per unit volume ***U*** (density, momentum component, energy, enthalpy, ...) in an arbitrary, fixed in space volume Ω having a closed boundary $\delta\Omega$.

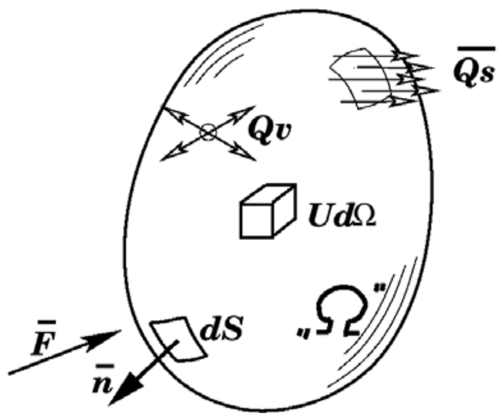


Fig. 2.1 Control volume to get conservation law for a quantity ***U***

Total amount of quantity ***U*** enclosed inside considered control volume is $\iiint_{\Omega} U \cdot d\Omega$ and varies in time due to:

- **inflow through the boundary from the surrounding (flux effect)**
- **due to sources of a quantity *U* inside control volume or on their boundary**

The flux vector of a quantity $\boldsymbol{U}$ can be expressed as a sum of convective flux (transportation of a quantity $\boldsymbol{U}$ with the motion velocity $\vec{\boldsymbol{V}}$):

$$\vec{\boldsymbol{F}}_{\boldsymbol{C}} = \vec{\boldsymbol{V}} \cdot \boldsymbol{U}$$

and diffusive flux, (due to the smoothing tendency in any non-homogeneous distribution of intensive quantity):

$$\vec{\boldsymbol{F}}_{\boldsymbol{D}} = -\kappa \cdot \rho \cdot \nabla(\boldsymbol{U} / \rho),$$

where κ - diffusivity constant.

Variation of a quantity $\boldsymbol{U}$ within the control volume Ω due to nonzero normal component of a flux:

$$-\oint\!\!\oint_{\partial\Omega} \vec{F} \cdot \vec{n} dS$$

and due to volume and surface sources: $\iiint_{\Omega} Q_V d\Omega + \oint\!\!\oint_{\partial\Omega} \vec{Q}_S \cdot \vec{n} dS$.

As a result general form of the conservation equation for the quantity $\boldsymbol{U}$ is:

$$\frac{\partial}{\partial \boldsymbol{t}} \iiint_{\Omega} \boldsymbol{U} d\Omega = -\oint\!\!\oint_{\partial\Omega} \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{n}} d\boldsymbol{S} + \iiint_{\Omega} \boldsymbol{Q}_V d\Omega + \oint\!\!\oint_{\partial\Omega} \vec{\boldsymbol{Q}}_{\boldsymbol{S}} \cdot \vec{\boldsymbol{n}} d\boldsymbol{S} \qquad (2.1a)$$

or:

$$\frac{\partial}{\partial t}\iiint_{\Omega} U d\Omega + \oiint_{\partial\Omega} \vec{F} \cdot \vec{n} dS = \iiint_{\Omega} Q_V d\Omega + \oiint_{\partial\Omega} \vec{Q}_S \cdot \vec{n} dS \tag{2.1a}$$

Applying Gauss (divergence) theorem for fluxes and surface sources we can get::

$$\iiint_{\Omega} \left(\frac{\partial U}{\partial t} + \nabla \cdot \vec{F} - Q_V - \nabla \cdot \vec{Q}_S \right) d\Omega = 0$$

Since the above equation must be true for any volume choice, the integrand must be equal zero, leading to differential form of the conservation law:

$$\frac{\partial \boldsymbol{U}}{\partial \boldsymbol{t}} + \nabla \cdot \left(\vec{\boldsymbol{F}} - \vec{\boldsymbol{Q}}_S \right) = \boldsymbol{Q}_V \tag{2.1b}$$

Equation (2.1a) is a general form of the conservation law in the integral form, applicable to any (fixed in time) control volume. Equation (2.1b) expresses the same conservation law in the differential form. Mathematically they are equivalent and can be used as a basis for various computational methods.

Mass conservation law

The quantity $\boldsymbol{U}$ is, in this case, the specific mass – density. $\boldsymbol{\rho}$. Equations (2.1a) and (2.1b) express the mass conservation law – known as continuity equation. There is no diffusive mass flux nor mass sources. Finally equations took the form:

$$\frac{\partial}{\partial t}\iiint_{\Omega}\rho d\Omega+\oiint_{\partial\Omega}\left(\rho\vec{V}\right)\cdot\vec{n}dS=0 \qquad (2.2a)$$

$$\frac{\partial\rho}{\partial t}+\nabla\cdot\left(\rho\vec{V}\right)=0 \qquad (2.2b)$$

Momentum conservation law

In this case under $\boldsymbol{U}$ we put $\rho\vec{V}$ (momentum for unit volume). Equations (2.1a) and (2.1b) express momentum conservation law. We must add expressions for volume and surface momentum sources. Surface momentum sources are: normal forces (pressure) and tangent stresses (due to viscosity). They are second order tensors:

$$\overline{\overline{\boldsymbol{Q}_S}} = \overline{\overline{\boldsymbol{\sigma}}} = -p\overline{\overline{\boldsymbol{I}}} + \overline{\overline{\boldsymbol{\tau}}}$$

$\overline{\overline{\boldsymbol{I}}}$ is unit tensor and tangent stress tensor for newtonian fluid is equal:

$$\tau_{ij} = \mu\left(\frac{\partial \boldsymbol{u}_i}{\partial \boldsymbol{x}_j} + \frac{\partial \boldsymbol{u}_j}{\partial \boldsymbol{x}_i}\right) - \frac{2}{3}\mu\frac{\partial \boldsymbol{u}_k}{\partial \boldsymbol{x}_k}\delta_{ij}$$

Volume momentum sources are mass forces (e.g. gravity) $\boldsymbol{Q}_V = \rho \cdot \vec{\boldsymbol{f}}$. As a result equations take the form:

$$\frac{\partial}{\partial t}\iiint_\Omega \left(\rho\vec{V}\right)d\Omega + \oiint_{\partial\Omega}\left(\rho\vec{V}\vec{V}\right)\cdot \vec{n}dS = \iiint_\Omega\left(\rho\vec{f}\right)d\Omega - \oiint_{\partial\Omega} p\vec{n}dS + \oiint_{\partial\Omega}\overline{\overline{\tau}}\cdot\vec{n}dS \tag{2.3a}$$

$$\frac{\partial\left(\rho\vec{V}\right)}{\partial t} + \nabla\cdot\left(\rho\vec{V}\vec{V}\right) = \rho\vec{f} - \nabla p + \nabla\cdot\overline{\overline{\tau}} \tag{2.3b}$$

known as Navier-Stokes equations.

Energy conservation law

In this case under quantity $\boldsymbol{U}$ we put product of density ρ and total energy $E = e + \left(\vec{V}\right)^2 / 2$:total energy for unit volume). Equations (2.1a) i (2.1b) express energy conservation law. The surface energy sources are the result of work of shear and normal stresses acting on the surface:

$$\overline{Q_S} = \left(-p\bar{\bar{I}} + \bar{\bar{\tau}}\right) \cdot \vec{v}$$

Volume energy sources are work of the volume forces and other heat sources (due to radiation, chemical reactions):

$$Q_V = q + \rho\vec{f} \cdot \vec{V} \; .$$

There are also diffusive heqt flux due to heat conduction:

$$\vec{\boldsymbol{F}}_{\boldsymbol{D}} = -\boldsymbol{k}\nabla \boldsymbol{T} \; ,$$

(***k*** is the thermal conductivity coefficient).

Equation (2.1a) takes the form:

$$\frac{\partial}{\partial t}\iiint_\Omega (\rho E) d\Omega + \oiint_{\partial\Omega} \rho E\vec{V} \cdot \vec{n} dS =$$
$$\iiint_\Omega \left(q + \rho\vec{f} \cdot \vec{V}\right) d\Omega + \oiint_{\partial\Omega} k\nabla T \cdot \vec{n} dS + \oiint_{\partial\Omega} \left(\left(-p\bar{\bar{I}} + \bar{\bar{\tau}}\right) \cdot \vec{V}\right) \cdot \vec{n} dS$$

or

$$\frac{\partial}{\partial t}\iiint_\Omega (\rho E) d\Omega + \oiint_{\partial\Omega} \rho H\vec{V} \cdot \vec{n} dS =$$
$$\iiint_\Omega \left(q + \rho\vec{f} \cdot \vec{V}\right) d\Omega + \oiint_{\partial\Omega} k\nabla T \cdot \vec{n} dS + \oiint_{\partial\Omega} \left(\bar{\bar{\tau}} \cdot \vec{V}\right) \cdot \vec{n} dS \qquad (2.4a)$$

Equation (2.1a) takes the form:

$$\frac{\partial}{\partial t}\iiint_{\Omega}(\rho E)d\Omega+\oiint_{\partial\Omega}\rho E\vec{V}\cdot\vec{n}dS=$$

$$\iiint_{\Omega}\left(q+\rho\vec{f}\cdot\vec{V}\right)d\Omega+\oiint_{\partial\Omega}k\nabla T\cdot\vec{n}dS+\oiint_{\partial\Omega}\left(\left(-p\bar{\bar{I}}+\bar{\bar{\tau}}\right)\cdot\vec{V}\right)\cdot\vec{n}dS$$

or

$$\frac{\partial}{\partial t}\iiint_{\Omega}(\rho E)d\Omega+\oiint_{\partial\Omega}\rho H\vec{V}\cdot\vec{n}dS=$$

$$\iiint_{\Omega}\left(q+\rho\vec{f}\cdot\vec{v}\right)d\Omega+\oiint_{\partial\Omega}k\nabla T\cdot\vec{n}dS+\oiint_{\partial\Omega}\left(\bar{\bar{\tau}}\cdot\vec{v}\right)\cdot\vec{n}dS \quad (2.4a)$$

and equation (2.1.b):

$$\frac{\partial(\rho E)}{\partial t}+\nabla\cdot\left(\rho E\vec{V}\right)=q+\rho\vec{f}\cdot\vec{V}+\nabla\cdot\left(k\nabla T\right)-\nabla\cdot(p\vec{V})+\nabla\cdot\left(\bar{\bar{\tau}}\cdot\vec{V}\right)$$

or

$$\frac{\partial(\rho E)}{\partial t}+\nabla\cdot\left(\rho H\vec{V}\right)=q+\rho\vec{f}\cdot\vec{V}+\nabla\cdot\left(k\nabla T\right)+\nabla\cdot\left(\bar{\bar{\tau}}\cdot\vec{V}\right) \quad (2.4b)$$

To close above equations we must add extra relations, expressing internal energy, density, viscosity and thermal conductivity:

$$e=e(T,p)$$
$$\rho=\rho(T,p)$$
$$\mu=\mu(T,p)$$
$$k=k(T,p)$$

and eventually mass force field ***f*** and heat sources ***q***.
Internal energy for the perfect gas is equal to:

$$e = C_v T = \frac{1}{\kappa - 1} RT = \frac{a^2}{\kappa(\kappa - 1)}$$

density:

$$\rho = \frac{p}{RT} = \frac{\kappa p}{a^2}$$

Dynamic viscosity coefficient is a fluid property and depend mainly on temperature. The thermal conductivity coefficient of a gas is closely related to dynamic viscosity:

$$k = \frac{\mu \cdot C_p}{\mathrm{Pr}}$$

Pr is Prandtl number.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \vec{V} \right) = 0$$

$$\vec{V} \cdot \nabla \rho + \rho \nabla \cdot \vec{V}$$

$$\underbrace{\frac{\partial \rho}{\partial t} + \vec{V} \cdot \nabla \rho}_{d\rho / dt} + \rho \nabla \cdot \vec{V} = 0$$

$$d\rho / dt + \rho \nabla \cdot \vec{V} = 0$$

$$\frac{\partial\left(\rho\vec{V}\right)}{\partial t}+\nabla\cdot\left(\rho\vec{V}\vec{V}\right)=\rho\vec{f}-\nabla p+\nabla\cdot\overline{\overline{\tau}}$$

$$\rho\frac{\partial\vec{V}}{\partial t}+\frac{\partial\rho}{\partial t}\vec{V} \qquad\qquad \nabla\cdot\left(\rho\vec{V}\right)\vec{V}+\left(\rho\vec{V}\cdot\nabla\right)\vec{V}$$

$$\rho\frac{\partial\vec{V}}{\partial t}+\rho\left(\vec{V}\cdot\nabla\right)\vec{V}+\cancel{\vec{v}\left(\frac{\partial\rho}{\partial t}+\nabla\cdot\left(\rho\vec{V}\right)\right)}=\rho\vec{f}-\nabla p+\nabla\cdot\overline{\overline{\tau}}$$

$$\rho\underbrace{\left(\frac{\partial\vec{\boldsymbol{V}}}{\partial \boldsymbol{t}}+\left(\vec{\boldsymbol{V}}\cdot\nabla\right)\vec{\boldsymbol{V}}\right)}_{\boldsymbol{d\overline{v}/dt}}=\rho\vec{\boldsymbol{f}}-\nabla\boldsymbol{p}+\nabla\cdot\overline{\overline{\boldsymbol{\tau}}}$$

$$\frac{d\vec{V}}{dt}=\vec{f}-\frac{1}{\rho}\nabla p+\frac{1}{\rho}\nabla\cdot\overline{\overline{\tau}}$$

$$\frac{\partial(\rho E)}{\partial t}+\nabla\cdot\left(\rho H\vec{V}\right)=q+\rho\vec{f}\cdot\vec{V}+\nabla\cdot\left(k\nabla T\right)+\nabla\cdot\left(\overline{\overline{\tau}}\cdot\vec{V}\right)$$

$$E = (\boldsymbol{H}-\boldsymbol{p}/\rho)$$

……

$$\rho\underbrace{\left(\frac{\partial \boldsymbol{H}}{\partial \boldsymbol{t}}+\left(\vec{\boldsymbol{V}}\cdot\nabla\right)\boldsymbol{H}\right)}_{\boldsymbol{dH}/\boldsymbol{dt}}=\frac{\partial \boldsymbol{p}}{\partial \boldsymbol{t}}+\boldsymbol{q}+\rho\vec{\boldsymbol{f}}\cdot\vec{\boldsymbol{V}}+\nabla\cdot\left(\boldsymbol{k}\nabla\boldsymbol{T}\right)+\nabla\cdot\left(\overline{\overline{\tau}}\cdot\vec{\boldsymbol{V}}\right)$$

$$\rho\frac{dH}{dt}=\frac{\partial p}{\partial t}+q+\rho\vec{f}\cdot\vec{V}+\nabla\cdot\left(k\nabla T\right)+\nabla\cdot\left(\overline{\overline{\tau}}\cdot\vec{V}\right)$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \vec{V} \right) = 0$$

$$\rho \left(\frac{\partial \overline{v}}{\partial t} + \left(\vec{V} \cdot \nabla \right) \vec{v} \right) = \rho \vec{f} - \nabla p + \nabla \cdot \overline{\overline{\tau}}$$

$$\rho \left(\frac{\partial H}{\partial t} + \left(\vec{v} \cdot \nabla \right) H \right) = \frac{\partial p}{\partial t} + q + \rho \vec{f} \cdot \vec{v} + \nabla \cdot \left(k \nabla T \right) + \nabla \cdot \left(\overline{\overline{\tau}} \cdot \vec{v} \right)$$

$$\tau_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \frac{2}{3} \mu \frac{\partial u_k}{\partial x_k} \delta_{ij}$$

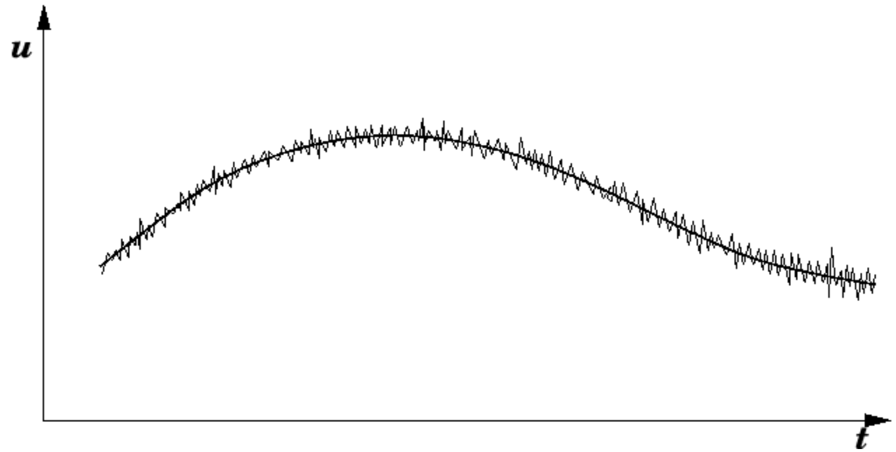
boundary conditions

second law of thermodynamics (? !)

SOLUTION

turbulence:

- **phenomena of chaotic fluid motion, variable in time and space (always three-dimensional) with characteristic chaotic changes of all flow parameters, but it is possible to specify some statistical mean values.**
- **there occur relocation (mixing) of entire [small] fluid mass elements in a chaotic manner, in contrast to relocation of single molecules in laminar flow**
- **typical at high Reynolds numbers – characteristic for aeronautics**



Prędkość średnia prędkość poboczna

$$u_{(t)} = \bar{u}_{(t)} + u'_{(t)}$$

$$p_{(t)} = \bar{p}_{(t)} + p'_{(t)}$$

$$\rho_{(t)} = \bar{\rho}_{(t)} + \rho'_{(t)}$$

$$\bar{u}_{(t)} = \frac{1}{T} \int_{t}^{t+T} u_{(\tau)} d\tau$$

$$\overline{u'} = \frac{1}{T}\int_{t}^{t+T} u'_{(\tau)}\,d\tau = 0$$

$$u'_{RMS} = \sqrt{\frac{1}{T}\int_{0}^{T}\left(u'_{(\tau)}\right)^2 d\tau}$$

$$u' = \sqrt{\frac{1}{3}\left(u_x'^{\,2} + u_y'^{\,2} + u_z'^{\,2}\right)} = \sqrt{\frac{2}{3}\left(\frac{u_x'^{\,2} + u_y'^{\,2} + u_z'^{\,2}}{2}\right)} = \sqrt{\frac{2}{3}k}$$

$$I = \frac{u'_{RMS}}{\overline{u}}$$

incompressible flow (ρ=*const*), averaged x-momentum equation:

$$\frac{\partial\left(\overline{\rho(\overline{u}+u')}\right)}{\partial t}+\frac{\partial\left(\overline{\rho(\overline{u}+u')(\overline{u}+u')}\right)}{\partial x}+\cdots=-\frac{\partial\left(\overline{\overline{p}+p'}\right)}{\partial x}+\mu\Delta\overline{(\overline{u}+u')}$$

$$\left(\overline{(\overline{u}+u')(\overline{v}+v')}\right)=\frac{1}{T}\int_{t}^{t+T}\left(\overline{u}\,\overline{v}+\overline{u}\,v'+\overline{v}\,u'+u'\,v'\right)d\tau=$$

$$\overline{u}\,\overline{v}+\overline{u}\,\overline{v'}+\overline{v}\,\overline{u'}+\overline{u'\,v'}=\overline{u}\,\overline{v}+\overline{u'\,v'}$$

finally:

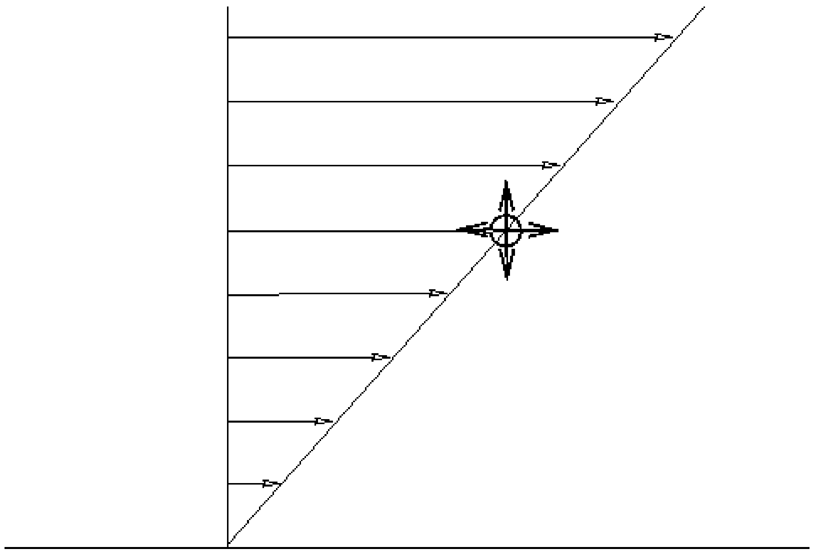
$$\rho\frac{\partial\overline{u}}{\partial t}+\rho\overline{u}\frac{\partial\overline{u}}{\partial x}+\rho\overline{v}\frac{\partial\overline{u}}{\partial y}+\rho\overline{w}\frac{\partial\overline{u}}{\partial z}=-\frac{\partial\overline{p}}{\partial x}+$$

$$\mu\Delta\overline{u}-\rho\frac{\partial\left(\overline{u'\,u'}\right)}{\partial x}-\rho\frac{\partial\left(\overline{u'\,v'}\right)}{\partial y}-\rho\frac{\partial\left(\overline{u'\,w'}\right)}{\partial z}$$

Reynolds Averaged Navier-Stokes Equations (RANS)

$$\tau_{ij} = \underbrace{\mu\left(\frac{\partial \overline{\boldsymbol{u}}_i}{\partial \boldsymbol{x}_j} + \frac{\partial \overline{\boldsymbol{u}}_j}{\partial \boldsymbol{x}_i}\right) - \frac{2}{3}\mu\frac{\partial \overline{\boldsymbol{u}}_k}{\partial \boldsymbol{x}_k}\delta_{ij}}_{lam} \quad \underbrace{-\rho\overline{\boldsymbol{u}'\boldsymbol{u}'} - \rho\overline{\boldsymbol{u}'\boldsymbol{v}'} - \rho\overline{\boldsymbol{u}'\boldsymbol{w}'}}_{turb}$$

concept of **turbulent viscosity**

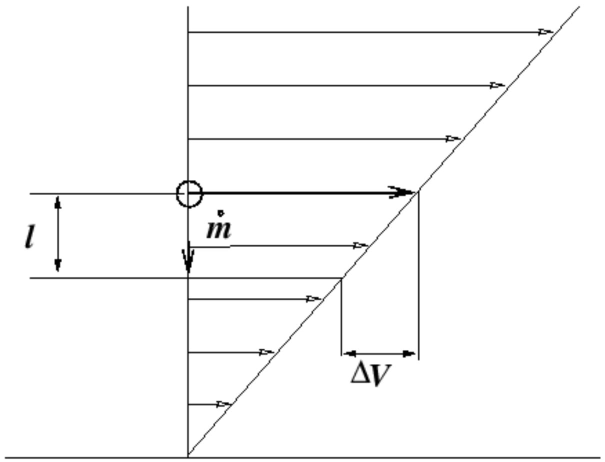


$$\tau_{xz} = \mu\frac{\partial \overline{\boldsymbol{u}}}{\partial z} \quad -\rho\overline{\boldsymbol{u}'\boldsymbol{w}'}$$

$$\Downarrow$$

$$\mu_T \frac{\partial \overline{\boldsymbol{u}}}{\partial z}$$

$$\tau_{xz} = (\mu + \mu_T)\frac{\partial \overline{\boldsymbol{u}}}{\partial z}$$



mixing length

$$\tau_T = \mathring{m}\cdot\underbrace{\Delta n\cdot\frac{\partial\bar{u}}{\partial n}}_{\Delta u} = \rho\cdot\underbrace{v_T}_{l\cdot\left|\frac{\partial\bar{u}}{\partial n}\right|}\cdot l\cdot\frac{\partial\bar{u}}{\partial n} = \underbrace{\rho\cdot l^2\cdot\left|\frac{\partial\bar{u}}{\partial n}\right|}_{\mu_T}\cdot\frac{\partial\bar{u}}{\partial n}$$

MOMENTUM LOSS ~ $K_1\cdot V_{ŚR}$ + $K_2\cdot V_{ŚR}{}^2$

laminar flow turbulent flow (fully developed)

Aerodynamics consider usually **steady flow equations, including turbulence via turbulent stresses**.

$$\frac{\partial \rho}{\partial t} = 0\,; \quad \frac{\partial \vec{V}}{\partial t} = 0\,; \quad \frac{\partial H}{\partial t} = 0\,; \quad \frac{\partial p}{\partial t} = 0\,;$$

$$\mu = \mu + \mu_T = \mu + \mu_\varepsilon$$

$$\nabla \cdot \left(\rho \vec{V}\right) = 0$$

$$\rho\left(\vec{V} \cdot \nabla\right)\vec{V} = -\nabla p + \nabla \cdot \overline{\overline{\tau}}$$

$$\rho\left(\vec{V} \cdot \nabla\right)H = q + \nabla \cdot \left(k \nabla T\right) + \nabla \cdot \left(\overline{\overline{\tau}} \cdot \vec{V}\right)$$

in a case of barotropic fluid $\rho = \rho(\boldsymbol{p})$ → ρ, μ do not depend (directly) on ***e*** ***(T)***

$$\nabla \cdot \left(\rho \vec{V}\right) = 0$$

$$\rho\left(\vec{V} \cdot \nabla\right)\vec{V} = -\nabla p + \nabla \cdot \overline{\overline{\tau}}$$

$\boldsymbol{p / \rho^{k} = const}$ **lub** $\rho = \rho_{\infty} = \boldsymbol{const}$ (instead of energy)

(isentropic) (incompressible)

Incompressible, viscous flow

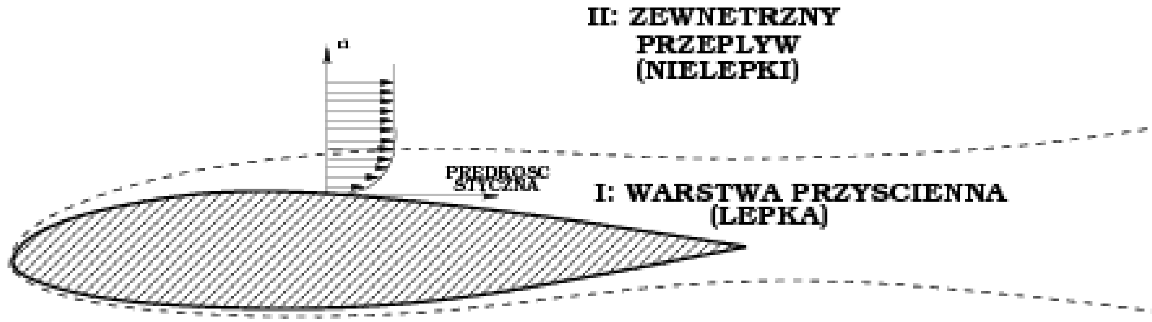
$$\nabla \cdot \vec{V} = 0$$

$$\left(\vec{V} \cdot \nabla\right)\vec{V} = -\frac{1}{\rho_\infty}\nabla p + \frac{1}{\rho_\infty}\nabla \cdot \overline{\overline{\tau}}$$

Navier-Stokes equations can be expressed in a nondimensional form:

$$\frac{d\vec{V}}{dt} + \frac{\nabla p}{\rho} = \frac{1}{\mathrm{Re}} \Delta \vec{V} \tag{2.5}$$

which suggest, that at **high Reynolds numbers** viscous term is very small, and could be neglected. In reality viscous terms are very important inside thin flow layer just near body surface: inside **boundary** layer. Outside this layer flow can be treated as inviscid.



Rys.2.2 Characteristic flow regions around body (Prandtla concept)

Concept of BOUNDARY LAYER (Prandtl)

Navier-Stokes Eq. For incompressible flow:

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

momentum in x-direction

$$\rho u \frac{\partial v}{\partial x} + \rho v \frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

momentum in y-direction

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

continuity eq.

$$0 \leq x \leq c$$

$$0 \leq y \leq \delta \qquad \underline{\delta \ll c}$$

$$u \sim O(U_\infty)$$

$$\frac{\partial u}{\partial x} \sim O\left(\frac{U_\infty}{c}\right) \qquad \frac{\partial u}{\partial y} \sim O\left(\frac{U_\infty}{\delta}\right)$$

$$\frac{\partial^2 u}{\partial x^2} \sim O\left(\frac{U_\infty}{c^2}\right) \qquad \frac{\partial^2 u}{\partial y^2} \sim O\left(\frac{U_\infty}{\delta^2}\right)$$

$$\Downarrow$$

$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} \sim O\left(\frac{U_\infty}{c}\right) \qquad v = \int_0^y \frac{\partial v}{\partial y} dy = -\int_0^y \frac{\partial u}{\partial x} dy \sim O\left(\frac{U_\infty \delta}{c}\right)$$

$$u\frac{\partial u}{\partial x} \sim O\left(\frac{U_\infty^2}{c}\right) \qquad v\frac{\partial u}{\partial y} \sim O\left(\frac{U_\infty \delta}{c} \cdot \frac{U_\infty}{\delta}\right) \sim O\left(\frac{U_\infty^2}{c}\right)$$

$$\rho u\frac{\partial u}{\partial x}+\rho v\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}+\mu\frac{\partial^2 u}{\partial y^2}+\mu\cancel{\frac{\partial^2 u}{\partial x^2}}$$

mass forces and friction forces MUST be of the same order:

<u>mass forces</u>

$$x: \quad u\frac{\partial u}{\partial x},\ v\frac{\partial u}{\partial y} \sim O\left(\frac{U_\infty^2}{c}\right)$$

$$y: \quad u\frac{\partial v}{\partial x},\ v\frac{\partial v}{\partial y} \sim O\left(\frac{U_\infty^2}{c}\cdot\frac{\delta}{c}\right)$$

<u>friction forces</u>

$$x: \quad \frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2} \sim O\left(\nu\frac{U_\infty}{\delta^2}\right)$$

$$y: \quad \frac{\mu}{\rho}\frac{\partial^2 v}{\partial y^2} \sim O\left(\nu\frac{U_\infty}{\delta^2}\cdot\frac{\delta}{c}\right)$$

in effect (in a case of laminar boundary layer):

$$x: \qquad \nu\frac{U_\infty}{\delta^2} \sim \frac{U_\infty^2}{c} \quad\Rightarrow\quad \frac{\delta}{c} \sim 1/\sqrt{U_\infty\cdot c/\nu} = 1/\sqrt{\mathrm{Re}}$$

all terms in $\boldsymbol{y}$ –momentum eq.:

$$\boldsymbol{y}: \quad \boldsymbol{u}\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}}, \ \boldsymbol{v}\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{y}}, \ \frac{\mu}{\rho}\frac{\partial^2 \boldsymbol{v}}{\partial \boldsymbol{x}^2} \sim \boldsymbol{O}\left(\frac{\boldsymbol{U}_\infty^2}{\boldsymbol{c}} \cdot \frac{\delta}{\boldsymbol{c}}\right)$$

are smaller then in $\boldsymbol{x}$ –momentum eq., so:

$$\frac{\partial \boldsymbol{p}}{\partial \boldsymbol{y}} \sim \boldsymbol{O}\left(\frac{\boldsymbol{U}_\infty^2}{\mathrm{c}} \cdot \frac{\delta}{\boldsymbol{c}}\right) \qquad \textbf{:}\,\text{small}$$

or: $$\frac{\partial \boldsymbol{p}}{\partial \boldsymbol{y}} = \boldsymbol{0} \quad \textbf{!!!}$$

so: $\boldsymbol{p}, \ \dfrac{\partial \boldsymbol{p}}{\partial \boldsymbol{x}}$ constant in $\boldsymbol{y}$-direction inside B.L.

one-dimensional Euler eq.:

$$\rho u \frac{\partial u}{\partial x} = -\frac{\partial p}{\partial x}$$

so for $y = \delta$ $\quad u = u|_{\delta} = u_e$

$$\rho u_e \frac{\partial u_e}{\partial x} = -\frac{\partial p}{\partial x}$$

finally two-dimensional, incompressible boundary layer eq. (Prandtl Eq.):

x-momentu eq.:

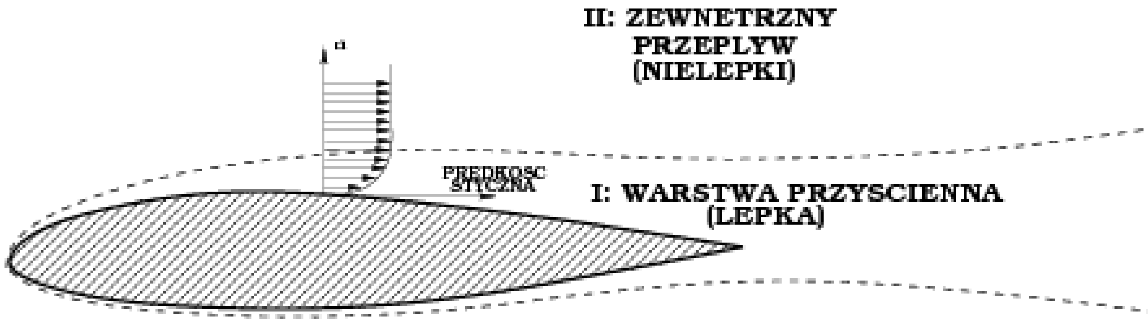
$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \rho_e u_e \frac{du_e}{dx} + \mu \frac{\partial^2 u}{\partial y^2}$$

y-momentum eq.:

$$\frac{\partial p}{\partial y} = 0, \quad p(y) = const$$

continuity eq.:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$



Removing less important terms in N-S eq. leads to the boundary layer eq. (Prandtl equations). In a case of two-dimensional flow momentum eq. take the following form (along surface and in normal direction):

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right)$$

$$p(y) = const \tag{2.6}$$

Outside boundary layer it is assumed that the flow is governed by inviscid equations: (**Euler equations**):

$$\rho\left(\vec{V} \cdot \nabla\right)\vec{V} = -\nabla p \tag{2.7}$$

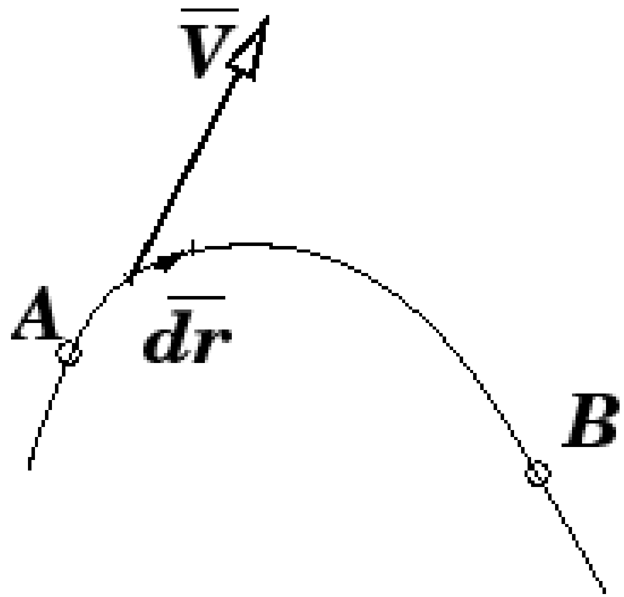
Both solutions must be joint together at a boundary layer edge.

Incompressible, inviscid flow:

$$\nabla \cdot \vec{V} = 0$$

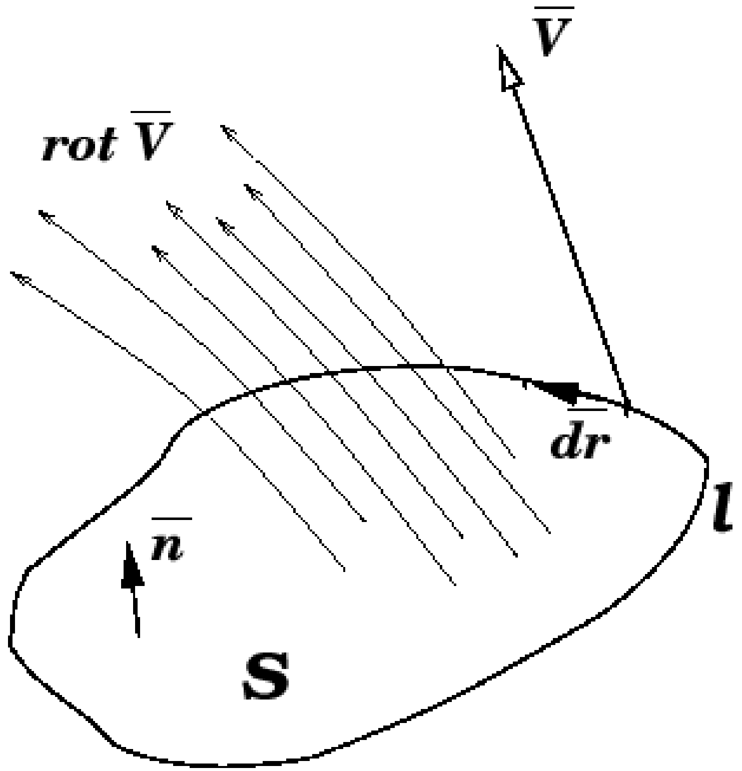
$$\left(\vec{V} \cdot \nabla\right)\vec{V} = -\frac{1}{\rho_\infty}\nabla p$$

Circulation



$$\Gamma_{AB} = \int_A^B \vec{V} \cdot d\vec{r}$$

Stokes theorem



$$\Gamma_l = \oint_l \vec{V} \cdot d\vec{r} = \iint_S \left(rot\, \vec{V} \right) \cdot \vec{n}\, dS$$

Kelvin's theorem (rate of circulation change along a closed curve)

$$\frac{d\Gamma}{dt} = \frac{d}{dt}\oint_l \vec{V}\cdot d\vec{r} \quad = \oint_l \frac{d\vec{V}}{dt}\cdot d\vec{r} + \oint_l \vec{V}\cdot\frac{d}{dt}\left(d\vec{r}\right) =$$

$$\left| \frac{d\vec{V}}{dt} = -\frac{\nabla p}{\rho} + \frac{\mu}{\rho}\Delta\vec{V} + \frac{1}{3}\frac{\mu}{\rho}\nabla\left(\nabla\cdot\vec{V}\right); \qquad \frac{d}{dt}\left(d\vec{r}\right) = d\left(\frac{d\vec{r}}{dt}\right) = d\vec{V} \right|$$

$$= \oint_l -\frac{\nabla p\cdot d\vec{r}}{\rho} + \oint_l \frac{\mu}{\rho}\left[\Delta\vec{V} + \frac{1}{3}\frac{\mu}{\rho}\nabla\left(\nabla\cdot\vec{V}\right)\right]\cdot d\vec{r} + \oint_l \vec{V}\cdot d\vec{V} =$$

$$= \oint_l -\frac{dp}{\rho} \quad + \quad \oint_l \frac{\mu}{\rho}\left[\Delta\vec{V} + \frac{1}{3}\nabla\left(\nabla\cdot\vec{V}\right)\right]\cdot d\vec{r} + \oint_l d\left(\frac{V^2}{2}\right) = \cdots$$

$$\frac{d\Gamma}{dt} = \oint_l -\frac{dp}{\rho} + \oint_l \frac{\mu}{\rho}\left[\Delta\vec{V} + \frac{1}{3}\nabla\left(\nabla\cdot\vec{V}\right)\right]\cdot d\vec{r}$$

$$= 0 \quad \textit{if} \quad \rho = \rho(p) \qquad (\textit{barotropic fluid})$$

$$= 0 \quad \textit{if} \quad \mu = 0 \qquad (\textit{inviscid fluid})$$

Γ=**const** ~ in a case of no viscosity and shock waves (isentropic flow);
flow initially irrotational – it will stay irrotational !!!

$$\Phi \left\{ scalar\ \ field\ \ function \right\} \quad \rightarrow \quad \vec{V} = \nabla\Phi \begin{Bmatrix} vector\ \ field\ \ function, \\ dependent\ on\ scalar\ \ field \end{Bmatrix}$$

is it true in the inverse case ???
$$\vec{V} \quad \rightarrow \quad \Phi \quad so \quad \vec{V} = \nabla\Phi$$

yes, if:

$$\left. \begin{matrix} \dfrac{\partial V_X}{\partial y} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial \Phi}{\partial x}\right) = \dfrac{\partial}{\partial x}\left(\dfrac{\partial \Phi}{\partial y}\right) = \dfrac{\partial V_Y}{\partial x} \\ \dots \\ \dots \end{matrix} \right\} \quad rot\vec{V} = \nabla \times \vec{V} = 0$$

Bernoulli equation

Integrating Euler equation (momentum eq. for inviscid flow)

$\frac{d\vec{V}}{dt} = -\frac{\nabla p}{\rho}$ along a stream-line $\int \frac{d\vec{V}}{dt} \cdot d\vec{r} = -\int \frac{\nabla p}{\rho} \cdot d\vec{r}$:

$$\int_A^B \frac{d\vec{V}}{dt} \cdot d\vec{r} = \int_A^B \frac{d\vec{V}}{dt} \cdot \vec{V}dt = \int_A^B \vec{V} \cdot d\vec{V} = \frac{1}{2}\left(V_B^2 - V_A^2\right) = -\int_A^B \frac{\nabla p}{\rho} \cdot d\vec{r} = -\int_A^B \frac{dp}{\rho} =$$

$$\left| \textit{isentropic flow}: \quad \frac{p}{\rho^k} = const; \quad \frac{1}{\rho} = \frac{p_\infty^{1/k}}{\rho_\infty} \cdot \frac{1}{p^{1/k}} \right| =$$

$$\cdots = -\frac{k}{k-1} \cdot \frac{p_\infty}{\rho_\infty} \cdot \left[\left(\frac{p_B}{p_\infty}\right)^{\frac{k-1}{k}} - \left(\frac{p_A}{p_\infty}\right)^{\frac{k-1}{k}} \right]$$

A $\rightarrow \infty$; B $\rightarrow$.

$$Cp = \frac{p - p_\infty}{\frac{1}{2}\rho_\infty V_\infty^2} = \cdots = \frac{\left\{1 + \frac{k-1}{2} Ma_\infty^2 \left(1 - \frac{V^2}{V_\infty^2}\right)\right\}^{\frac{k}{k-1}} - 1}{\frac{k}{2} Ma_\infty^2}$$

Relation between pressure and velocity, correct along stream-line in isentropic flow (~ no viscosity, no shock waves)

For incompressible fluid:

$$Cp = \frac{p - p_\infty}{\frac{1}{2}\rho_\infty V_\infty^2} = \cdots = 1 - \frac{V^2}{V_\infty^2}$$

SUMMARY:

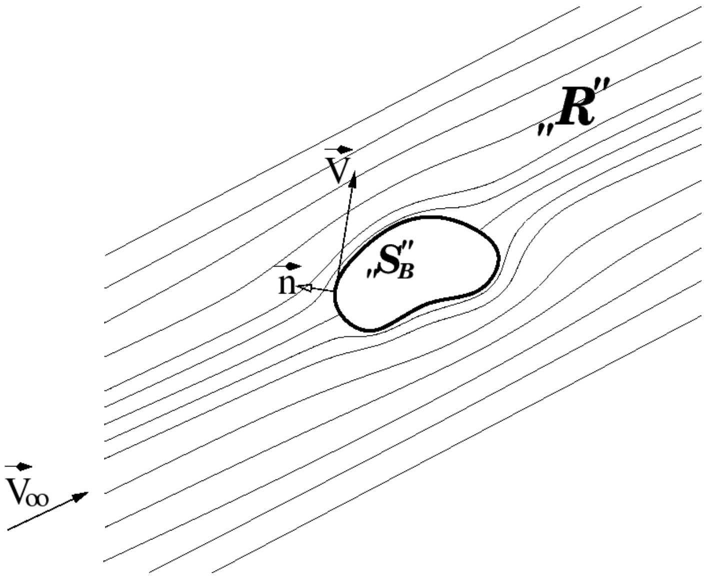
In a case:

1. Flow initially irrotational (uniform velocity field)
2. No viscosity, no shock waves (isentropic flow)

then:

1. Flow remains irrotational
2. There exist such a scalar field function Φ (velocity potential), that velocity field can be expressed as:: $\boldsymbol{V} = \nabla\Phi$
3. Momentum equations (Euler) can be a'priori integrated, result in relation between velocity and pressure

SOLUTION PROCEDURE FOR AN INCOMPRESSIBLE, IRROTATIONAL FLOW AROUND A BODY (DETERMINATION OF A FLOW FIELD, PRESSURE FIELD, FORCES AND MOMENTS ON A BODY)



Solve continuity equation:

$$\nabla \cdot \vec{V} = 0$$

and momentum equations (Euler):

$$\left(\vec{V} \cdot \nabla\right)\vec{V} = -\frac{\nabla p}{\rho_\infty}$$ in a flow field **„R"**

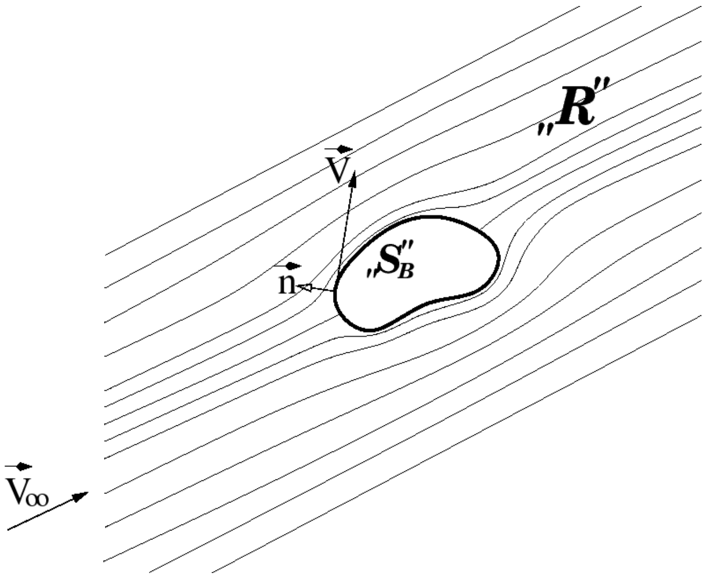
fulfill boundary conditions:

$$\vec{V} \cdot \vec{n} = V_n = 0$$ on the body surface **„S_B"**

and condition, $p \rightarrow p_\infty$ in infinity

($r \rightarrow \infty$)

EQUIVALENT SOLUTION PROCEDURE FOR AN INCOMPRESSIBLE, IRROTATIONAL FLOW AROUND A BODY (DETERMINATION OF A FLOW FIELD, PRESSURE FIELD, FORCES AND MOMENTS ON A BODY)



1. solve continuity equation

(Laplace eq. For velocity potential):

$$\nabla \cdot (\nabla \Phi) = \nabla^2 \Phi = 0$$ in a flow field ***„R"***

fulfill boundary conditions (Neumann):

$$\nabla \Phi \cdot \vec{n} = \frac{\partial \Phi}{\partial n} = V_n = 0$$ on the body surface ***„S_B"***

and conditions: $\Phi \rightarrow \Phi_\infty$ in infinity *($r \rightarrow \infty$)*

---

2. determine velocity field: $\vec{V} = \nabla \Phi$

---

3. determine pressure field using Bernouli eq.: $p = p(V)$

Flow field superposition, perturbation velocity, perturbation potential:

total velocity $\vec{V} = \vec{V}_{\infty} + \nabla\varphi$ ← perturbation potential

continuity equation $\nabla \cdot \left(\vec{V}_{\infty} + \nabla\varphi\right) = \quad \nabla^{2}\varphi = 0$

boundary condition on a body surface:

$$\left(\vec{V}_{\infty} + \nabla\varphi\right)\cdot \vec{n} = \vec{V}_{\infty} \cdot \vec{n} + \frac{\partial\varphi}{\partial \boldsymbol{n}} = V_{n} = 0 \quad \Rightarrow \quad \frac{\partial\varphi}{\partial \boldsymbol{n}} = -\vec{V}_{\infty} \cdot \vec{n}$$

far field condition

$$\nabla\phi \to 0 \quad \textit{or (equivalent)} \quad \phi \to 0$$

stream-line equation (2D):

$$\left.\begin{aligned} dx &= U \cdot dt \\ dy &= V \cdot dt \end{aligned}\right\} \quad \frac{dx}{U} = \frac{dy}{V} = dt$$

$$\frac{dx}{U} - \frac{dy}{V} = 0$$

$$-V \cdot dx + U \cdot dy = 0$$

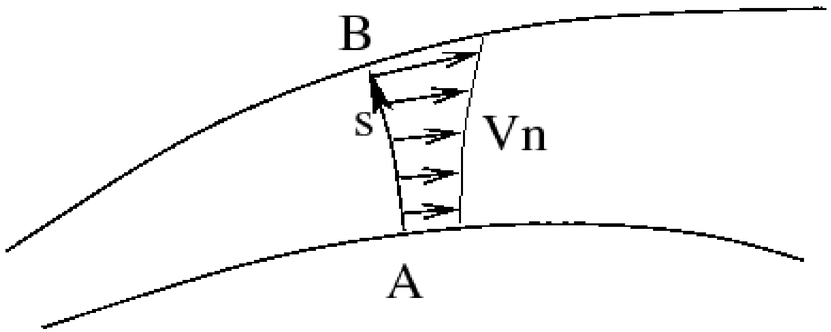
$$(-V) \cdot dx + (U) \cdot dy \quad = \frac{\partial \Psi}{\partial x} \cdot dx + \frac{\partial \Psi}{\partial y} \cdot dy = d\Psi$$

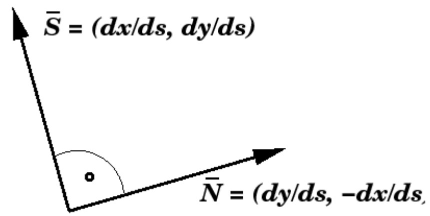
$$\frac{\partial}{\partial y}(-V) = \frac{\partial}{\partial y}\left(\frac{\partial \Psi}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial \Psi}{\partial y}\right) = \frac{\partial}{\partial x}(U) \quad \Rightarrow \quad div(\vec{V}) = 0$$

if ***div(V)=0*** there exist such a function ***Ψ*** (stream function), that

$$u = \frac{\partial \Psi}{\partial y}; \; v = -\frac{\partial \Psi}{\partial x}$$

function ***Ψ*** has constant value along a stream line





$$Q_{AB} = \int_A^B V_n \, ds = \int_A^B \vec{V} \cdot \vec{n} \, ds = \int_A^B \left(u \cdot \cos(n, Ox) + v \cdot \cos(n, Oy) \right) ds =$$

$$\int_A^B \left(\frac{\partial \Psi}{\partial y} \cdot \frac{dy}{ds} - \frac{\partial \Psi}{\partial x} \cdot \left(-\frac{dx}{ds} \right) \right) ds = \int_A^B d\Psi = \Psi_B - \Psi_A$$

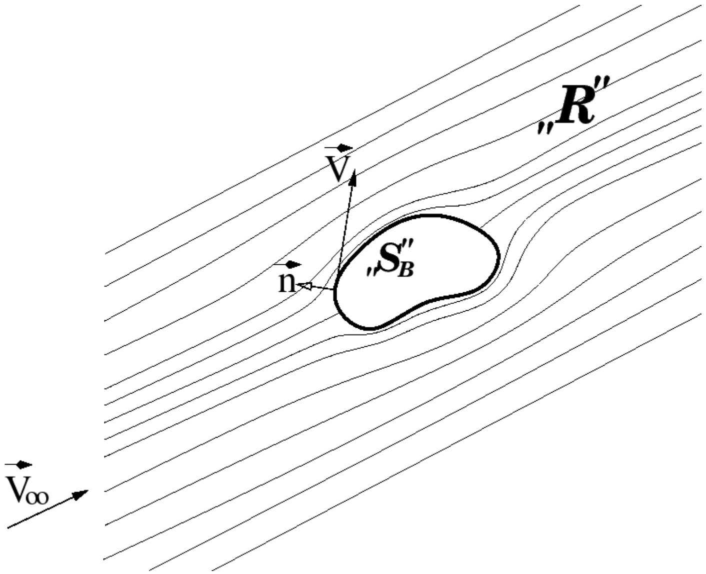
additionally, if ***rot(V)=0*** thus:

$$\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{y}} - \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}} = \frac{\partial}{\partial \boldsymbol{y}}\left(\frac{\partial \Psi}{\partial \boldsymbol{y}}\right) - \frac{\partial}{\partial \boldsymbol{x}}\left(-\frac{\partial \Psi}{\partial \boldsymbol{x}}\right) = \frac{\partial^2 \Psi}{\partial \boldsymbol{x}^2} + \frac{\partial^2 \Psi}{\partial \boldsymbol{y}^2} = \mathbf{0}$$

harmonic function

on a body surface Ψ=const

2. **EQUIVALENT** SOLUTION PROCEDURE FOR AN INCOMPRESSIBLE, IRROTATIONAL FLOW AROUND A BODY (DETERMINATION OF A FLOW FIELD, PRESSURE FIELD, FORCES AND MOMENTS ON A BODY) – **only 2D**



1. solve equation: ***rot(V)=0*** → potential flow (Laplace equation for stream function):

$$\nabla^2 \Psi = \mathbf{0}$$ in a flow field ***„R"***

condition (Dirichlet):

$$\Psi = \boldsymbol{const}$$ on a body contour ***„S_B"***

condition: $$\boldsymbol{V} \rightarrow \boldsymbol{V}_\infty$$ in infinity *($r\rightarrow\infty$)*

---

2. determine velocity field: $$\vec{V} = \left(\frac{\partial \Psi}{\partial y}, -\frac{\partial \Psi}{\partial x} \right)$$

---

3. determine pressure field using Bernouli eq.: $p = p(V)$