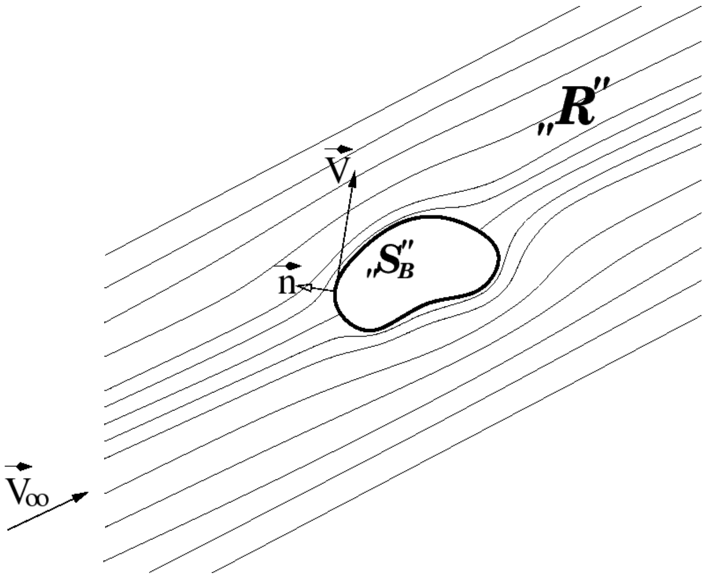
SOLUTION PROCEDURE FOR AN INCOMPRESSIBLE, IRROTATIONAL FLOW AROUND A BODY (DETERMINATION OF A FLOW FIELD, PRESSURE FIELD, FORCES AND MOMENTS ON A BODY)



Solve continuity equation:

$$\nabla \cdot \vec{V} = 0$$

and momentum equations (Euler):

$$\left(\vec{V} \cdot \nabla\right)\vec{V} = -\frac{\nabla p}{\rho_\infty}$$

in a flow field **„R"**

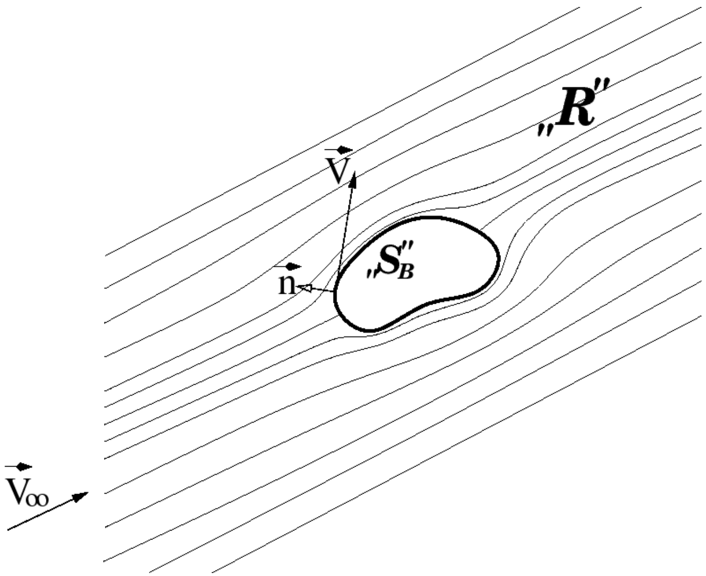
fulfill boundary conditions:

$$\vec{V} \cdot \vec{n} = V_n = 0$$

on the body surface **„S_B"**

and conditions, $\vec{V} \rightarrow \vec{V}_\infty, \; p \rightarrow p_\infty$ in infinity *($r \rightarrow \infty$)*

1. **EQUIVALENT** SOLUTION PROCEDURE FOR AN INCOMPRESSIBLE, IRROTATIONAL FLOW AROUND A BODY (DETERMINATION OF A FLOW FIELD, PRESSURE FIELD, FORCES AND MOMENTS ON A BODY)



1. solve continuity equation
(Laplace eq. For velocity potential):

$$\nabla \cdot (\nabla\Phi) = \nabla^2\Phi = \mathbf{0}$$ in a flow field ***„R"***

fulfill boundary conditions (Neumann):

$$\nabla\Phi \cdot \vec{\boldsymbol{n}} = \frac{\partial\Phi}{\partial \boldsymbol{n}} = \boldsymbol{V_n} = \mathbf{0}$$ on the body surface ***„S_B"***

and conditions: $\Phi \rightarrow \Phi_\infty$ in infinity *($r \rightarrow \infty$)*

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2. determine velocity field: $\vec{\boldsymbol{V}} = \nabla\Phi$

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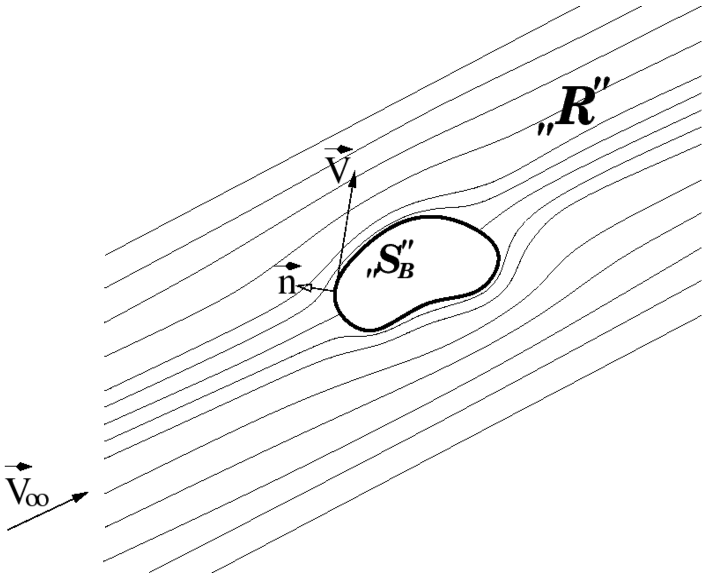
3. determine pressure field using Bernouli eq.: $p = p(V)$

2. **EQUIVALENT** SOLUTION PROCEDURE FOR AN INCOMPRESSIBLE, IRROTATIONAL FLOW AROUND A BODY (DETERMINATION OF A FLOW FIELD, PRESSURE FIELD, FORCES AND MOMENTS ON A BODY) – **only 2D**



1. solve equation: ***rot(V)=0*** → potential flow
(Laplace equation for stream function):

$$\nabla^2\Psi = \mathbf{0}$$ in a flow field **„*R*"**

condition (Dirichlet):

$$\Psi = \boldsymbol{const}$$ on a body contour **„*S*$_B$"**

condition: $$\boldsymbol{V} \rightarrow \boldsymbol{V}_\infty$$ in infinity *(r→∞)*

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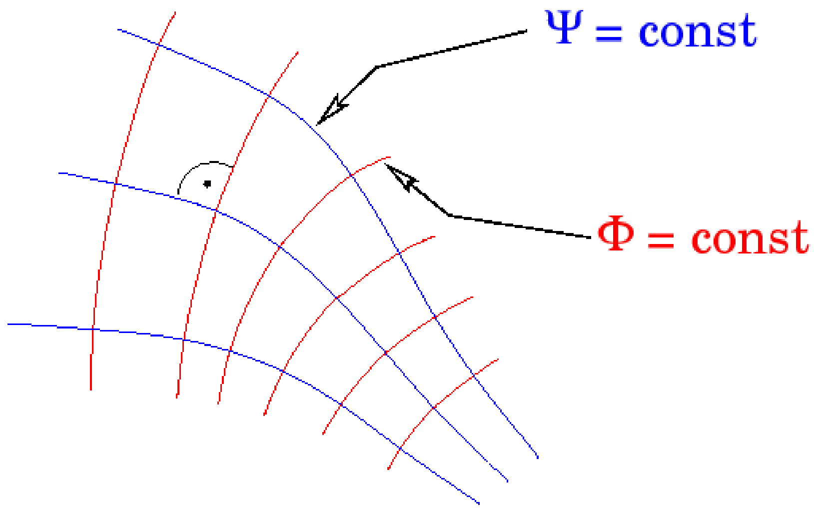
2. determine velocity field: $$\vec{\boldsymbol{V}} = \left(\frac{\partial \Psi}{\partial \boldsymbol{y}}, -\frac{\partial \Psi}{\partial \boldsymbol{x}} \right)$$

---

3. determine pressure field using Bernouli eq.: p = p(V)

$$\nabla\Phi\cdot\nabla\Psi=\left(\frac{\partial\Phi}{\partial x},\frac{\partial\Phi}{\partial y}\right)\cdot\left(\frac{\partial\Psi}{\partial x},\frac{\partial\Psi}{\partial y}\right)=(u,v)\cdot(-v,u)=0$$

$$\nabla\Phi\cdot\nabla\Psi=0 \quad \rightarrow \quad (\Phi=const)\perp(\Psi=const)$$



complex variable: $z = x + iy \left[= re^{i\vartheta} = r\left(\cos\vartheta + i\sin\vartheta\right)\right]$

complex function: $w(z) = \Phi(x, y) + i\Psi(x, y)$

derivative:

$$w'(z) = \frac{dw}{dz} = \frac{\partial\Phi}{\partial x} + i\frac{\partial\Psi}{\partial x} =$$

$$= \frac{\partial\Phi}{\partial(iy)} + i\frac{\partial\Psi}{\partial(iy)} = \frac{\partial\Psi}{\partial y} - i\frac{\partial\Phi}{\partial y} = u - i\,v$$

(complex velocity -
conjugate do velocity)

$$w'(z) = \frac{\partial \Phi}{\partial x} + i\frac{\partial \Psi}{\partial x} \quad = \frac{\partial \Psi}{\partial y} - i\frac{\partial \Phi}{\partial y}$$

$$\frac{\partial \Phi}{\partial x} = \frac{\partial \Psi}{\partial y} \qquad [= U]$$

$$\frac{\partial \Psi}{\partial x} = -\frac{\partial \Phi}{\partial y} \qquad [= -V]$$

Cauchy-Riemann conditions

$$z = x + i\,y = r \cdot e^{i\vartheta}$$

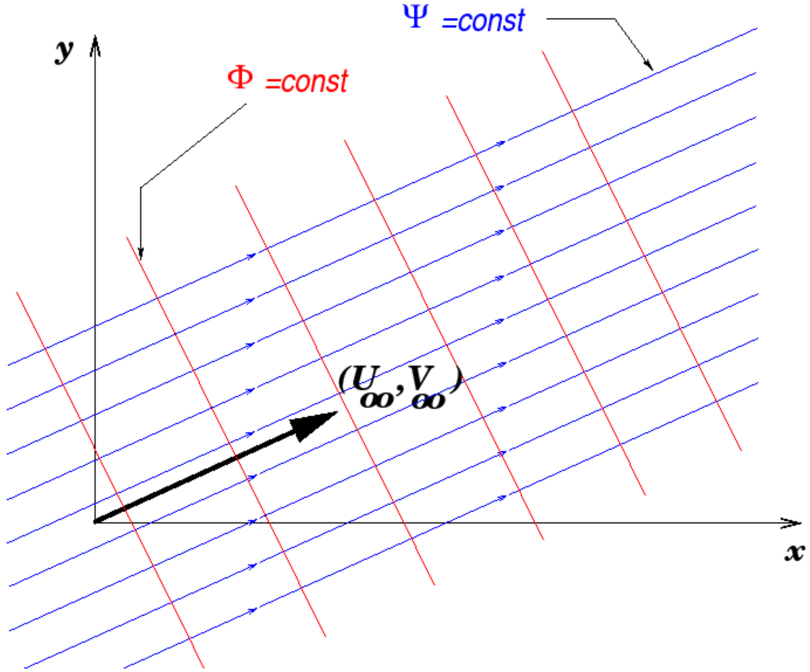
$$w'(z) = \sqrt{u^2 + v^2} \cdot e^{i\beta} = \tilde{V} \quad \text{(complex velocity – conjugate to } V\text{)}$$

$$V = u + i\,v = \sqrt{u^2 + v^2} \cdot e^{-i\beta}$$

Uniform flow field

$$w_\infty(z) = (U_\infty - i\,V_\infty)\cdot z$$

$$w'_\infty(z) = U_\infty - i\,V_\infty$$

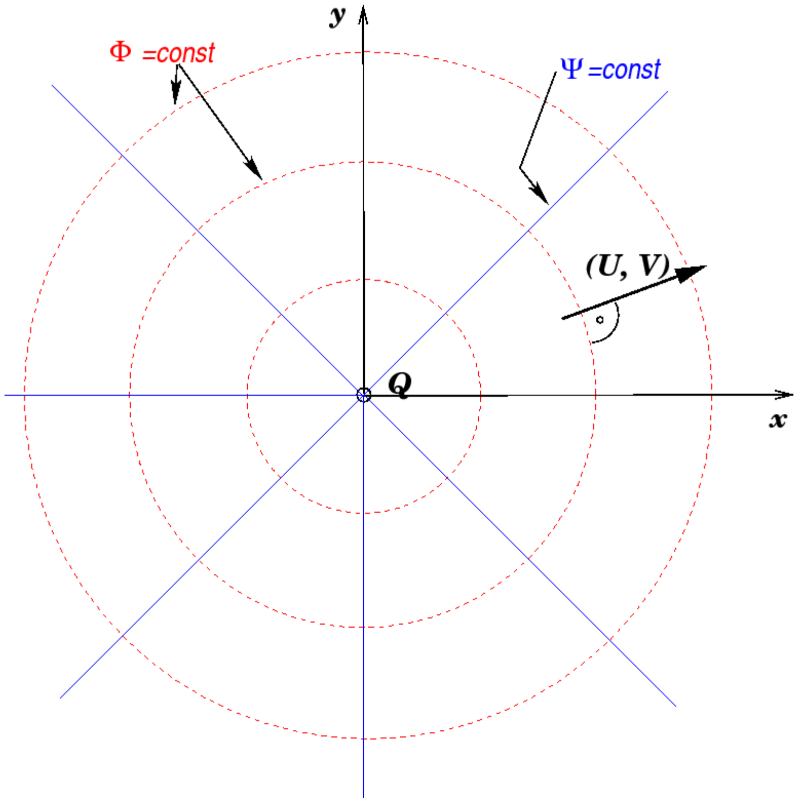


Point source

$$w_Q(z) = \frac{Q}{2\pi}\ln(z) = \frac{Q}{2\pi}\ln\left(r \cdot e^{i\Theta}\right) = \frac{Q}{2\pi}(\ln r + i\Theta)$$

$$w'_Q(z) = \frac{Q}{2\pi z} = \frac{Q}{2\pi}\frac{x - i\,y}{x^2 + y^2} = \frac{Q}{2\pi r}\frac{x - i\,y}{r} =$$

$$= V_r \cdot \frac{x - i\,y}{r}$$

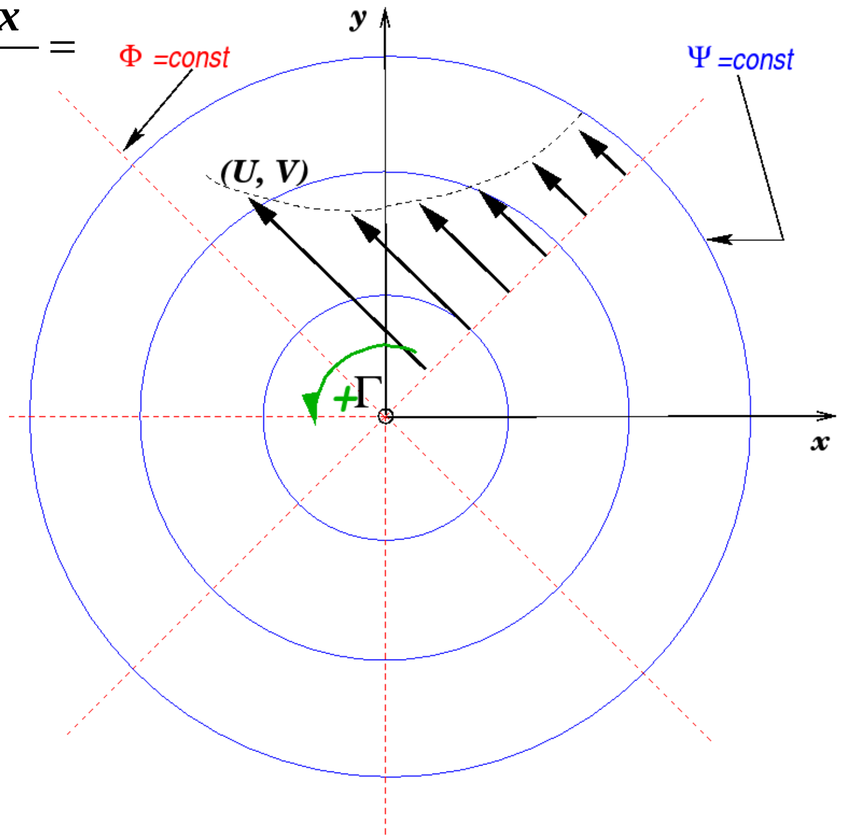


Potential vortex

$$w_\Gamma(z) = \frac{\Gamma}{2\pi i}\ln(z) = \frac{\Gamma}{2\pi i}\ln\left(r\cdot e^{i\Theta}\right) = \frac{\Gamma}{2\pi i}(\Theta - i\ln r)$$

$$w'_\Gamma(z) = \frac{\Gamma}{2\pi z i} = -\frac{\Gamma i}{2\pi}\frac{x - i y}{x^2 + y^2} = -\frac{\Gamma}{2\pi r}\frac{y + i x}{r} =$$

$$= -V_\Theta \cdot \frac{y + i x}{r}$$

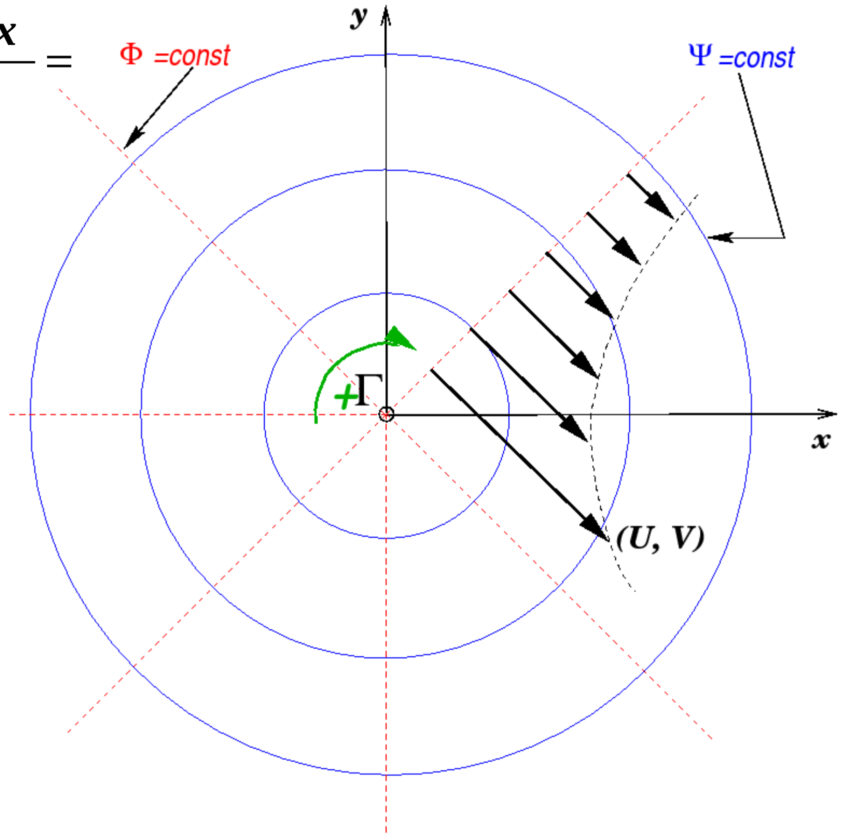


Potential vortex (-)

$$w_\Gamma(z) = \frac{-\Gamma}{2\pi i}\ln(z) = \frac{-\Gamma}{2\pi i}\ln\left(r\cdot e^{i\Theta}\right) = \frac{-\Gamma}{2\pi i}\left(\Theta - i\ln r\right)$$

$$w'_\Gamma(z) = \frac{-\Gamma}{2\pi\, z\, i} = +\frac{\Gamma i}{2\pi}\frac{x - i y}{x^2 + y^2} = +\frac{\Gamma}{2\pi r}\frac{y + i x}{r} =$$

$$= +V_\Theta \cdot \frac{y + i x}{r}$$

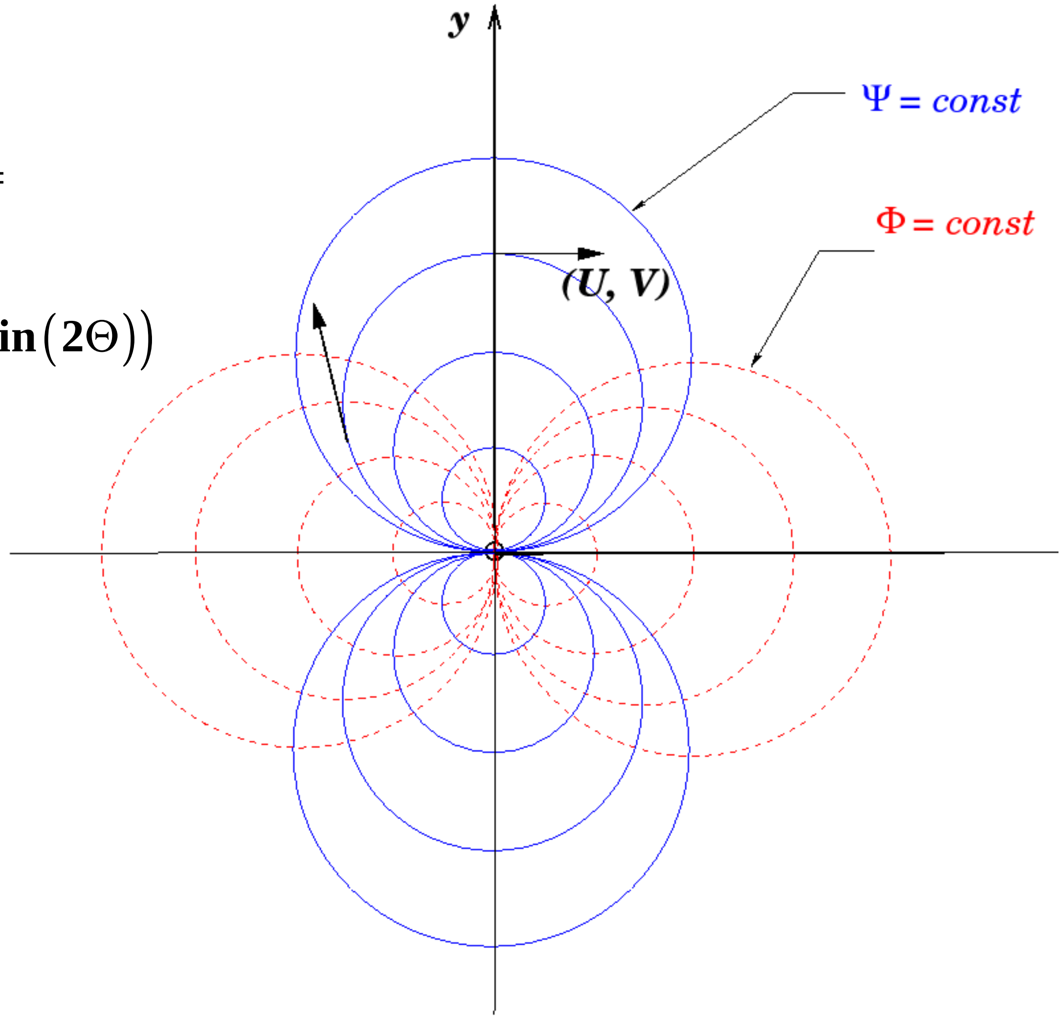


Doublet

$$w_D(z) = \frac{M}{2\pi}\frac{1}{z} = \frac{M}{2\pi}\frac{1}{r\,e^{i\Theta}} = \frac{M}{2\pi r}(\cos\Theta - i\sin\Theta)$$

$$w'_D(z) = \frac{-M}{2\pi}\frac{1}{z^2} = -\frac{M}{2\pi}\frac{1}{\left(r\,e^{i\Theta}\right)^2} =$$

$$= \frac{-M}{2\pi r^2}\left(\cos(2\Theta) - i\sin(2\Theta)\right)$$



$$W = W_1 + W_2 + W_3 + \cdots$$

$$2\pi U_\infty \cdot a^2$$

$$W(z) = U_\infty \cdot z + \frac{M}{2\pi z} + \frac{-\Gamma}{2\pi i}\ln z \quad = U_\infty\left(z + \frac{a^2}{z}\right) + \frac{\Gamma}{2\pi}\cdot i\,\ln z =$$

$$= U_\infty\left(re^{i\theta} + \frac{a^2}{re^{i\theta}}\right) + \frac{\Gamma}{2\pi}\cdot i\left(\ln r + i\theta\right) =$$

$$= U_\infty\left(re^{i\theta} + \frac{a^2}{r}e^{-i\theta}\right) - \frac{\Gamma}{2\pi}\left(\theta - i\ln r\right)$$

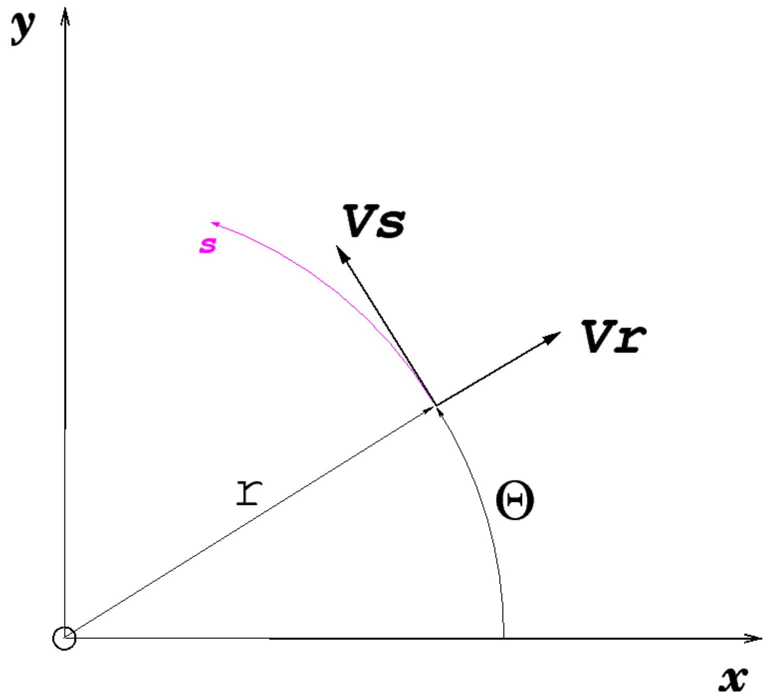
$$W(z) = \cdots$$

$$= U_\infty \left(r\left(\cos\theta + i\sin\theta\right) + \frac{a^2}{r}\left(\cos\theta + i\sin\theta\right) \right) - \frac{\Gamma}{2\pi}\left(\theta - i\ln r\right) =$$

$$= U_\infty \cos\theta \left(r + \frac{a^2}{r} \right) - \frac{\Gamma}{2\pi}\theta \quad + \qquad [= \Phi]$$

$$+ i\left[U_\infty \sin\theta \left(r - \frac{a^2}{r} \right) - \frac{\Gamma}{2\pi}\ln r \right] \qquad [= i\,\Psi]$$

$$V_r = \frac{\partial \Phi}{\partial r} = U_\infty \left(1 - \frac{a^2}{r^2}\right)\cos\Theta$$



$$s = r \cdot \Theta$$

$$V_s = \frac{\partial \Phi}{\partial s} = \frac{\partial \Phi}{\partial (r \cdot \Theta)} = \frac{1}{r}\frac{\partial \Phi}{\partial \Theta} = -U_\infty \left(1 + \frac{a^2}{r^2}\right)\sin\Theta - \frac{\Gamma}{2\pi r}$$

r = *const*

for $\boldsymbol{r = a}$

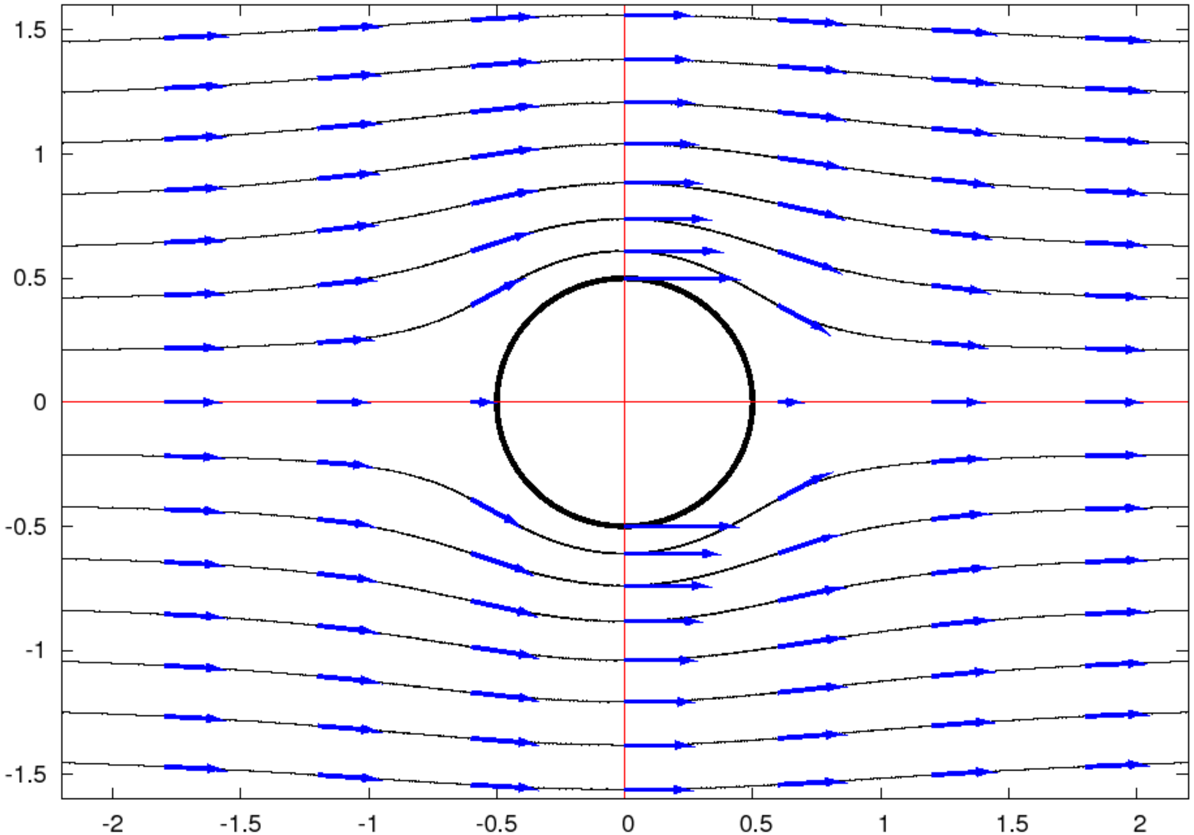
$$V_r = U_\infty \left(1 - \frac{a^2}{r^2}\right)\cos\Theta \quad = 0$$

$$V_s = -U_\infty \left(1 + \frac{a^2}{r^2}\right)\sin\Theta - \frac{\Gamma}{2\pi r} \quad = -2U_\infty \sin\Theta - \frac{\Gamma}{2\pi a}$$

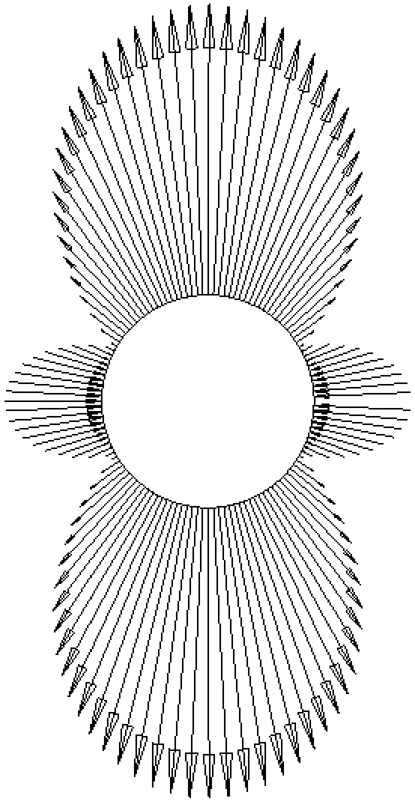
$$\Psi = U_\infty \sin\theta \left(r - \frac{a^2}{r}\right) - \frac{\Gamma}{2\pi}\ln r = -\frac{\Gamma}{2\pi}\ln a = const$$

$\Gamma = 0$

FLOW FIELD

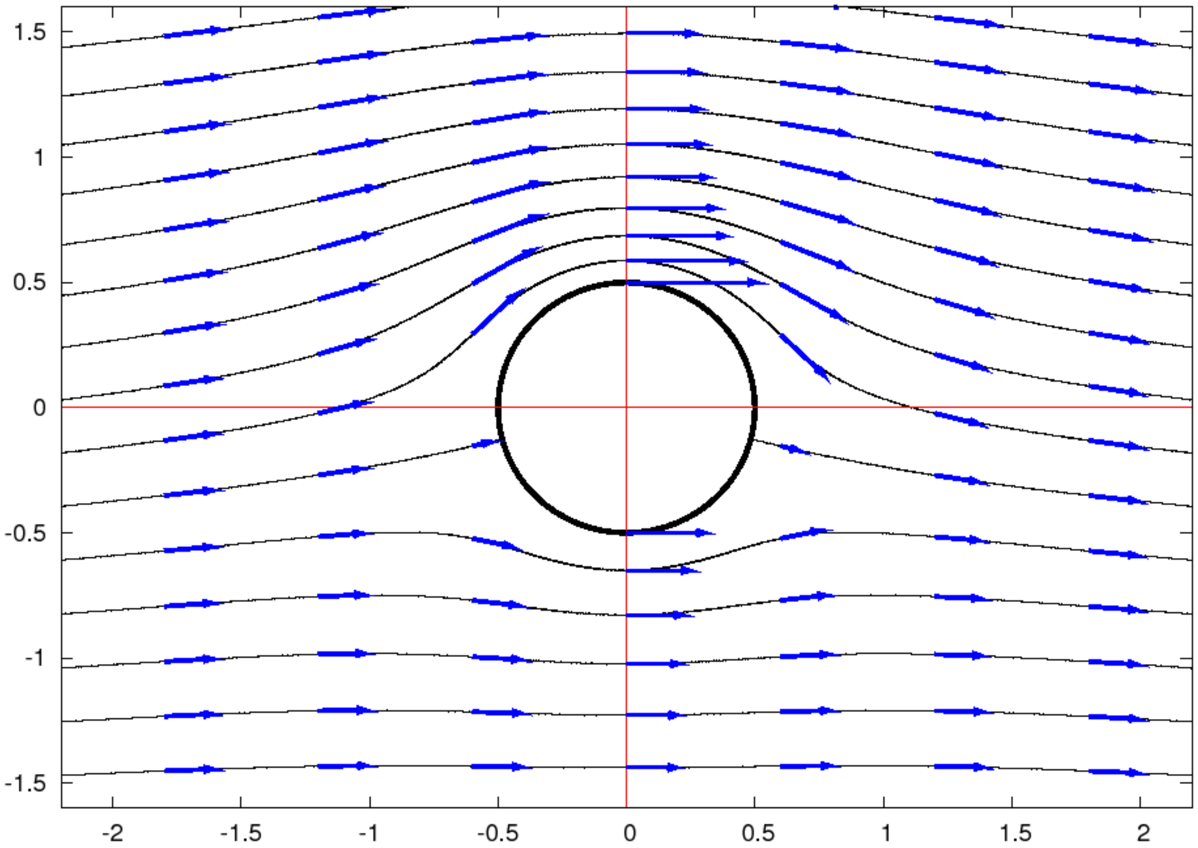


PRESSURE DISTRIBUTION

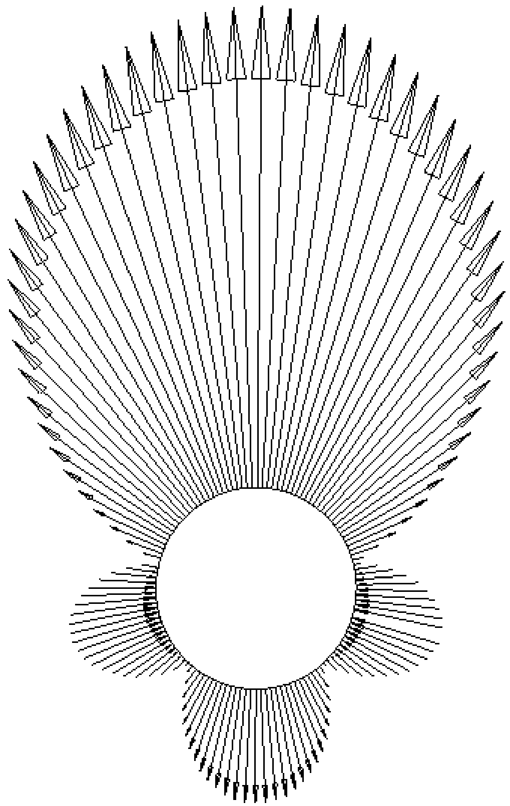


$$\Gamma = 0.25 \cdot \left(4\pi a U_{\infty} \right)$$

FLOW FIELD

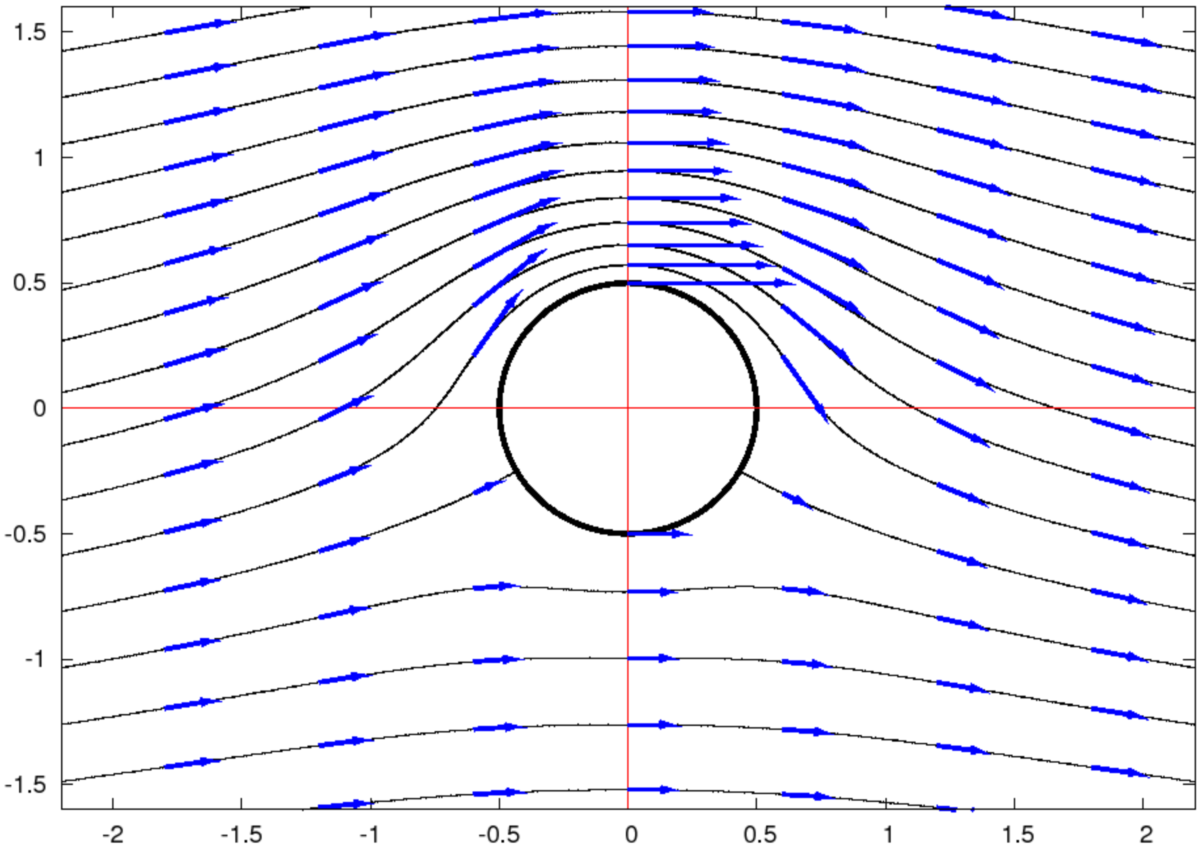


PRESSURE DISTRIBUTION

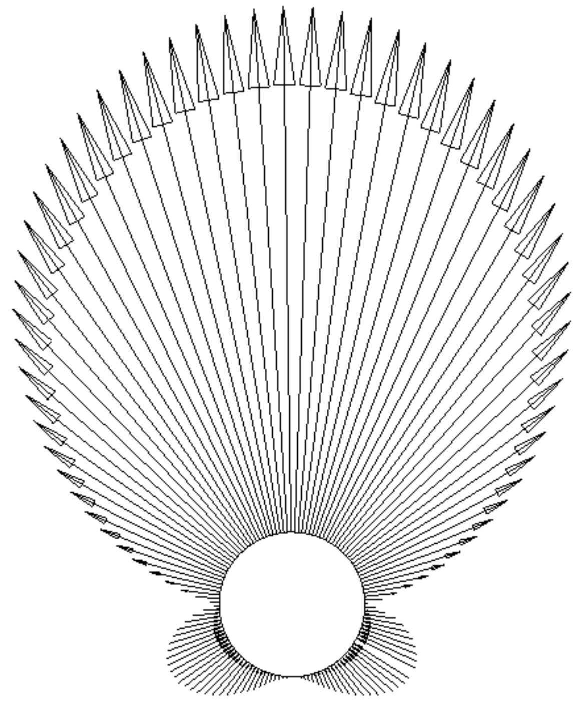


$$\Gamma = \mathbf{0.50} \cdot \left(\mathbf{4}\pi \boldsymbol{a U_{\infty}} \right)$$

FLOW FIELD

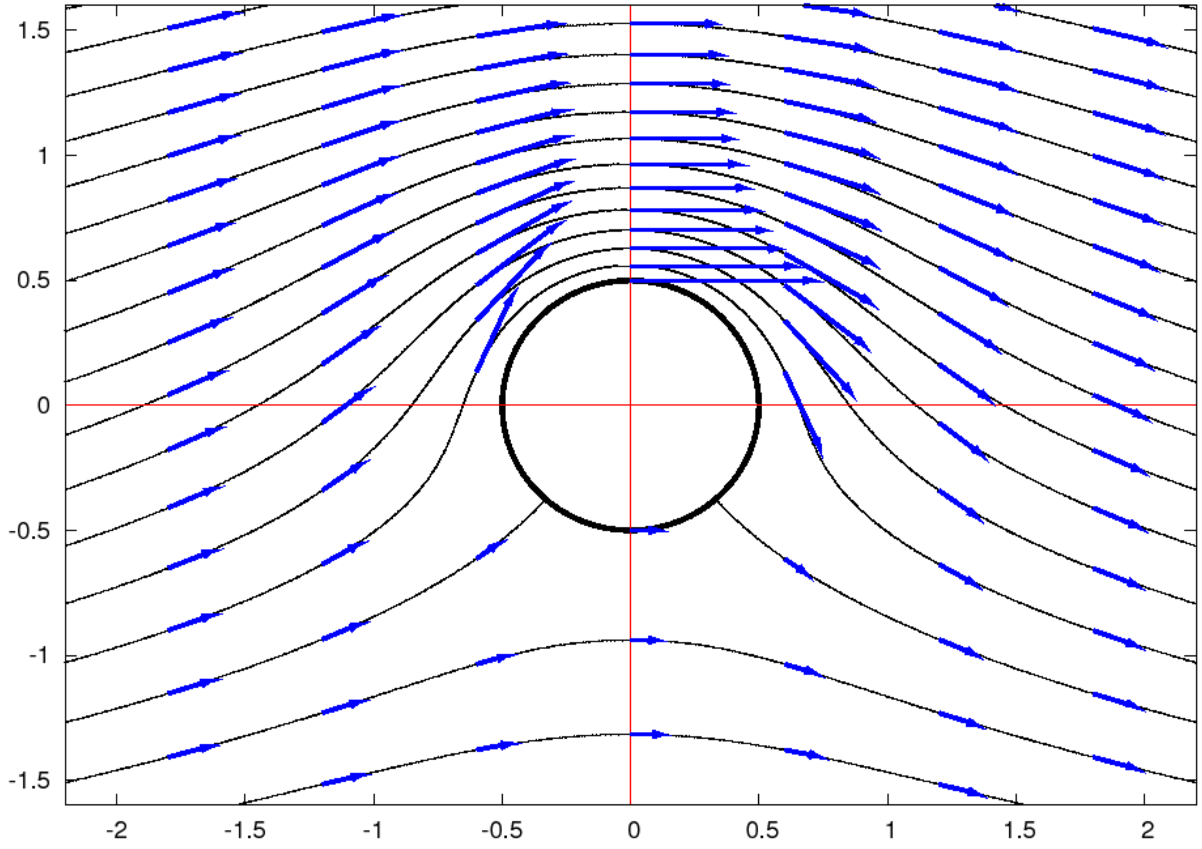


PRESSURE DISTRIBUTION

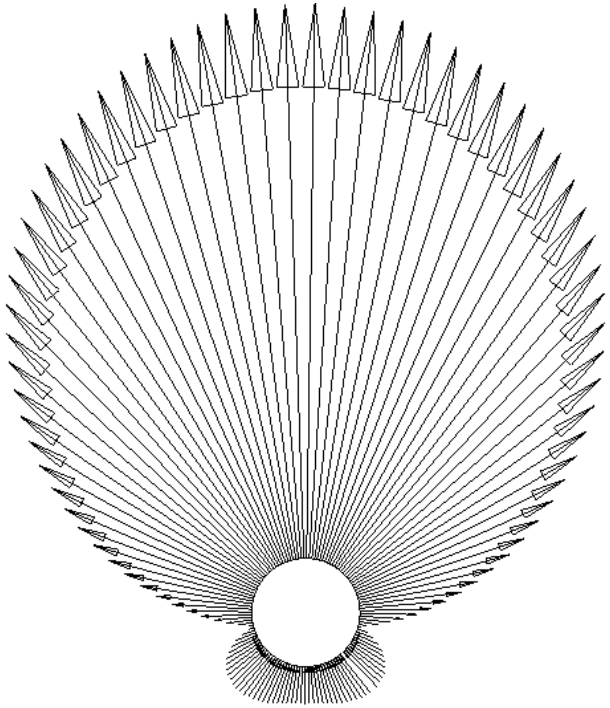


$$\Gamma = \mathbf{0.75} \cdot \left(\mathbf{4}\pi \boldsymbol{a} \boldsymbol{U}_\infty \right)$$

FLOW FIELD

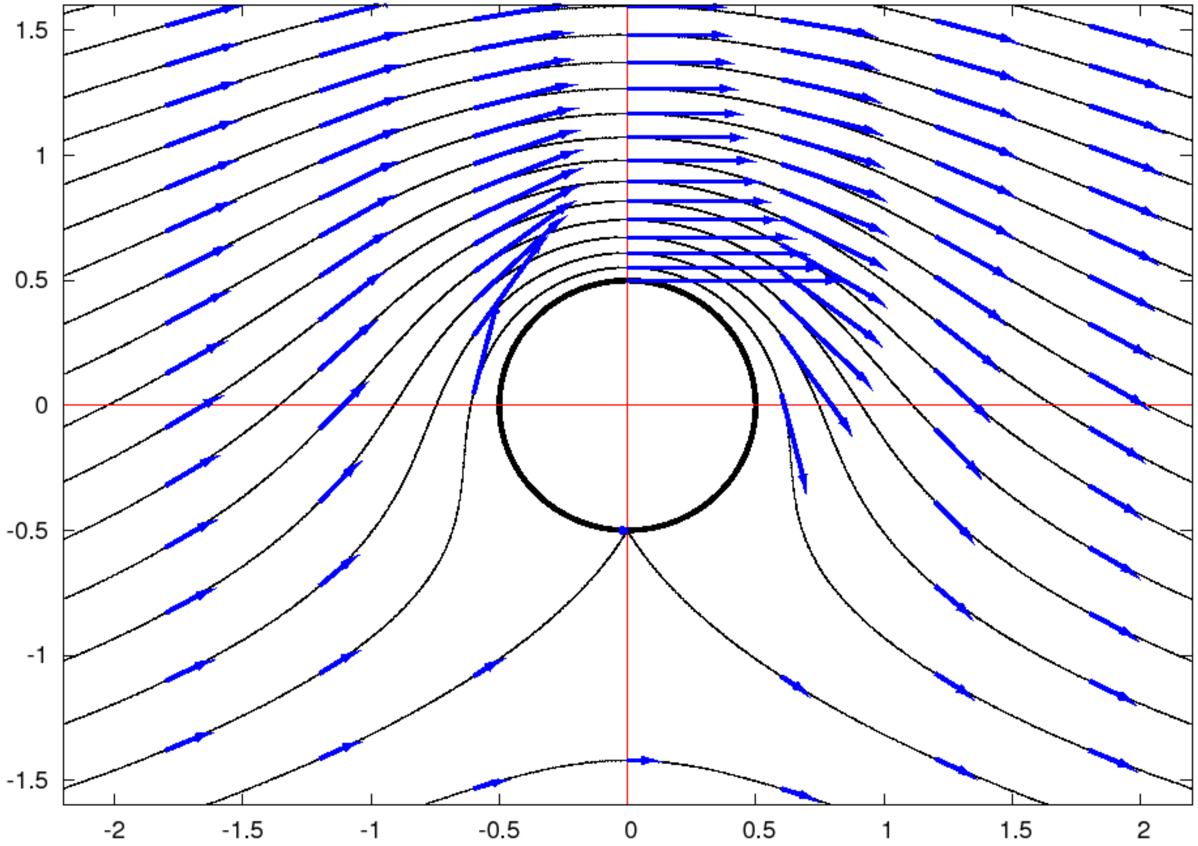


PRESSURE DISTRIBUTION

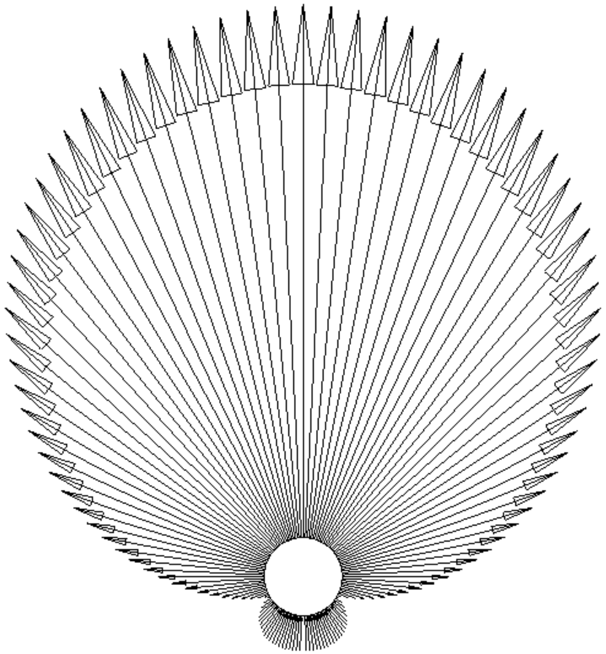


$$\Gamma = \mathbf{1.00} \cdot \left(\mathbf{4}\pi\, \boldsymbol{aU}_{\infty} \right)$$

FLOW FIELD

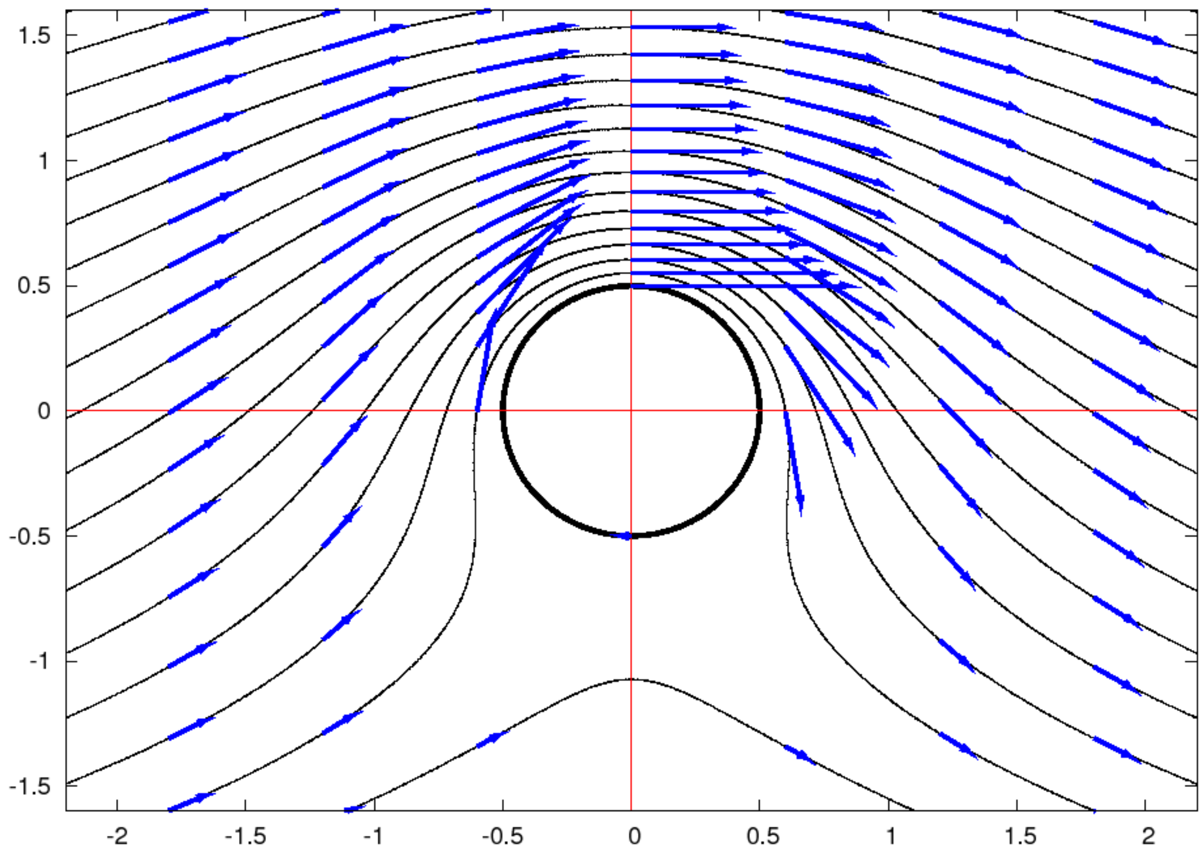


PRESSURE DISTRIBUTION



$$\Gamma = 1.10 \cdot \left(4\pi a U_{\infty} \right)$$

FLOW FIELD



DRAG FORCE

$$D = -\oint_l p \cdot dy = -\oint_l \left(p - p_\infty \right) \cdot dy = -\oint_l \left(\frac{\rho_\infty U_\infty{}^2}{2} - \frac{\rho_\infty U^2}{2} \right) \cdot d\left(a \sin\Theta \right) =$$

$$-\frac{\rho_\infty U_\infty{}^2}{2} \oint_l \left(1 - \left(2\sin\Theta + \frac{\Gamma}{2\pi a U_\infty} \right)^2 \right) \cdot a \cos\Theta \cdot d\Theta = \cdots = 0$$

LIFT FORCE

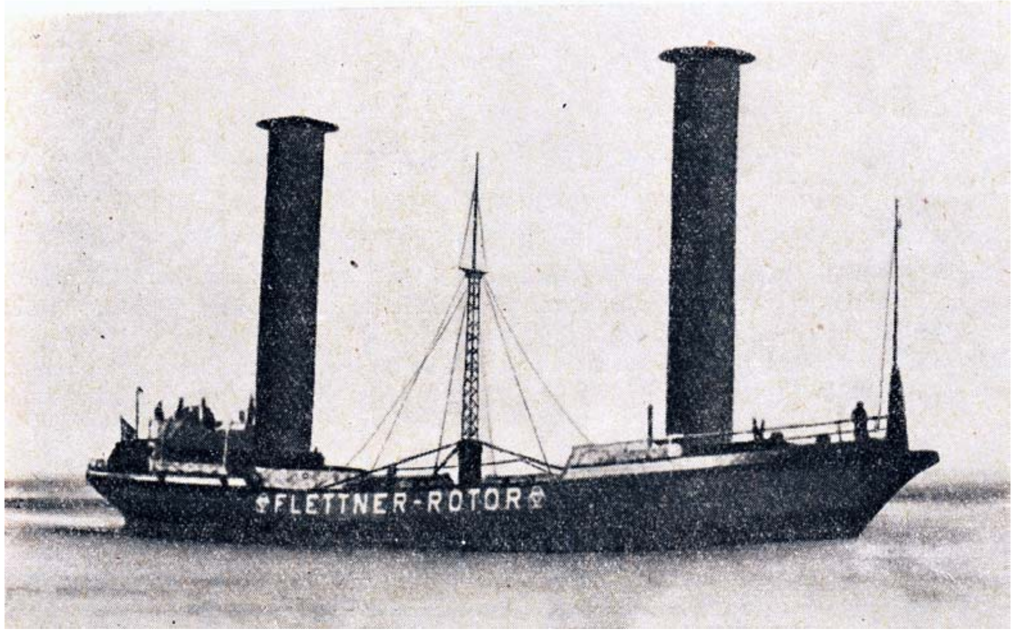
$$L = \oint_l p \cdot dx = \oint_l \left(p - p_\infty \right) \cdot dx = \oint_l \left(\frac{\rho_\infty U_\infty{}^2}{2} - \frac{\rho_\infty U^2}{2} \right) \cdot d\left(a \cos\Theta \right) =$$

$$- \frac{\rho_\infty U_\infty{}^2}{2} \oint_l \left(1 - \left(2 \sin\Theta + \frac{\Gamma}{2\pi a U_\infty} \right)^2 \right) \cdot a \sin\Theta \cdot d\Theta = \cdots = \rho_\infty \Gamma U_\infty$$

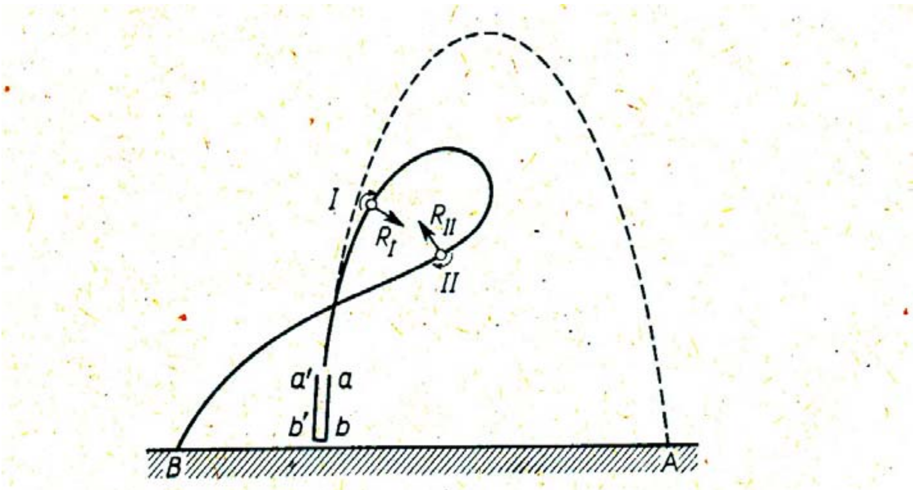
Joukowski formula for lift
(two-dimensional flow → unit span vortex)

$$L = \rho_\infty \cdot U_\infty \cdot \Gamma \cdot b$$

(b – span [length])



ship: Flettner rotor

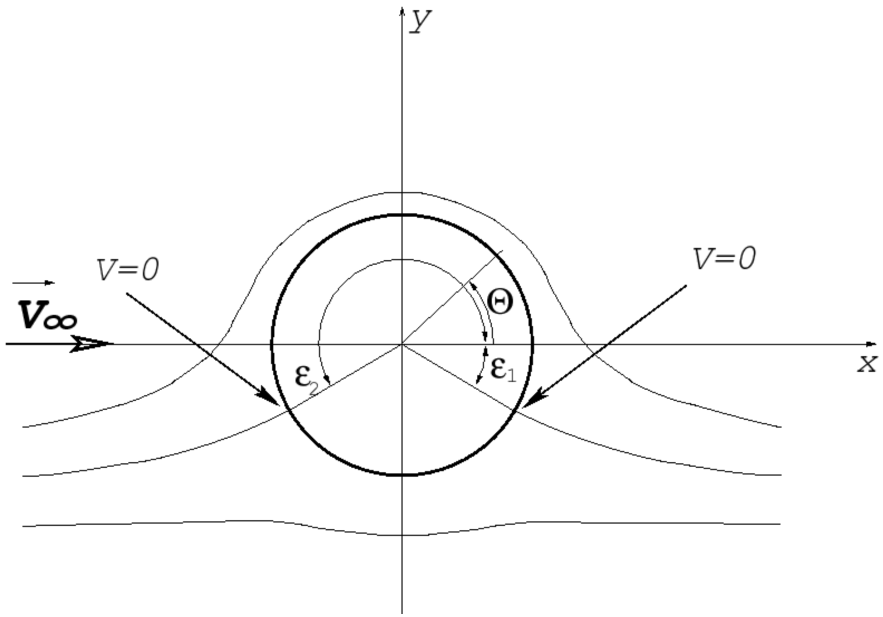


mortar (round shut ball)

Stagnation points

$$V_s = -2U_\infty \sin\Theta - \frac{\Gamma}{2\pi a} = 0$$

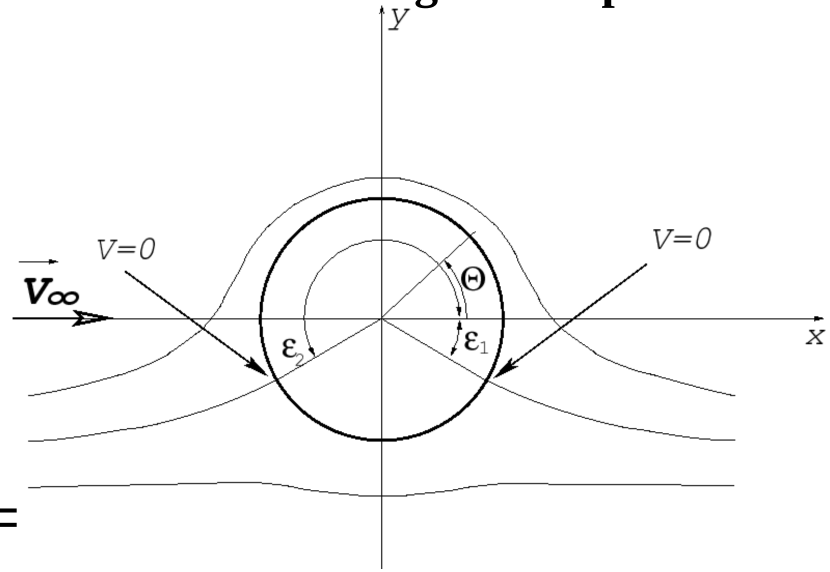
$$\rightarrow \quad \Theta = \varepsilon = -\arcsin\left(\frac{\Gamma}{4\pi a U_\infty}\right); \quad \varepsilon = \pi + \arcsin\left(\frac{\Gamma}{4\pi a U_\infty}\right)$$



Stagnation points – circulation –lift force

$$\Gamma = 4\pi\, aU_\infty \sin(-\varepsilon)$$

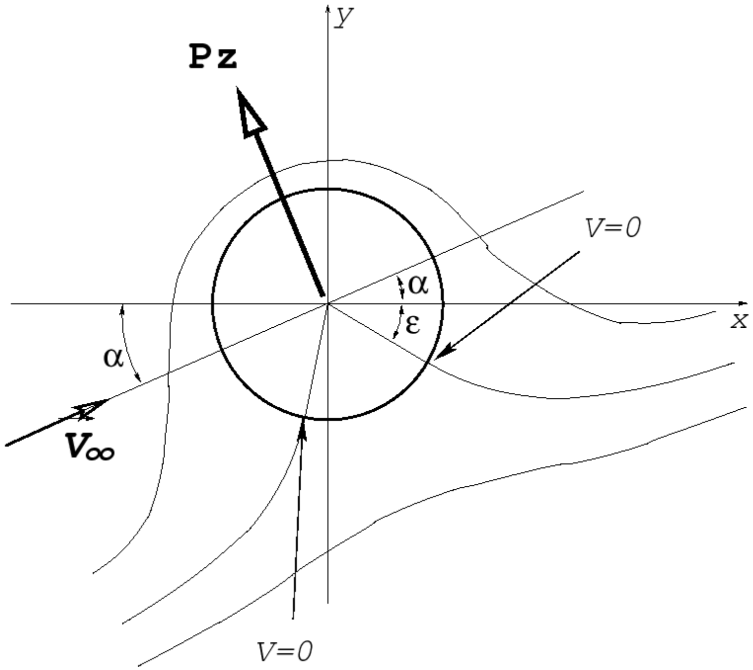
***(-ε)*: angle between undisturbed velocity direction and stagnation point direction**



$$L = (\rho_\infty)\cdot(U_\infty)\cdot(4\pi\, aU_\infty \sin(-\varepsilon))\cdot(b) =$$

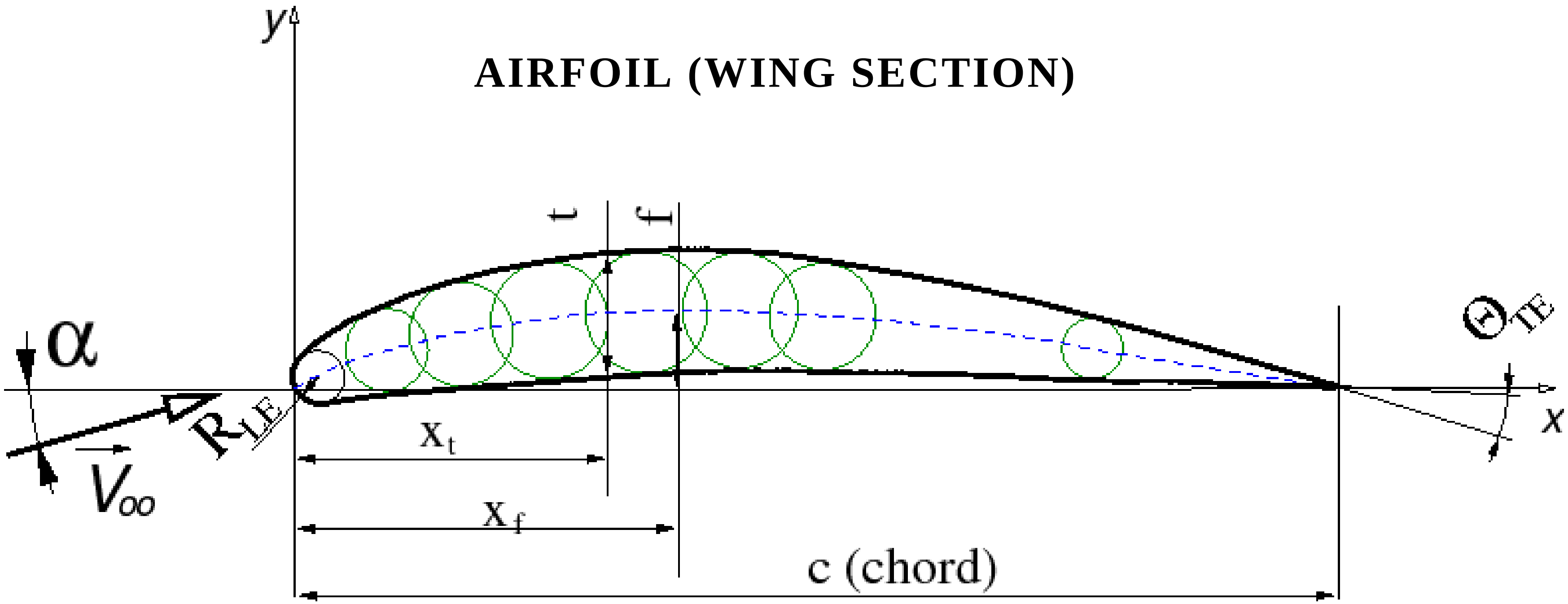
$$= \underbrace{4\pi \sin(-\varepsilon)}_{C_L}\cdot\underbrace{2ab}_{S}\cdot\underbrace{\rho_\infty \frac{U_\infty^{\;2}}{2}}_{q_\infty} \quad = \underbrace{4\pi \sin(-\varepsilon)}_{C_L}\cdot S\cdot q_\infty$$

Rotation of the undisturbed velocity (α)



$$L = \underbrace{4\pi \sin\left(\alpha - \varepsilon\right)}_{C_L} \cdot S \cdot q_\infty$$

AIRFOIL (WING SECTION)

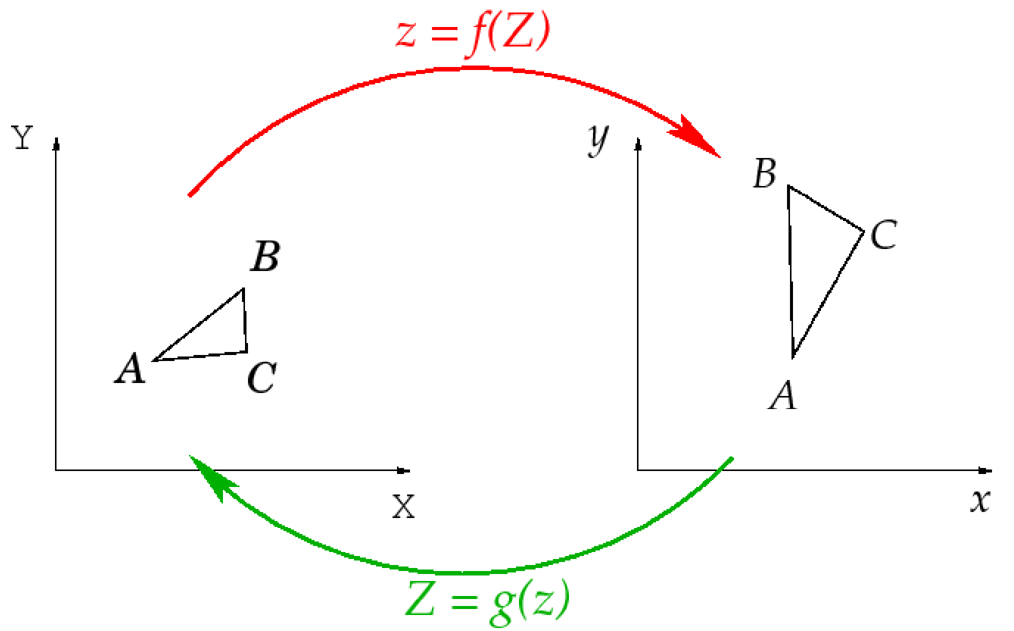


chord	c	
max. thickness	t	$\bar{t}$
max. thickness location	x_t	$\bar{x}_t$
max. camber	f	$\bar{f}$
max. camber location	x_f	$\bar{x}_f$
leading edge radius	R_{LE}	$\bar{R}_{LE}$
trailing edge angle	Θ_{TE}	
angle of attack	α	

Conformal mapping

$$\boldsymbol{Z = X + iY = \mathrm{R}\, e^{i\Theta}}$$

$$\boldsymbol{z = x + i\,y = r\,e^{i\vartheta}}$$



$$\boldsymbol{z = f(Z)}$$

$$\boldsymbol{Z = g(z)}$$

$$\frac{dz}{dZ} = f'(Z) = a \cdot e^{i\alpha}$$

$$dz = a\, e^{i\alpha} \cdot dZ$$

$$|dz| = a\,|dZ| = \left|\frac{dz}{dZ}\right| |dZ|$$

$$\arg(dz) = \arg(dZ) + \alpha$$

effect of the transformation of the very small line segment AB:

1. ***increase (decrease) length of the line segment by the factor |dz/dZ|*** ***(independent on its direction)***
2. ***rotation of the line segment through the angle arg(dz/dZ)***

true, if transformation not singular (a=0 or a=∞)

CONFORMAL MAPPING

$$W \rightarrow U, V \qquad \left(U - iV\right) = \frac{dW}{dZ}$$

$$\left(u - iv\right) = \frac{dW}{dz} = \frac{dW}{dZ}\frac{dZ}{dz} = \left(U - iV\right)\frac{dZ}{dz}$$

$$\left|\vec{v}\right| = \left|\vec{V}\right| \cdot \left|\frac{dZ}{dz}\right| = \left|\vec{V}\right| \frac{dR}{dr}$$

$$\Gamma_C = \oint_C \vec{V} \cdot \overrightarrow{dR} = \oint_C \left|\vec{V}\right| \cdot \cos\theta \cdot dR =$$

$$\oint_C \left(\left|\vec{v}\right| / \left|\frac{dZ}{dz}\right| \right) \cdot \cos\theta \cdot \left(\left|\frac{dZ}{dz}\right| \cdot dr \right) = \oint_C \left|\vec{v}\right| \cdot \cos\theta \cdot dr = \Gamma_c$$

additionally, if:

$$\frac{dz}{dZ} = f'(Z) \xrightarrow[r \to \infty]{} 1$$

thus

$$\vec{V}_\infty = \vec{v}_\infty$$

in both planes (spaces) there will act the same forces **(D=0, L=$\rho_\infty U_\infty \Gamma$)**

and:

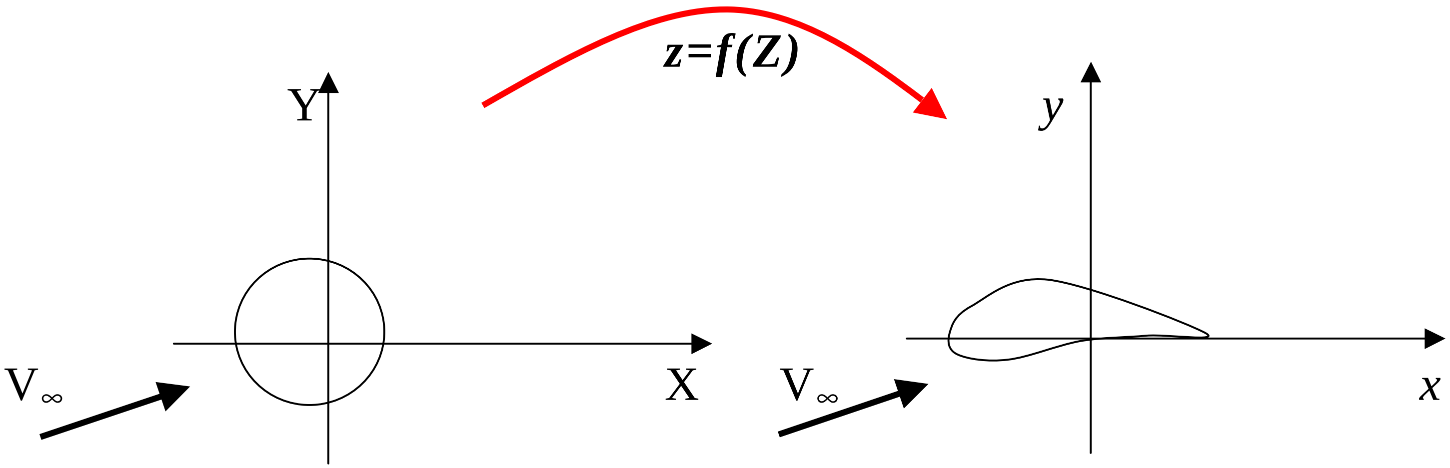
$$\frac{\partial^2 \Phi}{\partial X^2} + \frac{\partial^2 \Phi}{\partial Y^2} = 0 \quad \Rightarrow \quad \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0 \ ; \qquad \frac{\partial^2 \Psi}{\partial X^2} + \frac{\partial^2 \Psi}{\partial Y^2} = 0 \quad \Rightarrow \quad \frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} = 0$$

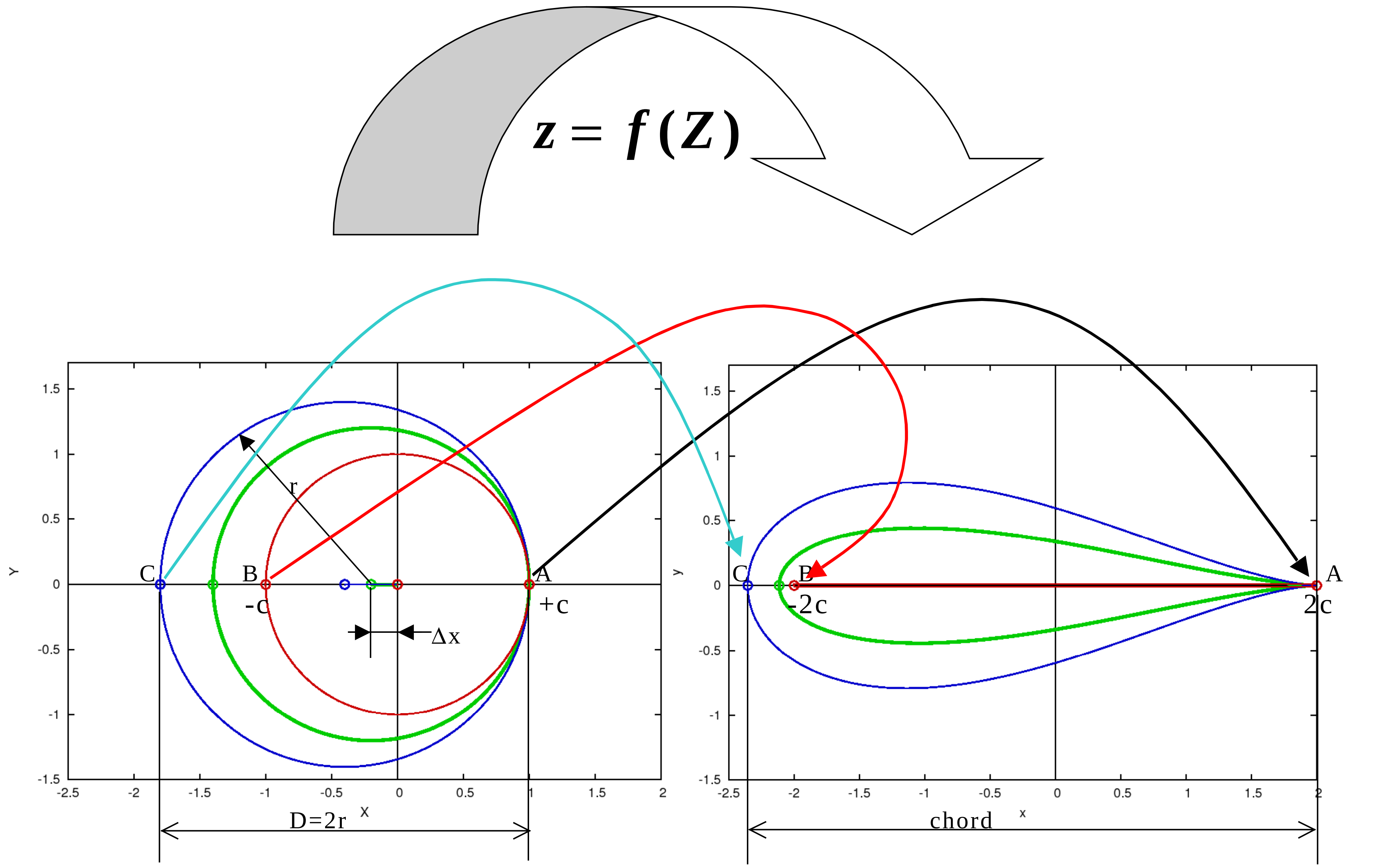
$$z = f(Z) \quad = Z + \frac{c^2}{Z} \qquad (JOUKOWSKY\ \ function)$$

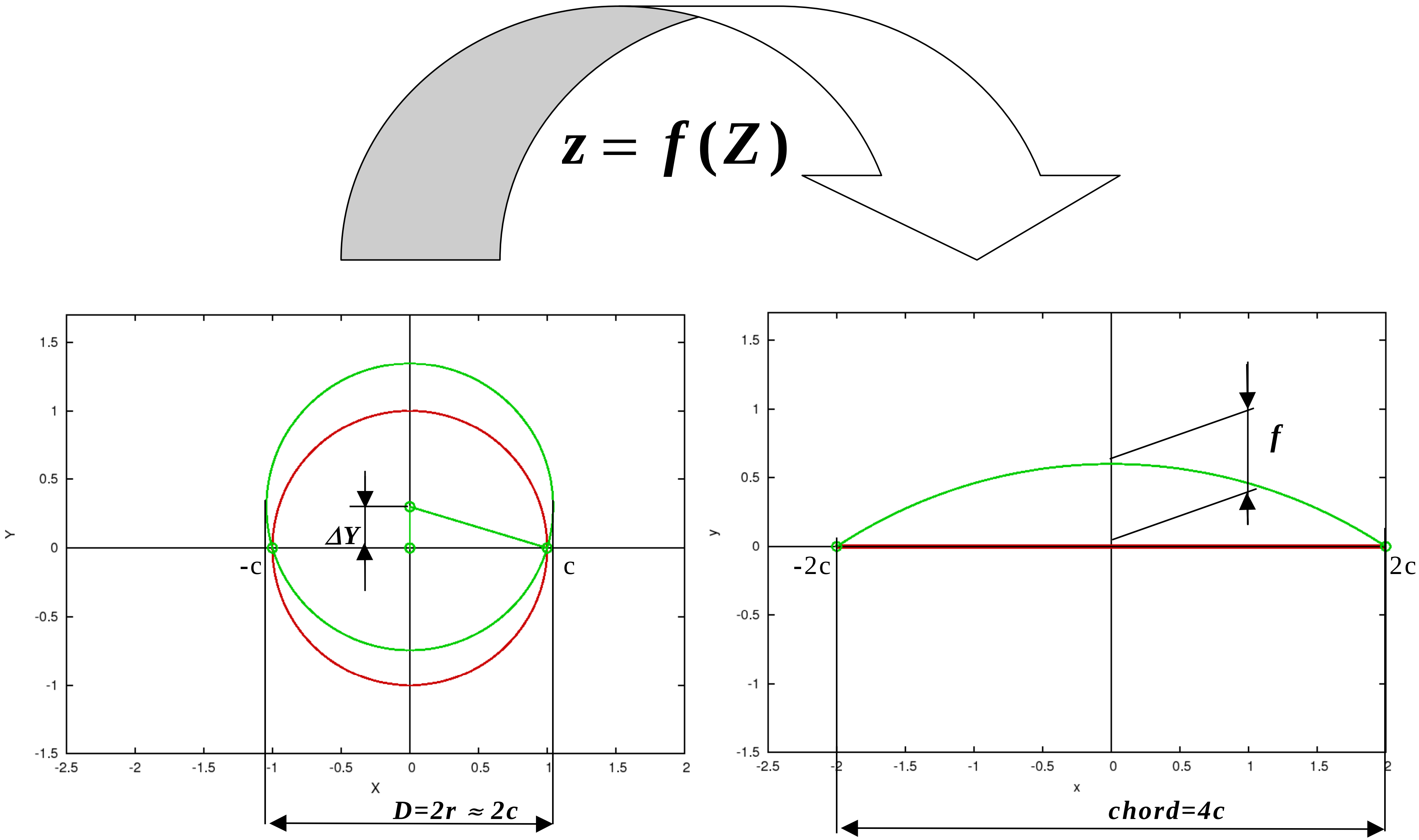
$$f'(Z) \quad = 1 - \frac{c^2}{Z^2}$$

$$z = \pm c \quad \Rightarrow \quad f' = 0$$

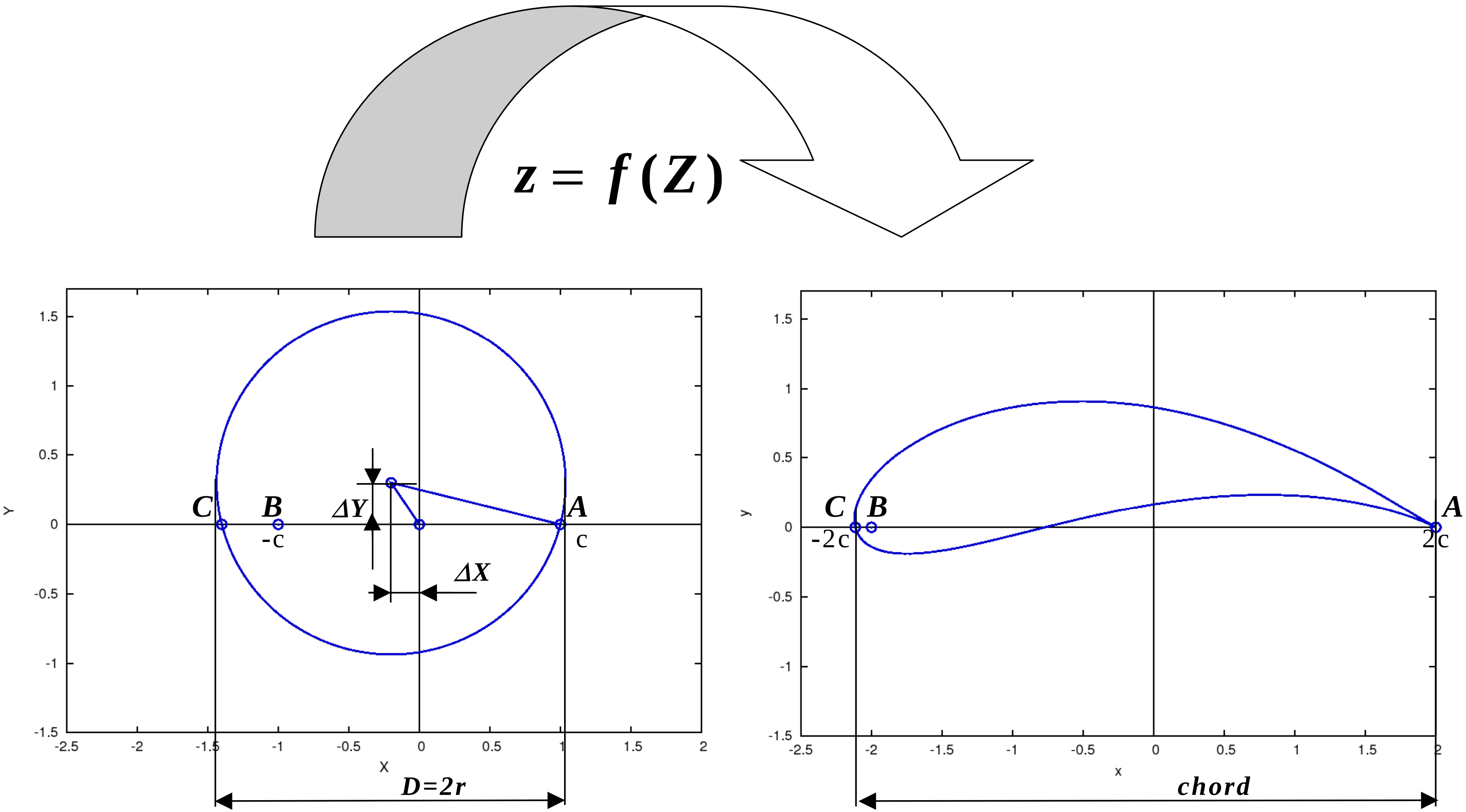
$$r \to \infty \quad \Rightarrow \quad f' = 1$$

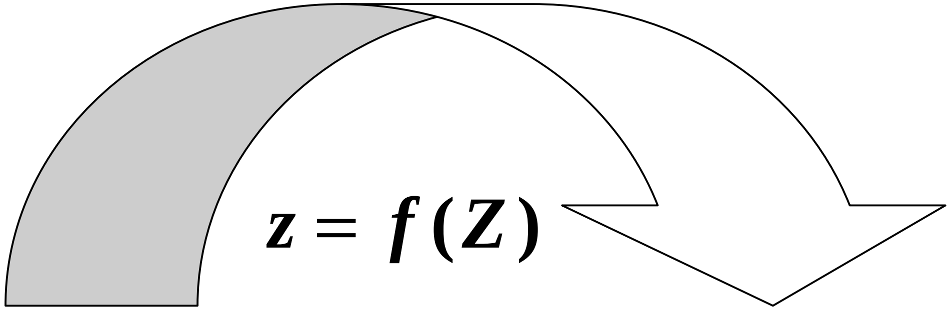



z = f(Z)
C
B
A
-c
+c
r
Δx
D=2r
X
Y
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
C
B
A
-2c
2c
chord
x
y

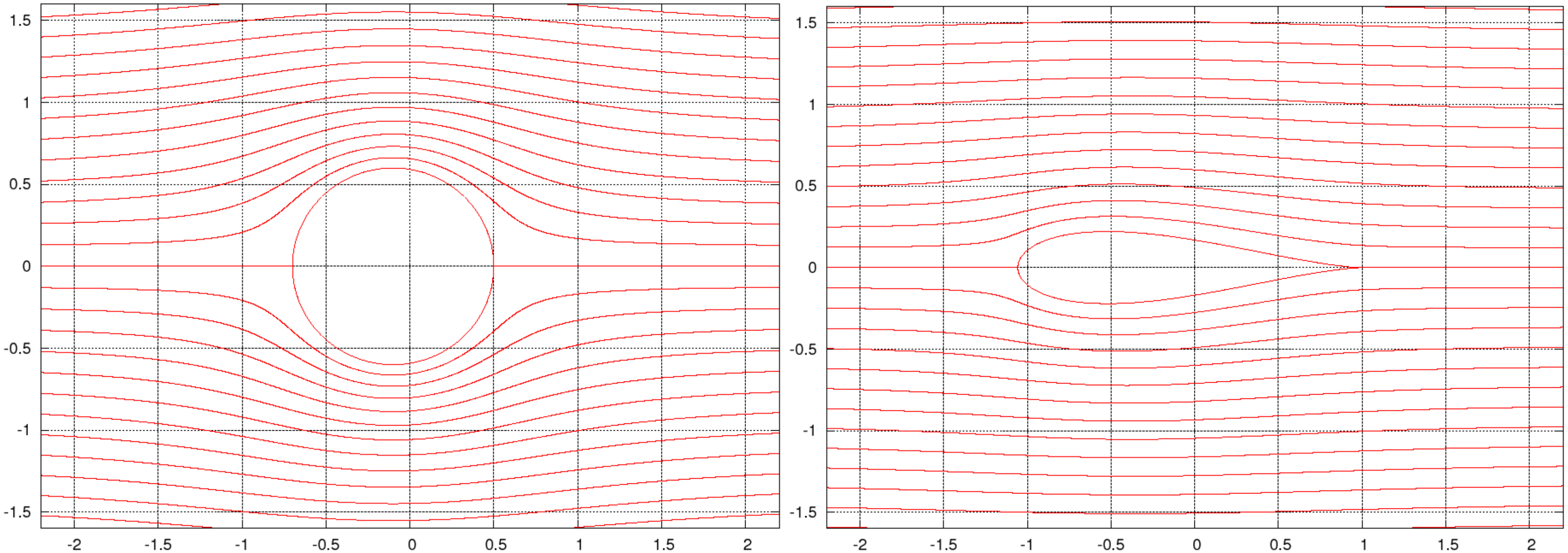


z = f(Z)
1.5
1
0.5
0
-0.5
-1
-1.5
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
X
Y
ΔY
-c
c
D=2r ≈ 2c
y
x
f
-2c
2c
chord=4c

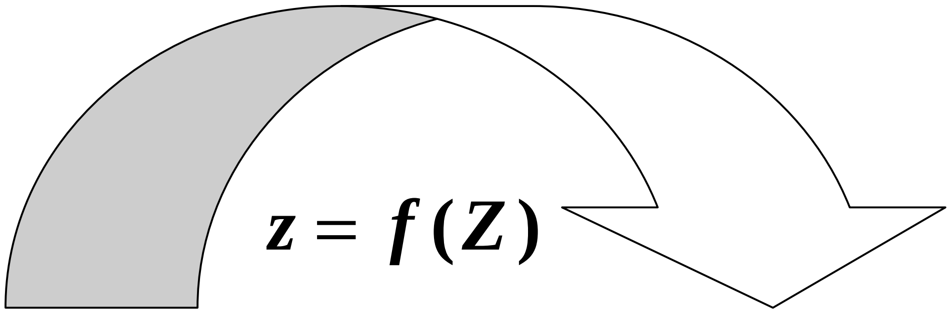

z = f (Z)
C
B
-c
ΔY
ΔX
A
c
X
Y
D=2r
C
B
-2c
A
2c
x
y
chord

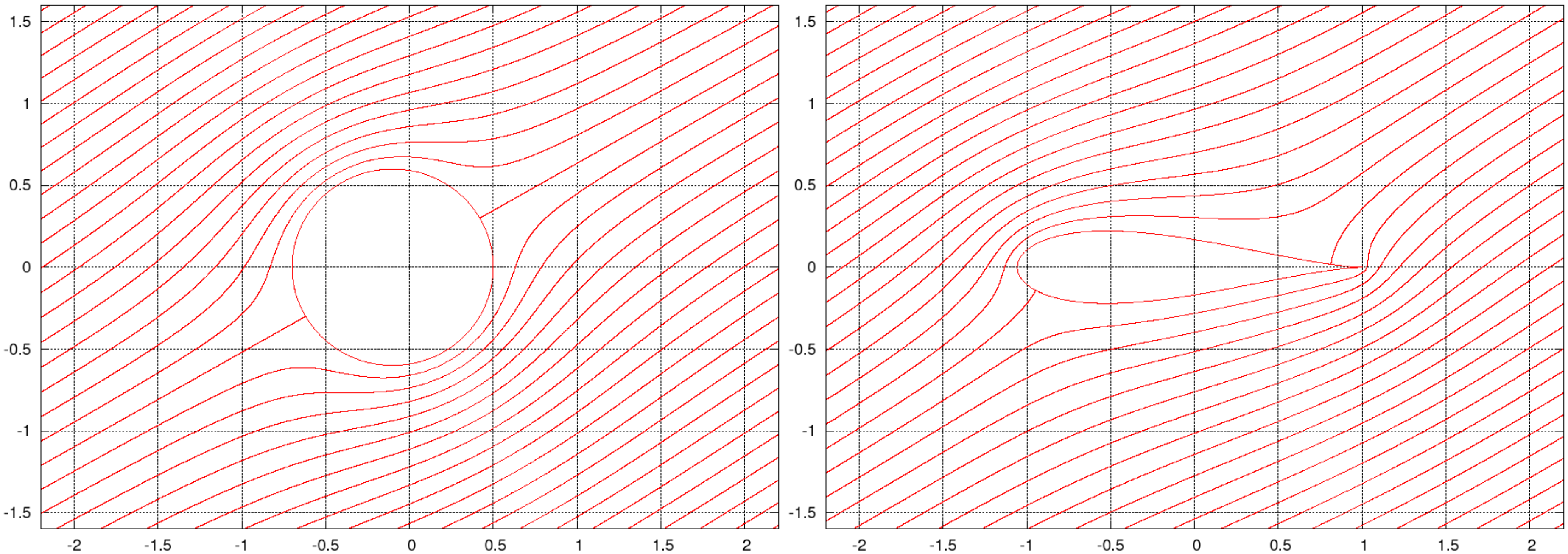


$$z = f(Z)$$

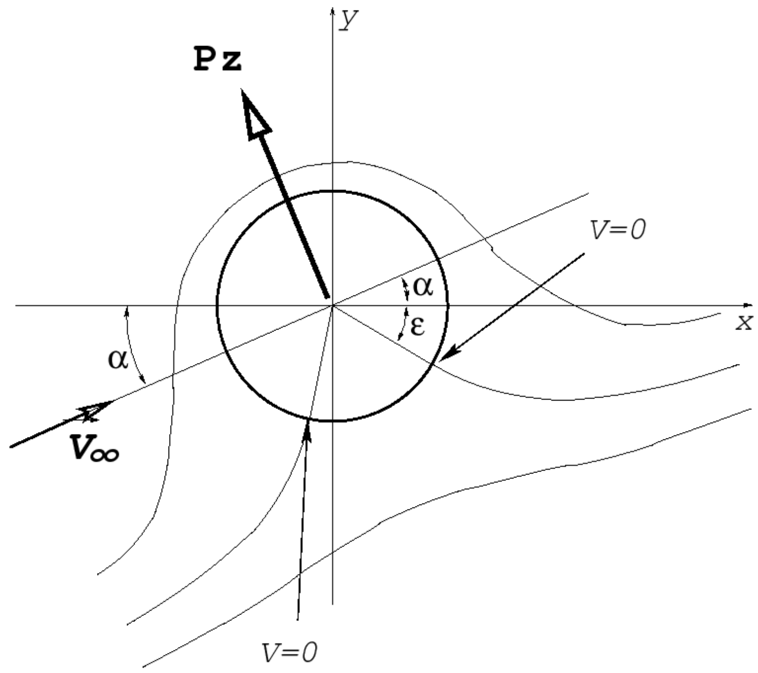


$\alpha = 0$ $\Gamma = 0$





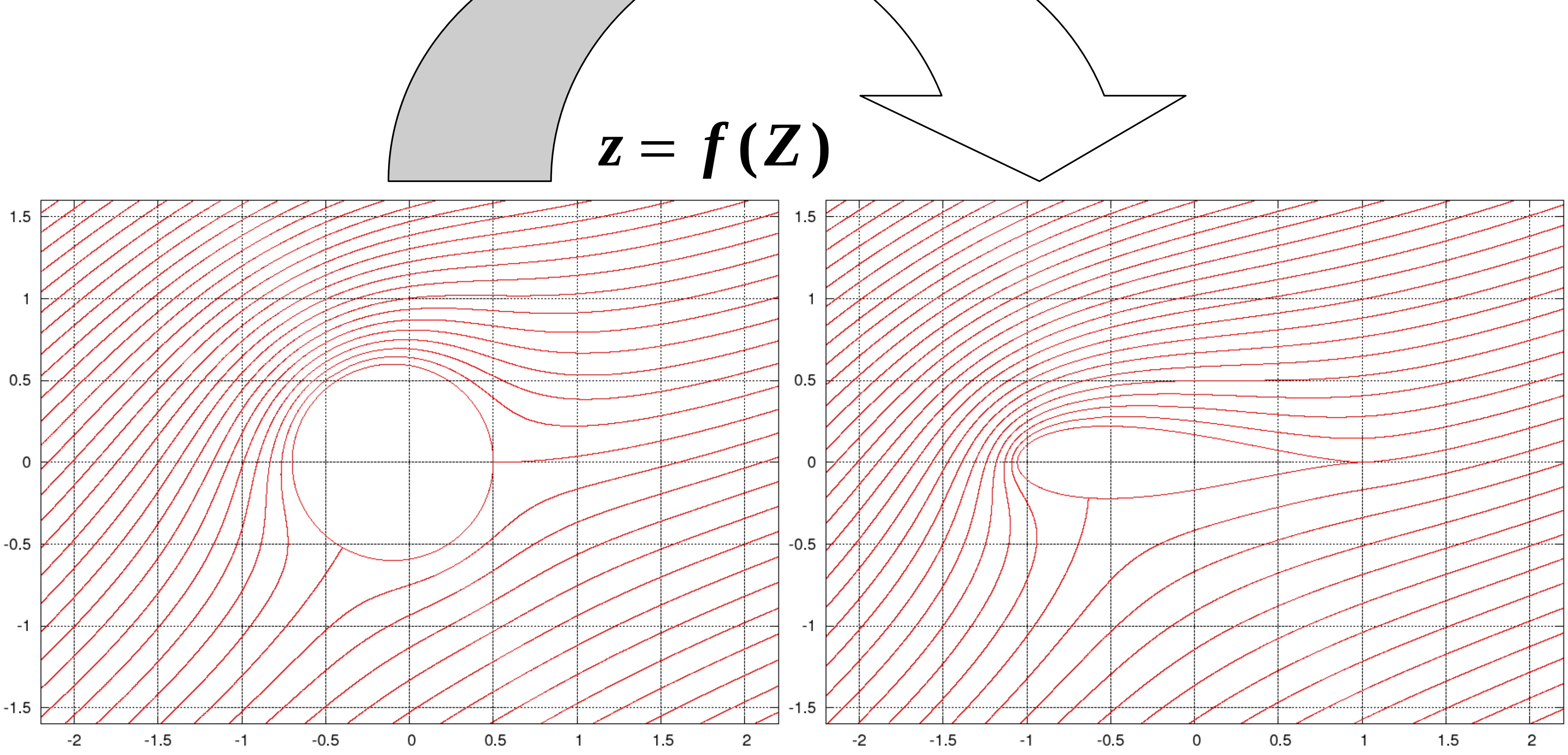
α=30^{O} Γ=0



$$\Gamma = 4\pi\, aU_{\infty} \sin\left(\alpha - \varepsilon\right)$$

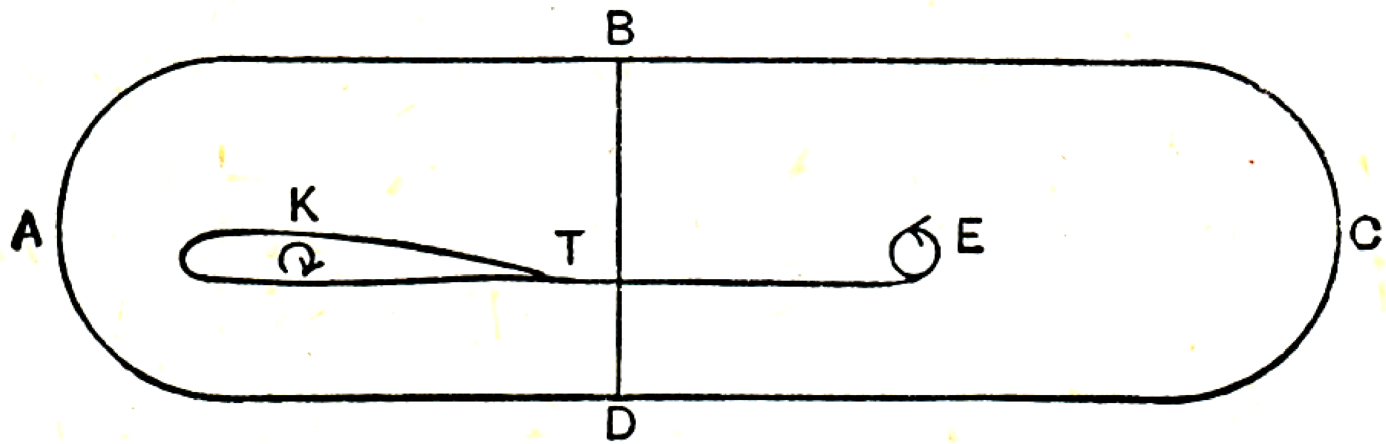
KUTTA-JOUKOWSKI HIPOTESES:
THE FLOW MUST LEAVE THE TRAILING EDGE SMOOTHLY

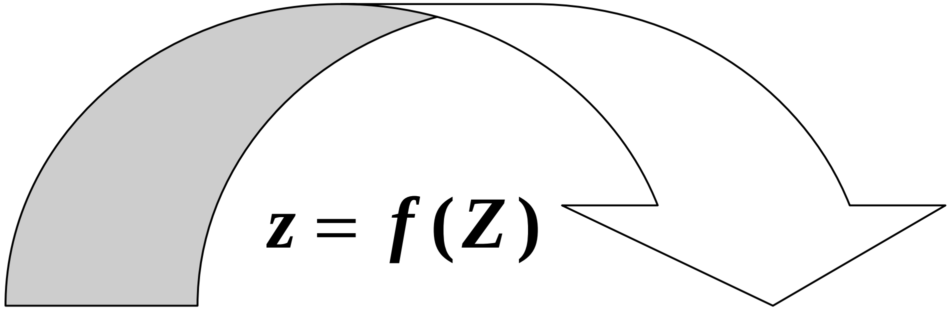
$$z = f(Z)$$

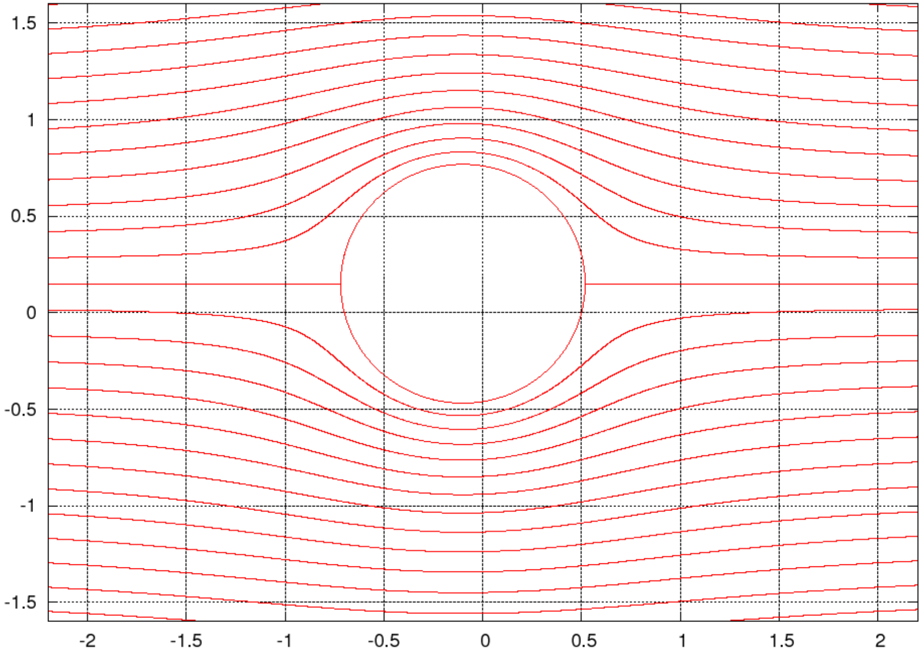


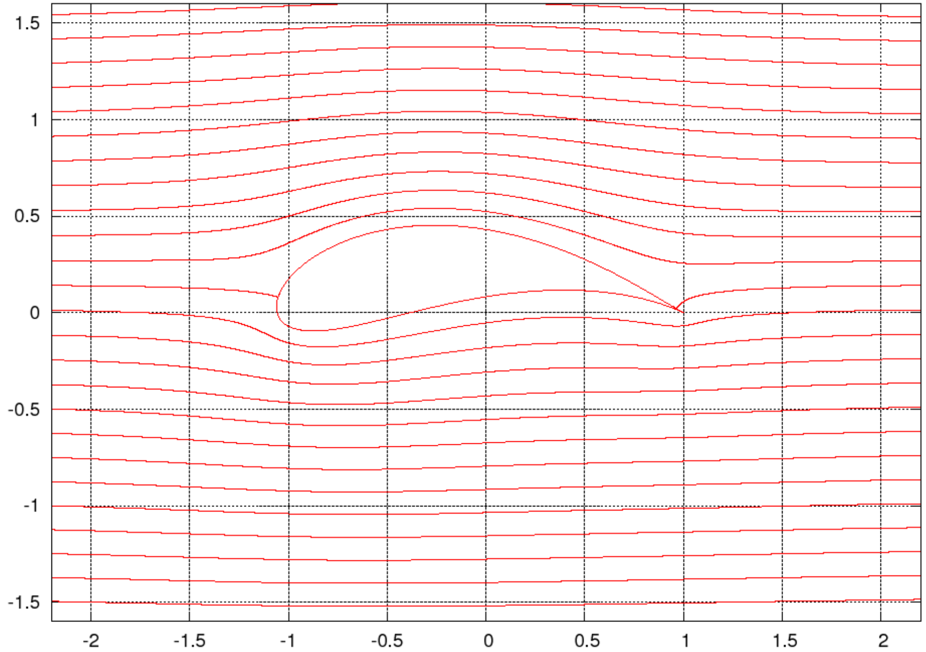
$$\alpha = 30^{O} \qquad \Gamma = 0.5 \cdot (4\pi a U_{\infty})$$

STARTING VORTEX



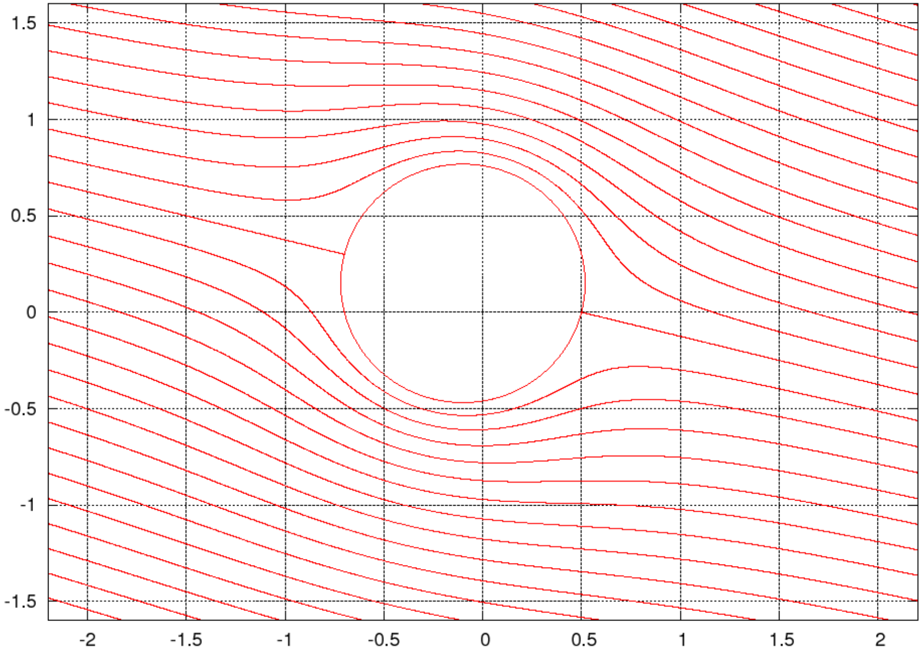


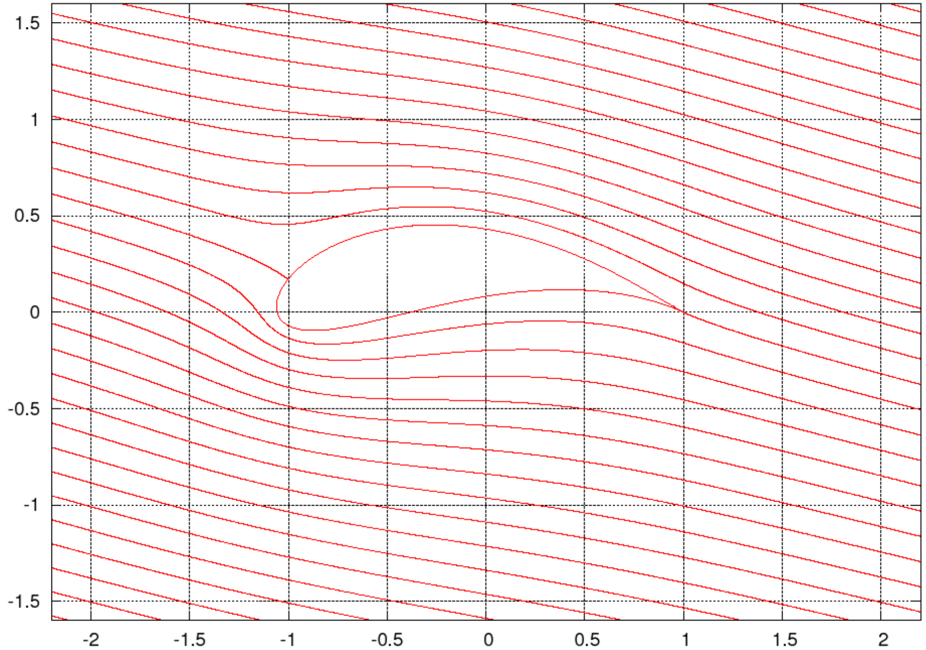




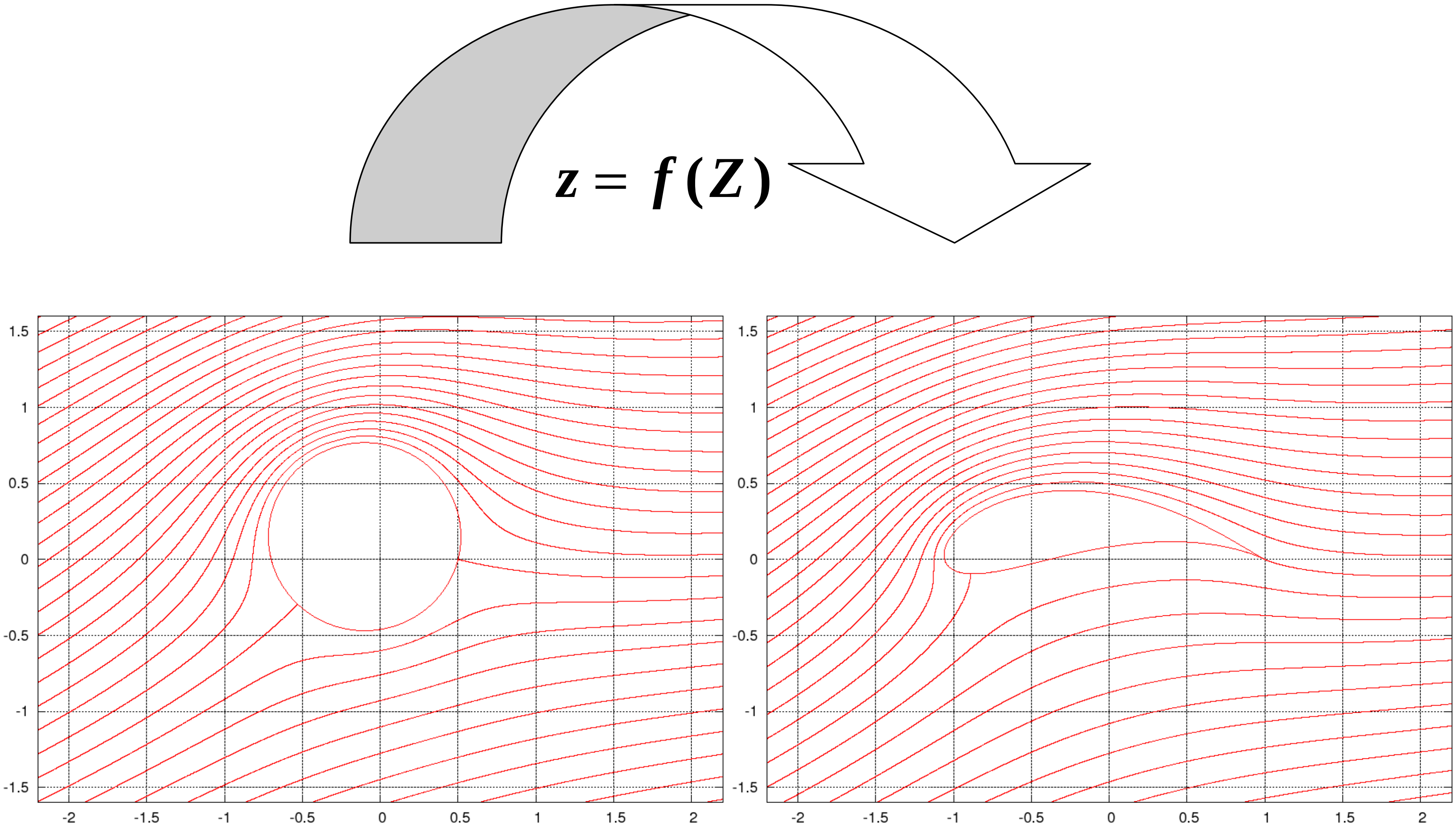
$\alpha = 0^{o}$ $\quad \Gamma = 0$

$$z = f(Z)$$





α=-13.7^{o} Γ=0



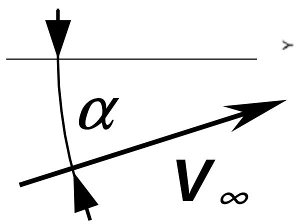
α=16^{O} Γ>0

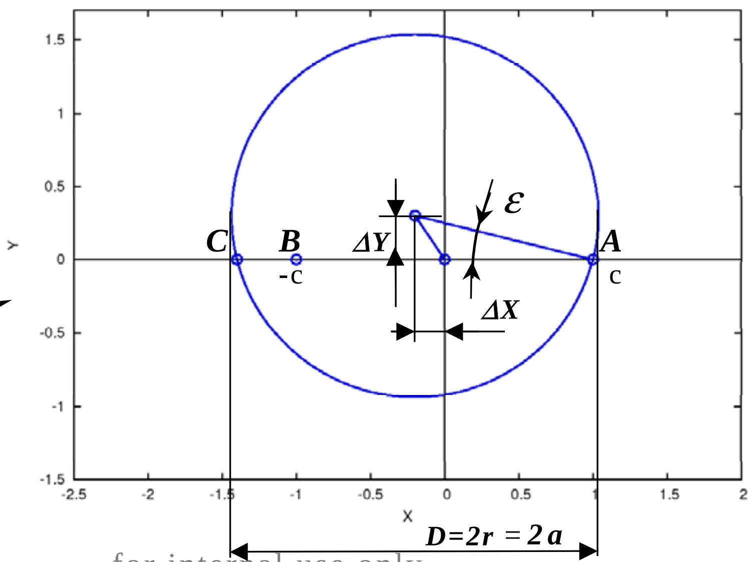
LIFT FORCE

CYLINDER:

$$L = (\rho_\infty)\cdot(U_\infty)\cdot\left(4\pi\, r U_\infty \sin(\alpha-\varepsilon)\right)\cdot(b) =$$

$$= \underbrace{4\pi\sin(\alpha-\varepsilon)}_{CL}\cdot\underbrace{2rb}_{S}\cdot\underbrace{\rho_\infty\frac{U_\infty^{2}}{2}}_{q_\infty} = \underbrace{4\pi\sin(\alpha-\varepsilon)}_{C_L}\cdot S\cdot q_\infty$$





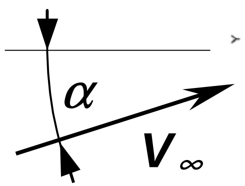
LIFT FORCE

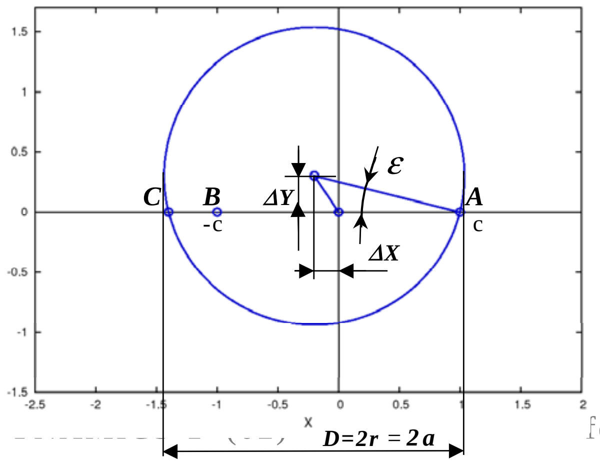
AIRFOIL:

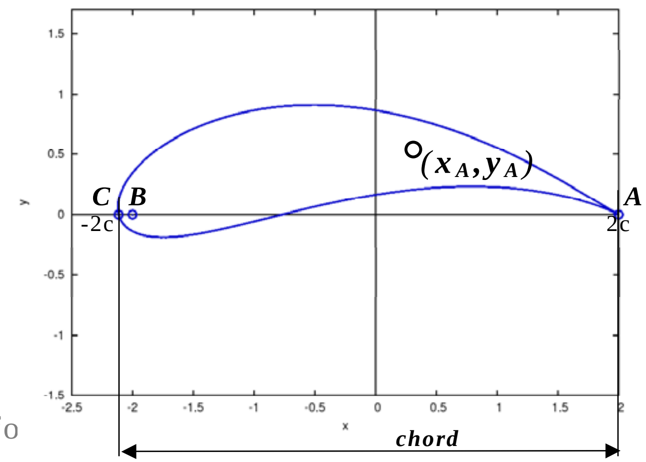
$$L_{AIRFOIL} = L_{CYLINDER} = (\rho_\infty)\cdot(U_\infty)\cdot\left(4\pi\, rU_\infty \sin(\alpha-\varepsilon)\right)\cdot(b) =$$

$$= \underbrace{4\pi \frac{2r}{chord}\sin(\alpha-\varepsilon)}_{C_L}\cdot\underbrace{chord\cdot b}_{S}\cdot\underbrace{\rho_\infty \frac{U_\infty^{\ 2}}{2}}_{q_\infty}$$

$$= \underbrace{4\pi \frac{2r}{chord}\cdot\sin(\alpha-\varepsilon)}_{C_L}\cdot S\cdot q_\infty$$

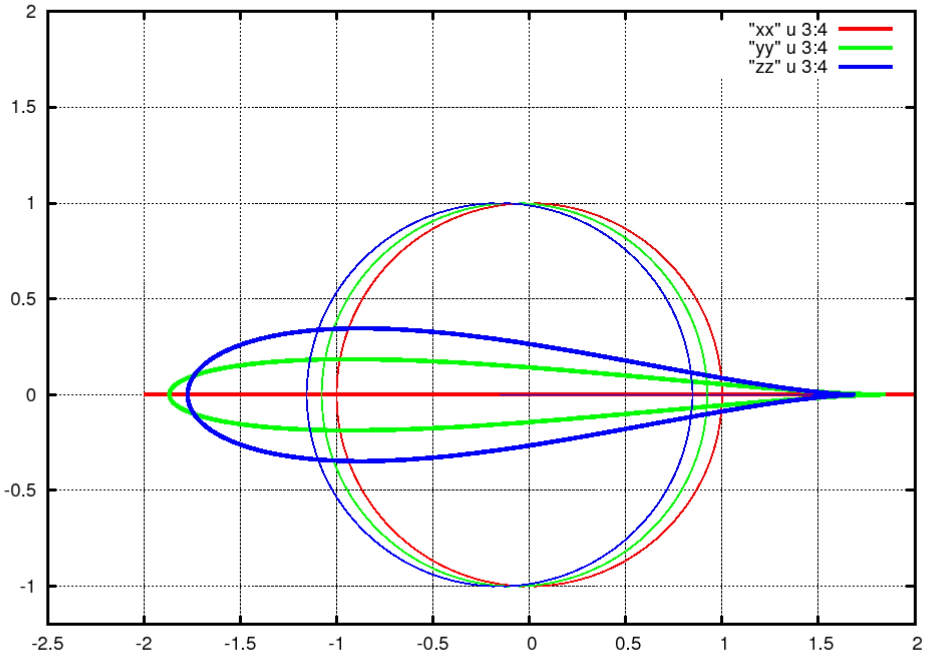


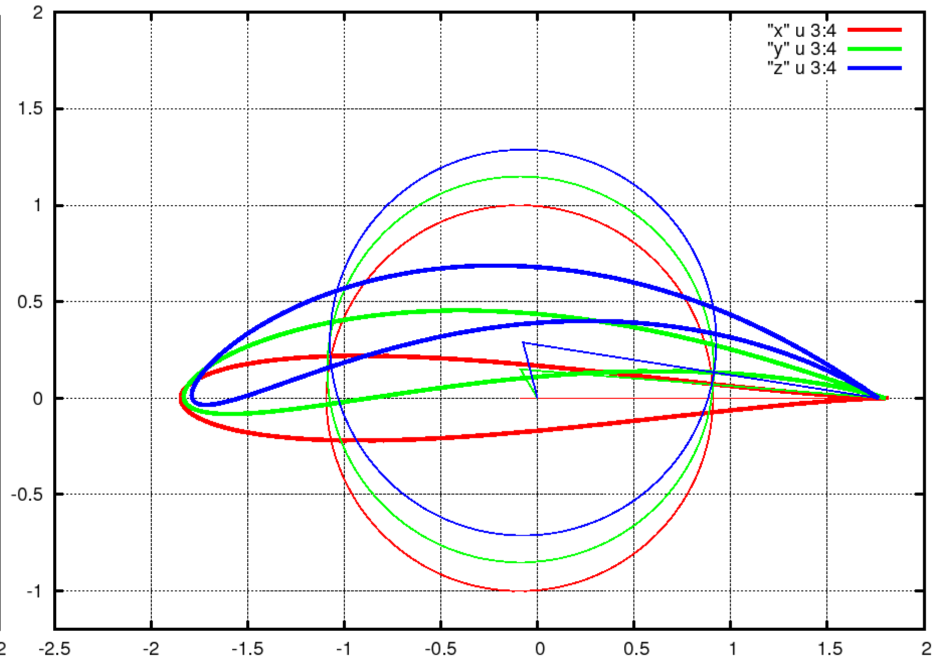




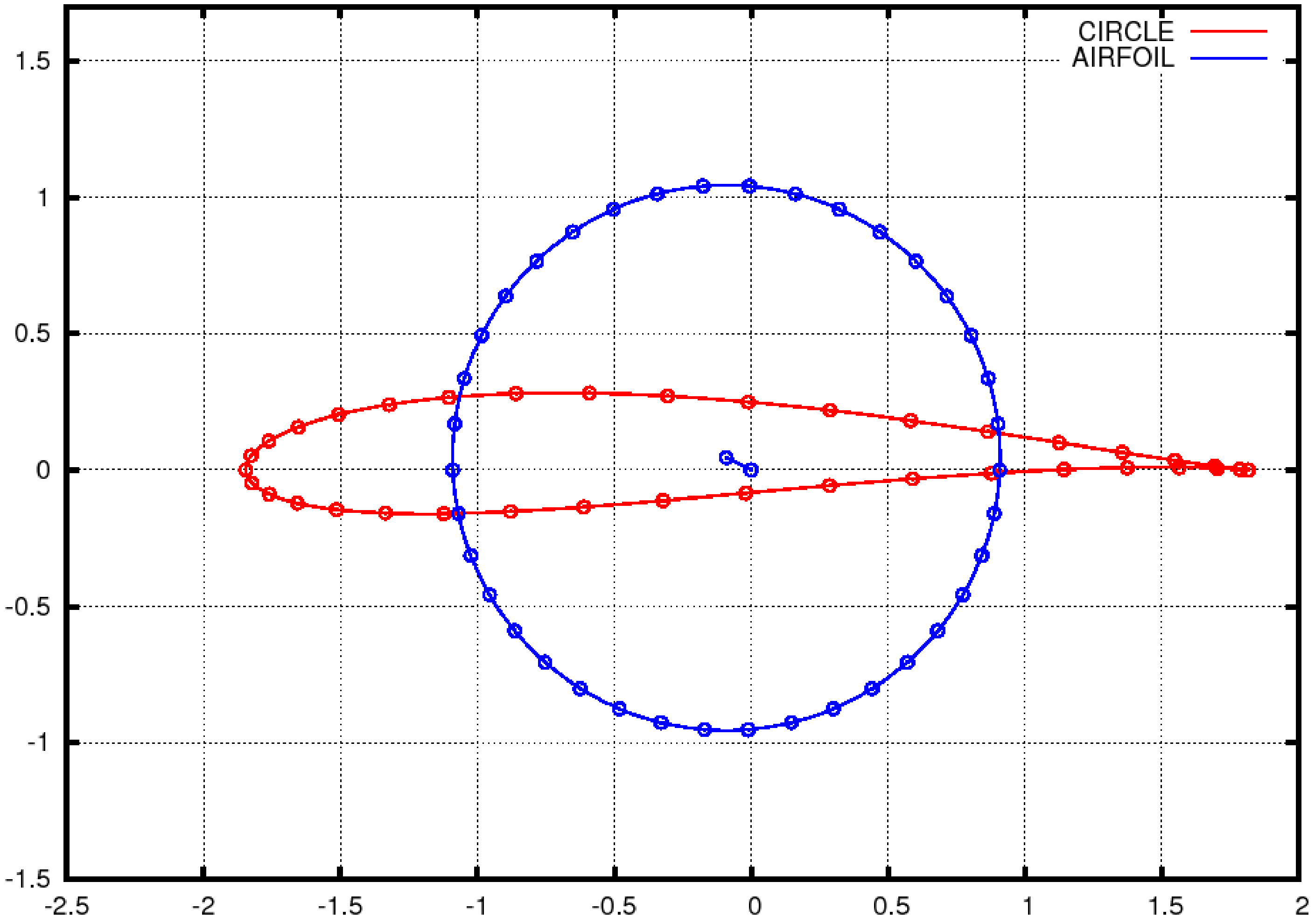
LIFT FORCE

$$(2r):(chord)\quad ?$$





above: 2r=1

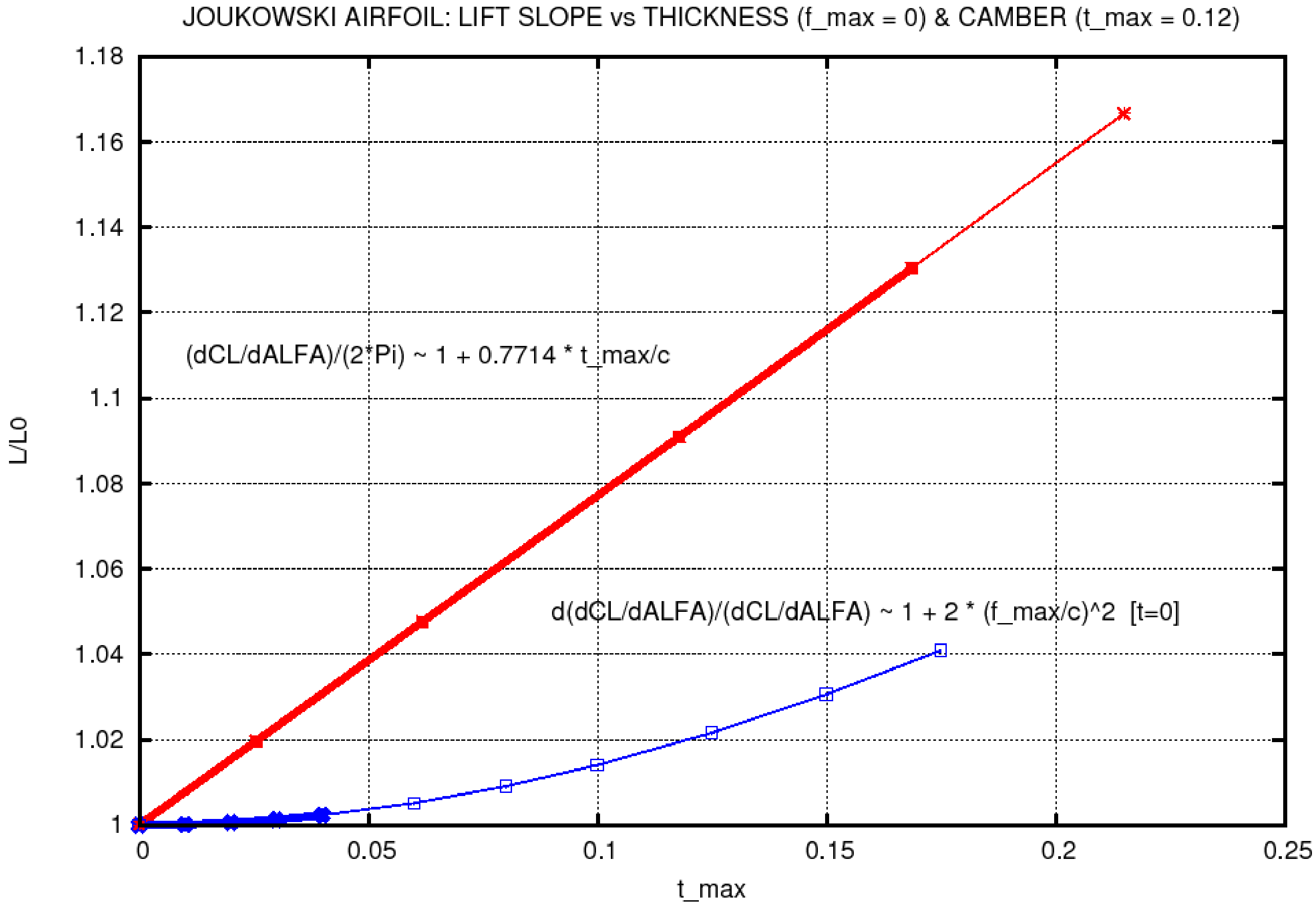

CIRCLE
AIRFOIL
1.5
1
0.5
0
-0.5
-1
-1.5
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2

FLAT PLATE (thickness t=0, camber f=0): $\varepsilon = 0$

$$\frac{2r}{chord} = 1/2 \quad \Rightarrow \quad C_L = 2\pi \cdot \sin(\alpha)$$

thickness t>0:

$$\frac{2r}{chord} > 1/2 \quad \Rightarrow C_L = 2\pi\left(1 + h\left(\bar{t}\right)\right) \cdot \sin(\alpha) \quad > 2\pi \cdot \sin(\alpha)$$


JOUKOWSKI AIRFOIL: LIFT SLOPE vs THICKNESS (f_max = 0) & CAMBER (t_max = 0.12)
1.18
1.16
1.14
1.12
1.1
1.08
1.06
1.04
1.02
1
L/L0
(dCL/dALFA)/(2*Pi) ~ 1 + 0.7714 * t_max/c
d(dCL/dALFA)/(dCL/dALFA) ~ 1 + 2 * (f_max/c)^2 [t=0]
0
0.05
0.1
0.15
0.2
0.25
t_max