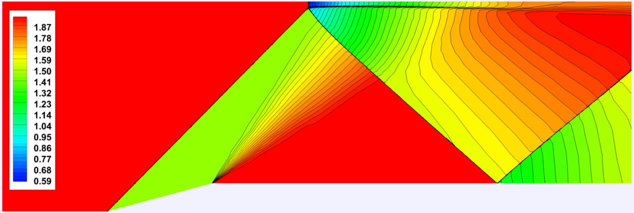
Aerodynamics I

Oblique shock waves and expansion waves



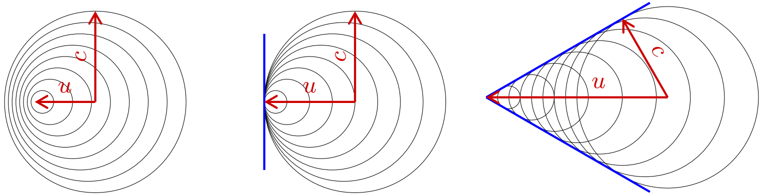
supersonic flow in a channel $M_\infty = 2$ *(Mach number field)*

19 maja 2014

Mach lines

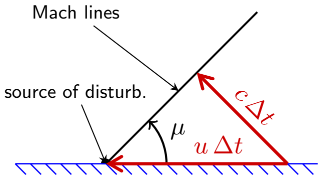
Previous lecture was dedicated to problems which describes 1D compressible flow phenomena. This lecture will introduce problems occuring in 2D flows.

Let us imagine a source of small disturbances moving with velocity $\mathbf{u}$. The disturbance are propagating with speed of sound c and are small enough in order to not change the state of gas significantly. (e.g., acoustic waves).



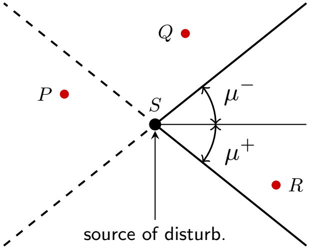
$u < c$ or $M < 1$ $u = c$ or $M = 1$ $u > c$ or $M > 1$

Mach lines



In supersonic flow the angle of the Mach lines can be computed from:

$$\mu = \arcsin\left(\frac{c}{u}\right) = \arcsin\left(\frac{1}{M}\right) \qquad (1.1)$$



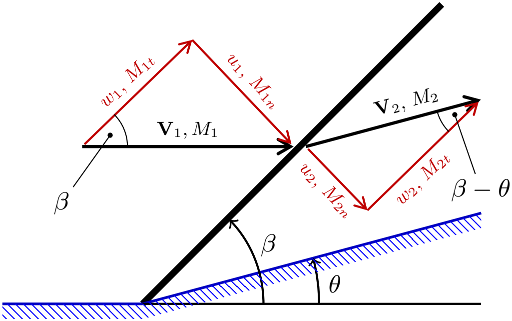
The Mach lines are bounding zones of influence and dependence.

- sate of gas in point P can influence the state of gas in S
- sate of gas in point R can depend on the state of gas in S
- sate of gas in point Q does not influence on the state of gas in S and does not depend on it

Oblique shock wave

Oblique shock wave

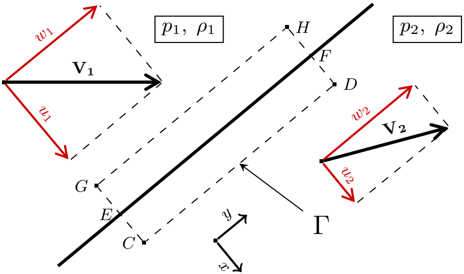
In supersonic flows the shock wave may be inclined to the flow velocity. Such shock wave is called oblique shock wave. Such shock wave occurs when the direction of flows is deflected by a certain angle θ (e.g. flow past the a concave corner)



Geometry of the oblique shock wave

Oblique shock wave

The conservation laws for an oblique shock wave can be written:



Continuity equation: $$\oint_\Gamma \rho\,\mathbf{v}\cdot\mathbf{n}\;d\Gamma = 0$$

Momentum equation: $$\oint_\Gamma \rho\,\mathbf{v}\,(\mathbf{v}\cdot\mathbf{n})\;d\Gamma = -\oint_\Gamma p\,\mathbf{n}\;d\Gamma$$

Energy equation: $$\oint_\Gamma \rho\,E\,\mathbf{v}\cdot\mathbf{n}\;d\Gamma = -\oint_\Gamma p\,\mathbf{v}\cdot\mathbf{n}\;d\Gamma$$

Oblique shock wave

Integrals for surface elements GE and HF and also EC and FD cancel each other. It is necessary to take into account only contrbution from surfaces GH and CD. The surface area of GH and CD is the same and is equal A.

Continuity equation:

$$\underbrace{-\rho_1 u_1 A}_{GH} + \underbrace{\rho_2 u_2 A}_{CD} = 0 \quad \rightarrow \quad \boxed{\rho_1 u_1 = \rho_2 u_2} \tag{2.1}$$

Momentum equation in direction tangent to the shock wave:

The component tangent to the shock wave of the normal vectors for GH and CD is equal 0. Therefore, the pressure integrals are also equal 0:

$$\underbrace{-\rho_1 u_1 w_1}_{GH} + \underbrace{\rho_2 u_2 w_2}_{CD} = 0 \quad \rightarrow \quad \rho_1 u_1 w_1 = \rho_2 u_2 w_2 \quad \xrightarrow{(2.1)} \quad \boxed{w_1 = w_2} \tag{2.2}$$

Oblique shock wave

Momentum equation in direction normal to the shock wave:

$$\underbrace{-\rho_1 u_1^2 A - p_1 A}_{GH} + \underbrace{\rho_2 u_2^2 + p_2 A}_{CD} \quad \rightarrow \quad \rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2 \qquad (2.3)$$

Energy equation:

$$\underbrace{-\rho_1 u_1 \left(e_1 + \frac{u_1^2 + w_1^2}{2} \right) A - u_1 p_1 A}_{GH} + \underbrace{\rho_2 u_2 \left(e_2 + \frac{u_2^2 + w_2^2}{2} \right) A + u_2 p_2 A}_{CD} = 0$$

Using relation (2.1):

$$e_1 + \frac{p_1}{\rho_1} + \frac{u_1^2 + w_1^2}{2} = e_2 + \frac{p_2}{\rho_2} + \frac{u_2^2 + w_2^2}{2}$$

Since $w_1 = w_2$ (2.2) it can be written as follows:

$$e_1 + \frac{p_1}{\rho_1} + \frac{u_1^2}{2} = e_2 + \frac{p_2}{\rho_2} + \frac{u_2^2}{2} \quad \rightarrow \quad h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2} \qquad (2.4)$$

Oblique shock wave

The equations (2.1), (2.3) and (2.4) are exactly the same as the normal shock wave equations written in direction normal to the oblique shock wave. We can use then the relations that were already derived:

$$M_{2n}^2 = \frac{2 + (k-1)\ M_{1n}^2}{2\ k\ M_{1n}^2 - (k-1)} \tag{2.5}$$

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{(k+1)\ M_{1n}^2}{2 + (k-1)\ M_{1n}^2} \tag{2.6}$$

$$\frac{p_2}{p_1} = 1 + \frac{2\ k}{k+1}\ (M_{1n}^2 - 1) \tag{2.7}$$

Using trigonometrical relations:

$$\frac{u_1}{w_1} = \mathrm{tg}(\beta) \qquad \frac{u_2}{w_2} = \mathrm{tg}(\beta - \theta) \quad \rightarrow \quad \frac{\mathrm{tg}(\beta-\theta)}{\mathrm{tg}(\beta)} = \frac{u_2}{u_1} \tag{2.8}$$

Oblique shock wave

Using equations (2.6), (2.8) and $M_{1n} = M_1 \ \sin(\beta)$ we can obtain:

$$\frac{\mathrm{tg}(\beta-\theta)}{\mathrm{tg}(\beta)} = \frac{2+(k-1)\ M_1^2 \ \sin^2(\beta)}{(k+1)\ M_1^2 \ \sin^2(\beta)} \tag{2.9}$$

The equation above can be transform into a explicit relation for θ:

$$\mathrm{tg}(\theta) = 2\,\mathrm{ctg}(\beta)\frac{M_1^2 \ \sin^2(\beta)-1}{M_1^2 \ (k+\cos(2\beta))+2} \tag{2.10}$$

This equation is known as a θ-β-M equation.

Oblique shock wave

Using:

$$M_{2n} = M_2 \ \sin(\beta - \theta) \qquad M_{1n} = M_2 \ \sin(\beta) \tag{2.11}$$

and equations (2.5), (2.7) and (2.6) we can obtain relations for change of Mach number, density and pressure:

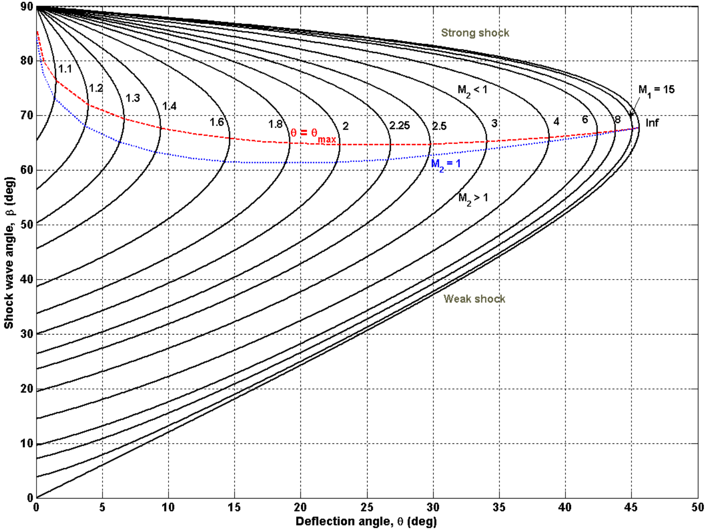
$$M_2^2 = \frac{1}{\sin^2(\beta - \theta)} \ \frac{2 + (k-1)\ M_1^2 \ \sin^2(\beta)}{2\ k\ M_1^2 \ \sin^2(\beta) - (k-1)} \tag{2.12}$$

$$\frac{\rho_2}{\rho_1} = \frac{(k+1)\ M_1^2 \ \sin^2(\beta)}{2 + (k-1)\ M_1^2 \ \sin^2(\beta)} \tag{2.13}$$

$$\frac{p_2}{p_1} = 1 + \frac{2\ k}{k+1} \ (M_1^2 \ \sin^2(\beta) - 1) \tag{2.14}$$

If β is equal $\pi/2$, the equations above are identical to the equations of the normal shock wave.

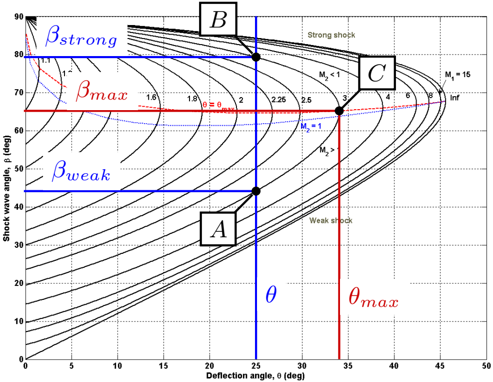
Oblique shock wave – equation θ-β-M



źródło: http://en.wikipedia.org

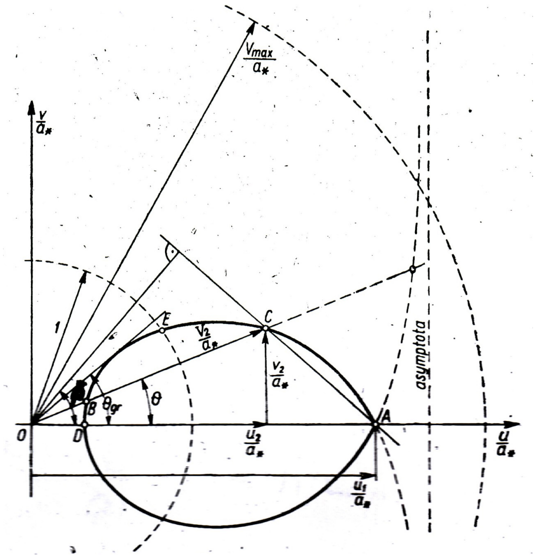
Oblique shock wave – equation θ-β-M

Example:



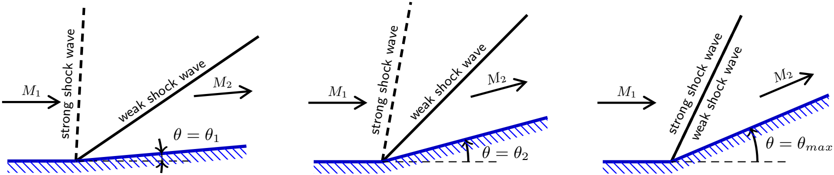
For flow $M_1 = 3$ past a concave corner with $\theta = 25^\circ$ we can get two possible solutions for shock angle β: A (weak shock wave) and B (strong shock wave). Maximal possible oblique shock angle β_{max} can also be found for corresponding angle θ_{max} (C).

Shock polars



Strong and weak oblique shock waves

Equations allows existence of two types of oblique shock waves: strong and weak.



$$\theta_1 < \theta_2 < \theta_{max}$$

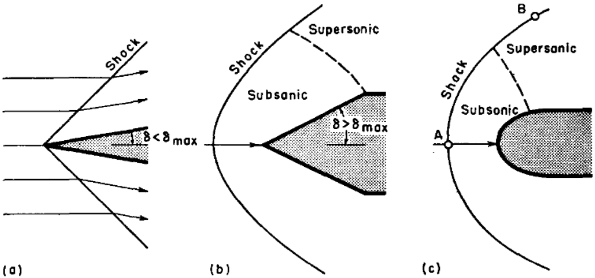
weak shock wave – $\beta_{weak} < \beta_{max}$, $M_2 > 1$ except when near θ_{max}
strong shock wave – $\beta_{strong} > \beta_{max}$, $M_2 < 1$

Change of gas state parameters p_2/p_1, ρ_2/ρ_1, T_2/T_1 i $s_2 - s_1$ is greater for the strong shock wave.

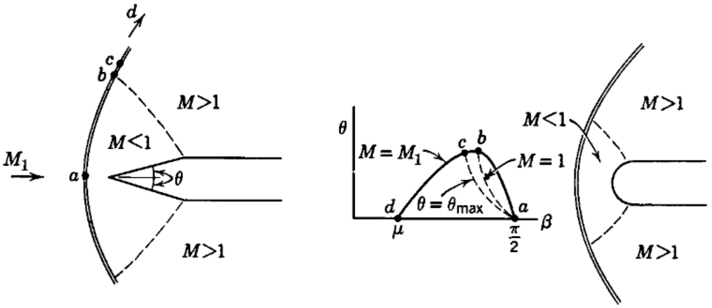
When $\theta \rightarrow 0$ then $\beta_{strong} \rightarrow \pi/2$ oraz $\beta_{weak} \rightarrow \mu = \arcsin(1/M_1)$
When $\theta = \theta_{max}$ then $\beta_{strong} = \beta_{weak} = \beta_{max}$

In physical flows, when $\theta < \theta_{max}$ typically, weak shock waves are present.

Detached shock wave

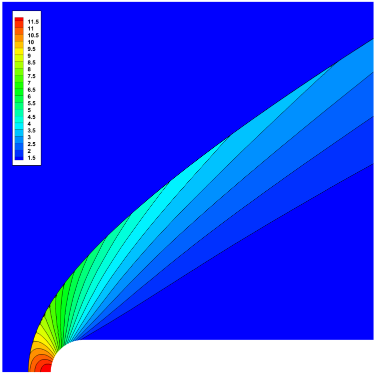


Detached shock wave

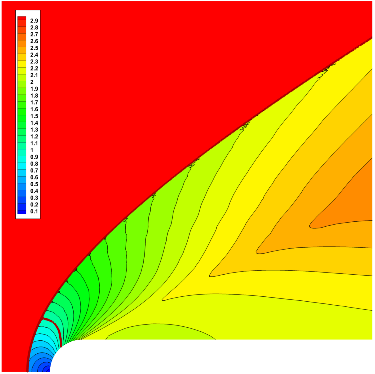


(a) (b) (c)

Flow past a blunt object - $M_\infty = 3$

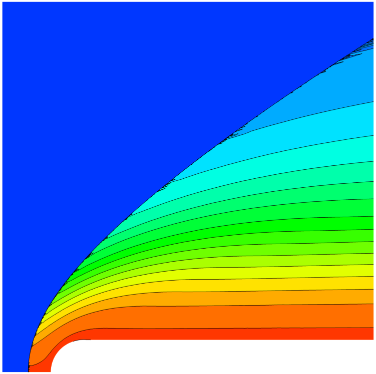


Pressure distribution

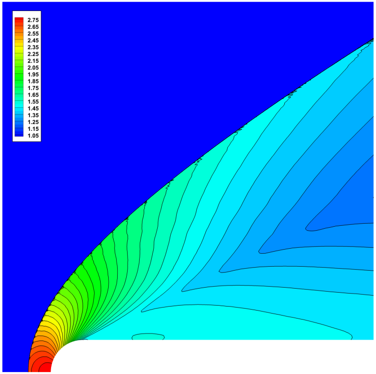


Mach number distribution

Flow past a blunt object - $M_\infty = 3$



Entropy distribution

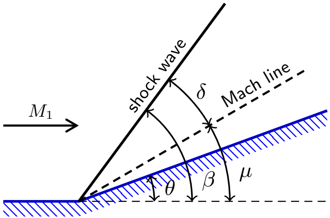


Distribution of the temperature change - T/T_∞

Smooth concave corner

In order to describe the flow past a smooth concave corner let us imagine a corner which is assembled from N corners with angle $\Delta\theta = \theta/N$ where $N \to \infty$.

What is a difference between the flow past single and multiple corner? Especially, what is the difference in the change of entropy?



For single corner, if $\Delta\theta \to 0$ then $\beta \to \mu$. It will be useful to use $\beta = \mu + \delta$. If $\Delta\theta \to 0$ then:

$$\begin{aligned} &\mathrm{tg}(\Delta\theta) \approx \Delta\theta \\ &\delta \to 0 \\ &\sin(\delta) \approx \mathrm{tg}(\delta) \approx \delta \\ &\cos(\delta) \approx 1 \end{aligned} \tag{2.15}$$

Smooth concave corner

Change of entropy for gas passing through the oblique shock wave:

$$\Delta s = s_2 - s_1 = \frac{R}{k-1} \ \ln\left[\frac{p_2}{p_1}\left(\frac{\rho_1}{\rho_2}\right)^k\right] =$$
$$= \frac{R}{k-1} \ \ln\left[\left[1 + \frac{2\,k}{k+1}\,(M_1^2\,\sin^2(\beta) - 1)\right]\left[\frac{(k+1)\,M_1^2\,\sin^2(\beta)}{2+(k-1)\,M_1^2\,\sin^2(\beta)}\right]^{-k}\right] \tag{2.16}$$

The equation above can be simplified using $\theta \to 0$ and $M_{1n} \to 1$. Then equation (2.16) can be expanded using Taylor series:

$$\Delta s = \frac{2}{3}\,\frac{k\,(k-1)}{(k+1)^2}\,(M_1^2\,\sin^2(\beta) - 1)^3 + \mathcal{O}\left((M_1^2\,\sin^2(\beta) - 1)^4\right) \tag{2.17}$$

Smooth concave corner

Let us find the relation of β on θ. To do this we will use (2.10) and approximations (2.15). The result can be transformed in order to obtain relation for δ and expanded using Taylor series:

$$\beta = \mu + \delta = \arcsin\left(\frac{1}{M_1}\right) + \frac{1}{4}\,\frac{(k+1)\,M_1^2}{M_1^2 - 1}\,\Delta\theta + \mathcal{O}(\Delta\theta^2) \qquad (2.18)$$

The approximated relation for β can be substituted into (2.17). After expanding using Taylor series once again we can obtain result:

$$\Delta s = \frac{1}{12}\,\frac{k\,M_1^6\,(k^2-1)}{\sqrt{(M_1^2-1)^3}}\,\Delta\theta^3 + \mathcal{O}(\Delta\theta^4) \qquad (2.19)$$

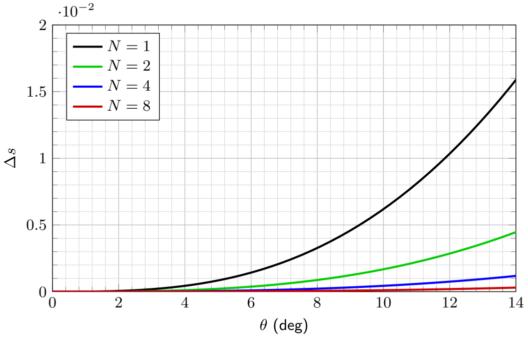
Smooth concave corner

For multiple concave corner with N deflections:

$$\Delta s \sim N\,\Delta\theta^3 = N\left(\frac{\theta}{N}\right)^3 = \frac{1}{N^2}\,\theta^3 \tag{2.20}$$

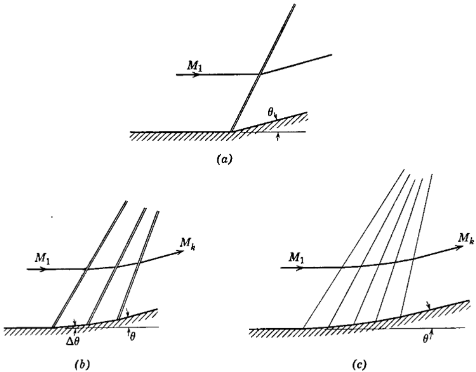
For single concave corner $(N = 1)$:

$$\Delta s \sim \theta^3 \tag{2.21}$$

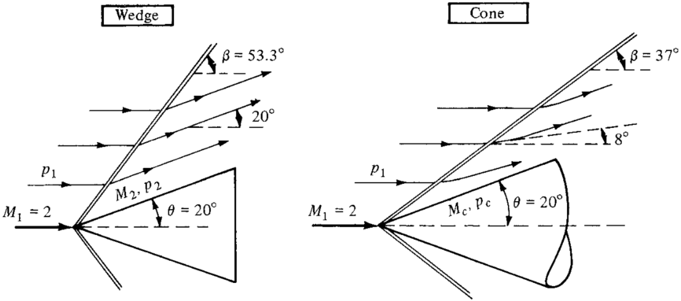


Plot obtained for $M_1 = 2$ using expansion up to $\mathcal{O}(\Delta\theta^{11})$

Smooth concave corner



Supersonic flow past a cone

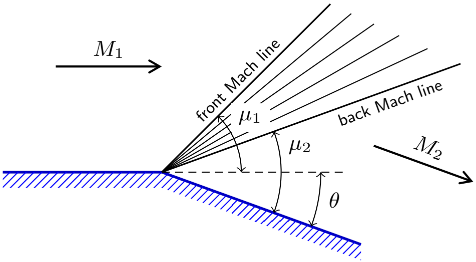


Expansion waves

Flow past convex corner

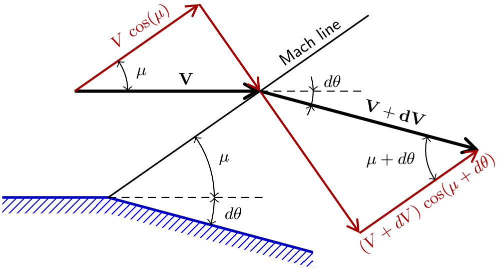
In supersonic flow past convex corner we can define the equations of conservation just like for concave corner. One of the solutions is a expansion shock wave. However, for such solution $\Delta s < 0$ thus it is not realistic.

Another solution is an isentropic process which results in a fan of weak expansion waves.



Expansion waves

Let us try to analyze the flow past corner with angle $d\theta \to 0$. From the tip of the corner, the Mach line can be drawn. After passing through the line the direction of the flow changes by $d\theta$. Geometry of such flow can be shown:



Expansion waves

Similarly to the oblique shock wave the tangent component of velocities on both sides of the wave must be equal:

$$(V + dV)\ \cos(\mu + d\theta) = V\ \cos(\mu) \quad \rightarrow \quad \frac{V + dV}{V} = \frac{\cos(\mu)}{\cos(\mu + d\theta)} \tag{3.1}$$

$$\frac{V + dV}{V} = \frac{\cos(\mu)}{\cos(\mu)\ \cos(d\theta) - \sin(\mu)\ \sin(d\theta)} \tag{3.2}$$

For $d\theta \rightarrow 0$ we can use approximations: $\cos(d\theta) \approx 1$, $\sin(d\theta) \approx \mathrm{tg}(d\theta) \approx d\theta$

$$1 + \frac{dV}{V} = \frac{\cos(\mu)}{\cos(\mu) - \sin(\mu)\, d\theta} = \frac{1}{1 - \mathrm{tg}(\mu)\, d\theta} = 1 + \mathrm{tg}(\mu)\, d\theta + \dots \tag{3.3}$$

$$\mathrm{tg}(\mu) = \frac{\sin(\mu)}{\cos(\mu)} = \frac{1}{\sqrt{M^2 - 1}} \tag{3.4}$$

After substituting (3.4) inot (3.3) we can get following:

$$d\theta = \sqrt{M^2 - 1}\ \frac{dV}{V} \tag{3.5}$$

Expansion waves

In order to obtain the relation for θ which can not be approximated using small angle assumptions it is necessary to integrate the (3.5):

$$\int_{\theta_1}^{\theta_2} d\theta = \int_{M_1}^{M_2} \sqrt{M^2-1}\,\frac{dV}{V} \tag{3.6}$$

The RHS must be transformed to be a function only of the Mach number:

$$V = M\,c \quad \rightarrow \quad dV = dM\,c + M\,dc \quad \rightarrow \quad \frac{dV}{V} = \frac{dM}{M} + \frac{dc}{c} \tag{3.7}$$

Using isentropic relations:

$$\left(\frac{c_0}{c}\right)^2 = \frac{T_0}{T} = 1 + \frac{k-1}{2}M^2 \quad \rightarrow \quad c = c_0\left(1 + \frac{k-1}{2}M^2\right)^{-\frac{1}{2}}$$

$$\frac{dc}{c} = -\frac{k-1}{2}\,M\,\left(1 + \frac{k-1}{2}M^2\right)^{-1} \tag{3.8}$$

After substituting (3.7) and (3.8) into (3.6) we can obtain:

$$\int_{\theta_1}^{\theta_2} d\theta = \int_{M_1}^{M_2} \frac{2\sqrt{M^2-1}}{2+(k-1)\,M^2}\,dM \tag{3.9}$$

Expansion waves - Prandtl-Meyer function

Using equation (3.9) a new function can be introduced:

$$\nu(M) = \int \frac{2\sqrt{M^2-1}}{2+(k-1)\,M^2}\,dM \tag{3.10}$$

After integration we can obtain algebraic relation:

$$\nu(M) = \sqrt{\frac{k+1}{k-1}}\,\operatorname{arc\,tg}\left[\sqrt{\frac{k-1}{k+1}\,(M^2-1)}\right] - \operatorname{arc\,tg}\left(\sqrt{M^2-1}\right)$$

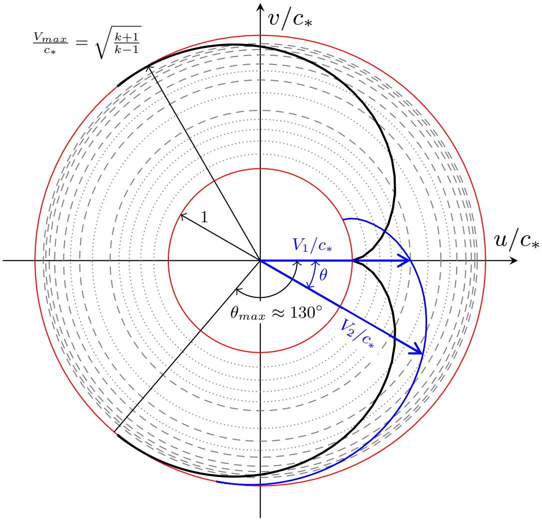
The function is named Prandtla-Meyera function. (3.11)

In order to solve flow past convex corner with angle θ and with Mach number M_1, it is necessary to solve nonlinear equation for unknown value of M_2:

$$\theta = \nu(M_2) - \nu(M_1) \tag{3.12}$$

Once the M_2 is known, other parameters can be found using isentropic relations..

Prandtl-Meyer function - polar plot



Prandtl-Meyer function

During derivation process of Prandtl-Meyer function no assumption on sign of $d\theta$ were made.

The function defines not only flow past convex corner but also concave corner as long as the flow satisfies isentropic process (e.g., smooth corners)

Summary

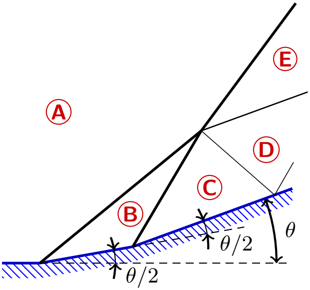
normal shock wave – Velocity vector is normal to the plane of the shock wave; $M_2 < M_1$ and $M_2 < 1$; $p_2 > p_1$; $\Delta s > 0$

oblique shock wave – Velocity vector is not normal to the plane of the shock wave; after passing the velocity vector is deflected by angle θ; $M_2 < M_1$; typically $M_2 > 1$; $p_2 > p_1$; $\Delta s > 0$

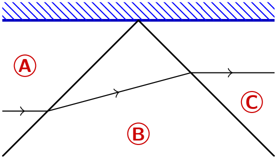
expansion waves – Present as a fan of characteristics (Mach lines); after passing through the fan the velocity vecter is deflected by angle θ; $M_2 > M_1$; $p_2 < p_1$; $\Delta s = 0$

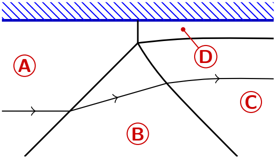
slip line – velocity vector is tangent (streamline); separates two regions with different velocities; $M_2 \neq M_1$; $p_2 = p_1$

Double concave corner

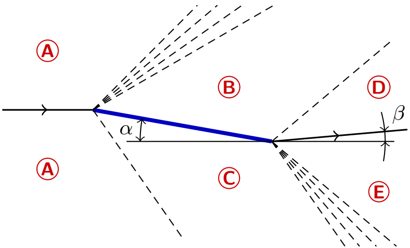


Reflected shock wave





Flat plate



Example:

Flow past flat plate $\alpha = 10°$, $M_\infty = 2$ i $p_\infty = 1$

$M_A = M_\infty = 2$, $p_A = p_\infty = 2$

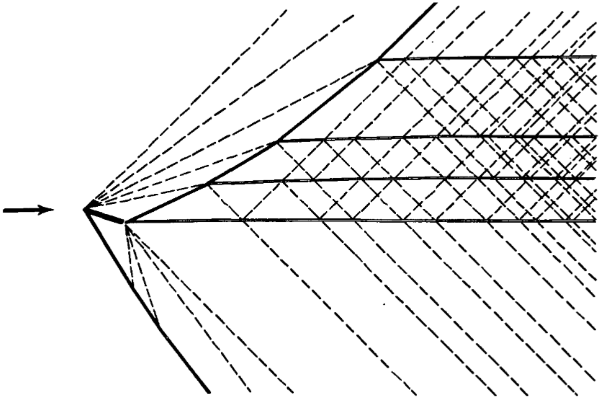
$M_B = 2.385$, $p_B = 0.548$ $M_C = 1.641$, $p_B = 1.707$

$M_D = 1.985$, $p_D = 1.001$ $M_E = 1.989$, $p_E = 1.001$

$\beta = 0.028°$

$C_L = 0.408$, $C_D = 0.0719$

Flat plate



Schematic drawing of characteristic present in supersonic flow past a flat plate.