



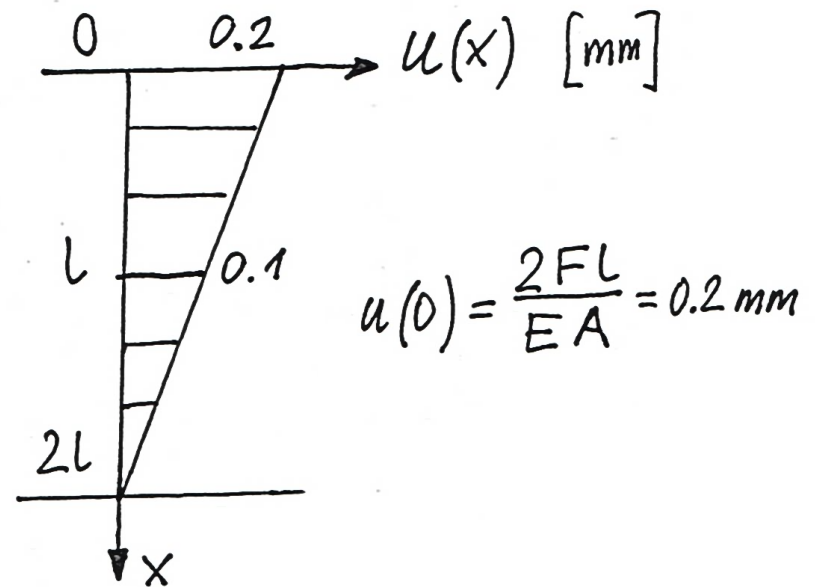
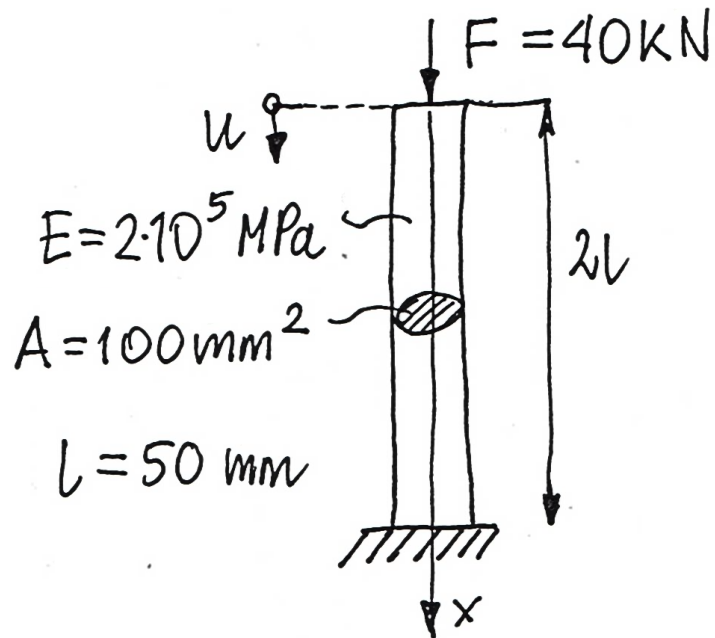
Institute of Aeronautics and Applied Mechanics

Finite element method 2 (FEM 2)

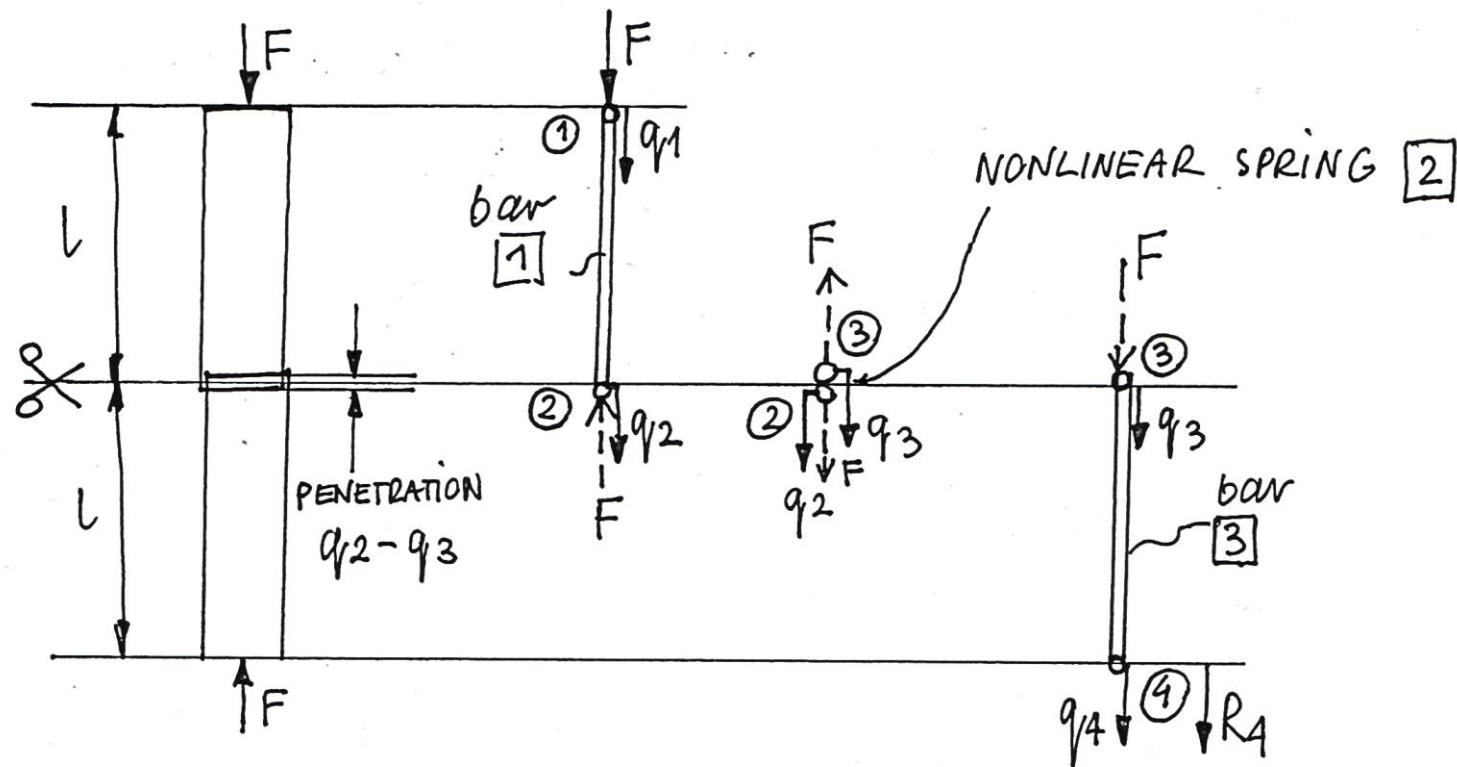
Contact problem

CONTACT PROBLEM

1°) LINEAR SOLUTION OF A COLUMN



2°) NONLINEAR SOLUTION (COLUMN DIVIDED INTO TWO PARTS)

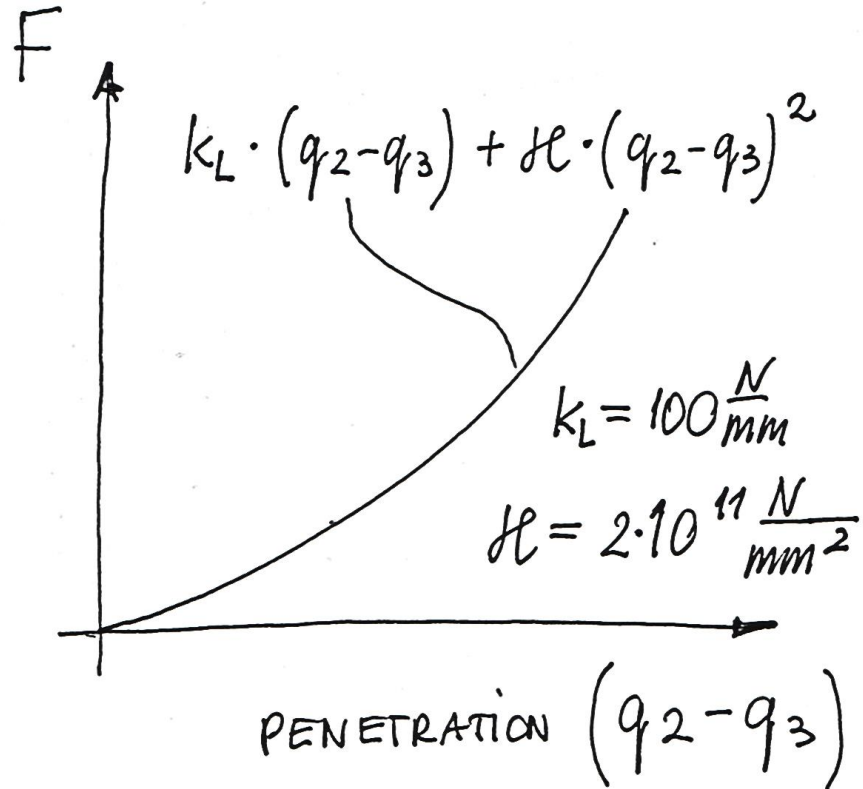
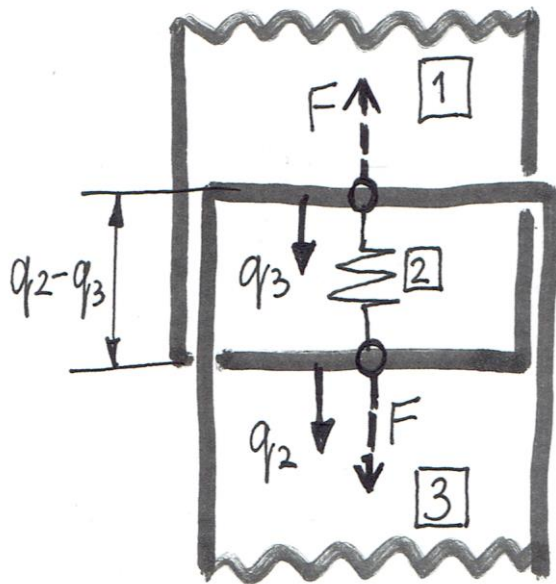


A SPRING IS ADDED TO STOP THE PENETRATION.

THE CONTACT AREA IS USUALLY A NONLINEAR FUNCTION OF LOAD.
 IN THE EXAMPLE THE AREA IS CONSTANT, SO INSTEAD WE ASSUME
 THAT THE SPRING HAS A NONLINEAR CHARACTERISTICS.

FORCE IN THE SPRING :

$$F = k \cdot (q_2 - q_3) \quad ; \quad \text{where } k = k_L + \mathcal{H} (q_2 - q_3)$$



global vector of nodal parameters:

$$\underset{1 \times 4}{[q]} = [q_1, q_2, q_3, q_4]$$

global load vector:

$$\underset{1 \times 4}{[F]} = [F, 0, 0, R_4]$$

local stiffness matrices:

$$\underset{2 \times 2}{[k]}_1 = \underset{2 \times 2}{[k]}_3 = \begin{bmatrix} a & -a \\ -a & a \end{bmatrix} ; \quad a = \frac{EA}{L} = 4 \cdot 10^5 \frac{N}{mm}$$

$$\underset{2 \times 2}{[k(q)]}_2 = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} ; \quad k = k_L + \mathcal{H}(q_2 - q_3)$$

global stiffness matrix :

$$[K(q)]_{4 \times 4} = \begin{bmatrix} [k]_1 & c & 0 & 0 \\ 0 & [k]_2 & 0 & 0 \\ 0 & 0 & [k]_3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} a & -a & 0 & 0 \\ -a & a+k & -k & 0 \\ 0 & -k & a+k & -a \\ 0 & 0 & -a & a \end{bmatrix}$$

set of nonlinear equations :

$$\begin{matrix} [K(q)] & \cdot & \{q\} & = & \{F\} \\ 4 \times 4 & & 4 \times 1 & & 4 \times 1 \end{matrix}$$

boundary condition $q_4 = 0 \Rightarrow N = \text{NDOF} - \text{NOF} = 4 - 1 = 3$

$$\begin{matrix} [K(q)] & \cdot & \{q\} & = & \{F\} \\ 3 \times 3 & & 3 \times 1 & & 3 \times 1 \end{matrix}$$

$$\begin{bmatrix} a & -a & 0 \\ -a & a+k & -k \\ 0 & -k & a+k \end{bmatrix} \cdot \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} F \\ 0 \\ 0 \end{Bmatrix} ; \quad k = f(q_2, q_3)$$

(Newton-Raphson method)

tangent matrix :

$$[K_T]_{3 \times 3} = \frac{\partial \{F\}_{3 \times 1}}{\partial [q]_{1 \times 3}} = \left[\frac{\partial \{F\}_{3 \times 1}}{\partial q_1}, \frac{\partial \{F\}_{3 \times 1}}{\partial q_2}, \frac{\partial \{F\}_{3 \times 1}}{\partial q_3} \right]$$

$$\frac{\partial \{F\}}{\partial q_1} = \frac{\partial ([K(q)] \cdot \{q\})}{\partial q_1} = \frac{\partial [K(q)]}{\partial q_1} \cdot \{q\} + [K(q)] \cdot \frac{\partial \{q\}}{\partial q_1} = [K(q)] \cdot \begin{Bmatrix} \frac{\partial q_1}{\partial q_1} \\ \frac{\partial q_2}{\partial q_1} \\ \frac{\partial q_3}{\partial q_1} \end{Bmatrix} =$$

$$\begin{matrix} \parallel \\ [0] \\ 3 \times 3 \end{matrix}$$

$$= \begin{bmatrix} a & -a & 0 \\ -a & a+k & -k \\ 0 & -k & a+k \end{bmatrix} \cdot \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} a \\ -a \\ 0 \end{Bmatrix} \quad \text{1st row of } [K]$$

$$\frac{\partial \{F\}}{\partial q_2} = \frac{\partial ([K(q)] \cdot \{q\})}{\partial q_2} = \frac{\partial [K(q)]}{\partial q_2} \cdot \{q\} + [K(q)] \cdot \frac{\partial \{q\}}{\partial q_2} =$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{\partial k}{\partial q_2} & -\frac{\partial k}{\partial q_2} \\ 0 & -\frac{\partial k}{\partial q_2} & \frac{\partial k}{\partial q_2} \end{bmatrix} \cdot \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} + [K(q)] \cdot \begin{Bmatrix} \frac{\partial q_1}{\partial q_2} \\ \frac{\partial q_2}{\partial q_2} \\ \frac{\partial q_3}{\partial q_2} \end{Bmatrix} =$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & h & -h \\ 0 & -h & h \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} + \begin{bmatrix} a & -a & 0 \\ -a & a+k & -k \\ 0 & -k & a+k \end{bmatrix} \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} =$$

$$= \begin{Bmatrix} 0 \\ h(q_2 - q_3) \\ -h(q_2 - q_3) \end{Bmatrix} + \begin{Bmatrix} -a \\ a+k \\ -k \end{Bmatrix} = \begin{Bmatrix} -a \\ a + k_L + 2h(q_2 - q_3) \\ -(k_L + 2h(q_2 - q_3)) \end{Bmatrix}$$

2nd row of $[k]$

$$\frac{\partial \{F\}}{\partial q_3} = \frac{\partial ([K(q)] \cdot \{q\})}{\partial q_3} = \frac{\partial [K(q)]}{\partial q_3} \cdot \{q\} + [K(q)] \cdot \frac{\partial \{q\}}{\partial q_3} =$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{\partial k}{\partial q_3} & -\frac{\partial k}{\partial q_3} \\ 0 & -\frac{\partial k}{\partial q_3} & \frac{\partial k}{\partial q_3} \end{bmatrix} \cdot \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} + [K(q)] \cdot \begin{Bmatrix} \frac{\partial q_1}{\partial q_3} \\ \frac{\partial q_2}{\partial q_3} \\ \frac{\partial q_3}{\partial q_3} \end{Bmatrix} =$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & -j\epsilon & j\epsilon \\ 0 & j\epsilon & -j\epsilon \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} + \begin{bmatrix} a & -a & 0 \\ -a & a+k & -k \\ 0 & -k & a+k \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} =$$

$$= \begin{Bmatrix} 0 \\ -j\epsilon(q_2 - q_3) \\ j\epsilon(q_2 - q_3) \end{Bmatrix} + \begin{Bmatrix} 0 \\ -k \\ a+k \end{Bmatrix} = \begin{Bmatrix} 0 \\ -(k_L + 2j\epsilon(q_2 - q_3)) \\ a + k_L + 2j\epsilon(q_2 - q_3) \end{Bmatrix}$$

3rd row of $[K]$

Thus :

$$[K_T]_{3 \times 3} = \begin{bmatrix} a & -a & 0 \\ -a & a + k_L + 2k(q_2 - q_3) & -(k_L + 2k(q_2 - q_3)) \\ 0 & -(k_L + 2k(q_2 - q_3)) & a + k_L + 2k(q_2 - q_3) \end{bmatrix}$$

initial solution :

$$\{q\}_0 = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}_{3 \times 1} ; [K(q)]_0 = \begin{bmatrix} a & -a & 0 \\ -a & a + k_L & -k_L \\ 0 & -k_L & a + k_L \end{bmatrix}$$

$$[K_T]_1 = \frac{\frac{\partial \{F\}}{\partial [q]}_{N \times 1}}{\frac{\partial [q]}{\partial [q]}_0}_{1 \times 3} = [K(q)]_0$$

SOLUTION AT ITERATION " i " :

$$[K_T]_i = \frac{\partial \{F\}}{\partial \{q\}_{i-1}} = \begin{bmatrix} a & -a & 0 \\ -a & a + k_L + 2k_e(q_2 - q_3) & -(k_L + 2k_e(q_2 - q_3)) \\ 0 & -(k_L + 2k_e(q_2 - q_3)) & a + k_L + 2k_e(q_2 - q_3) \end{bmatrix}_{i-1}$$

$$\{R\}_i = \{F\} - [K(q)]_{i-1} \cdot \{q\}_{i-1}$$

$\begin{matrix} 3 \times 1 & & 3 \times 1 & & 3 \times 3 & & 3 \times 1 & & 3 \times 1 \end{matrix}$

$$\{\Delta q\}_i = [K_T]_i^{-1} \cdot \{R\}_i \quad (i = 1, \dots, n)$$

$\begin{matrix} 3 \times 1 & & 3 \times 3 & & 3 \times 1 \end{matrix}$

$$\{q\}_i = \{q\}_{i-1} + \{\Delta q\}_i$$

$\begin{matrix} 3 \times 1 & & 3 \times 1 & & 3 \times 1 \end{matrix}$

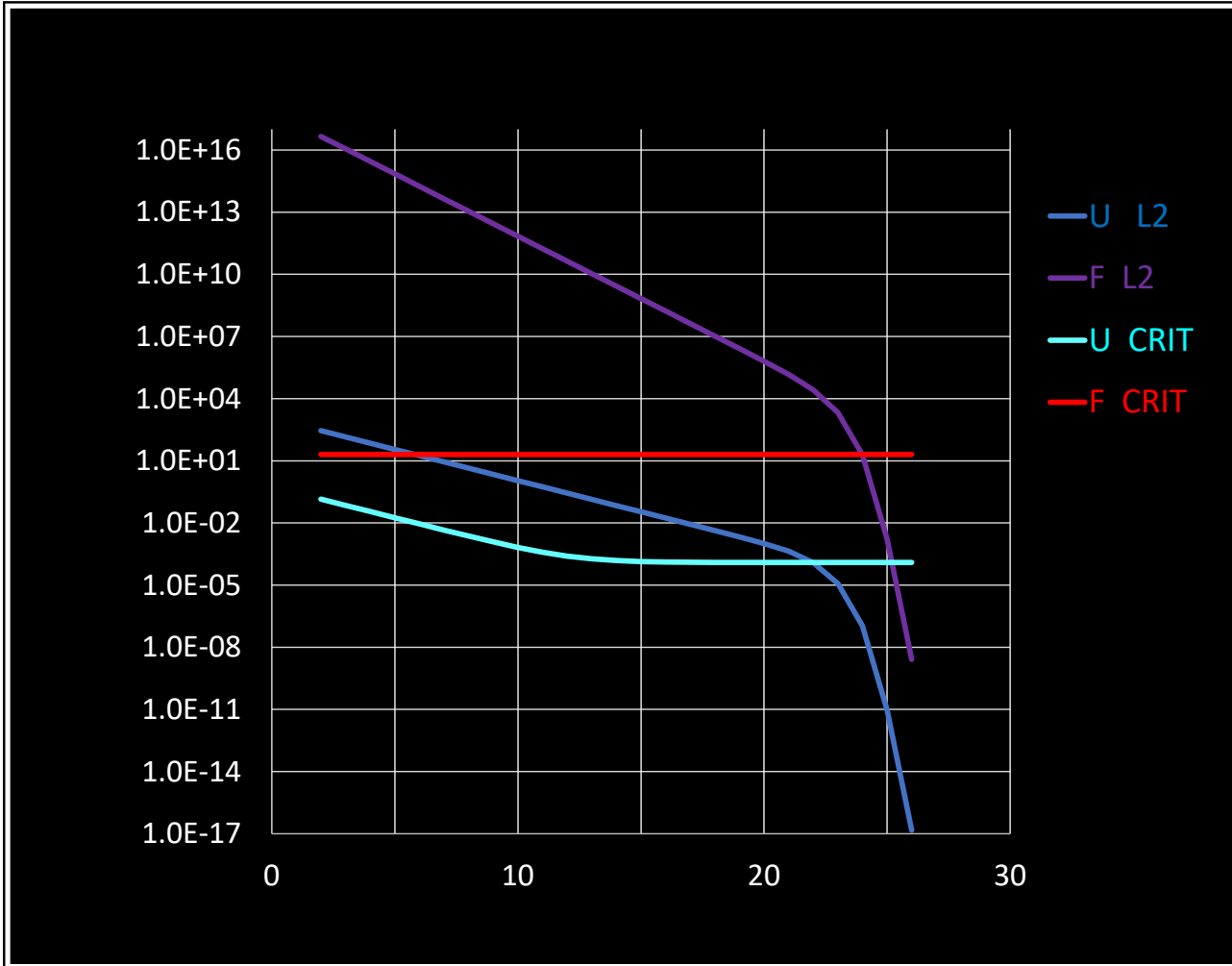
convergence criteria :

displacement criterion : $U2_{NORM_i} = \frac{\| \{ \Delta q \}_i \|_2}{\| \{ q \}_i \|_2} \leq \epsilon$
 $\epsilon = 0.0005$

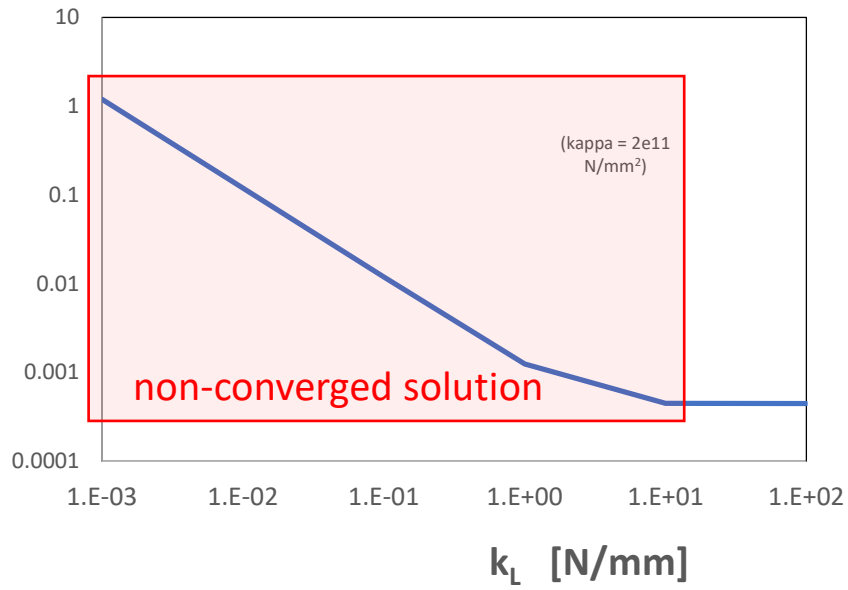
force criterion : $F2_{NORM_i} = \frac{\| \{ R \}_{3 \times 1} \|_2}{\| \{ F \}_{3 \times 1} \|_2} \leq \delta$
 $\delta = 0.0005$

FORCE IN THE SPRING : (CONTACT ELEMENT)

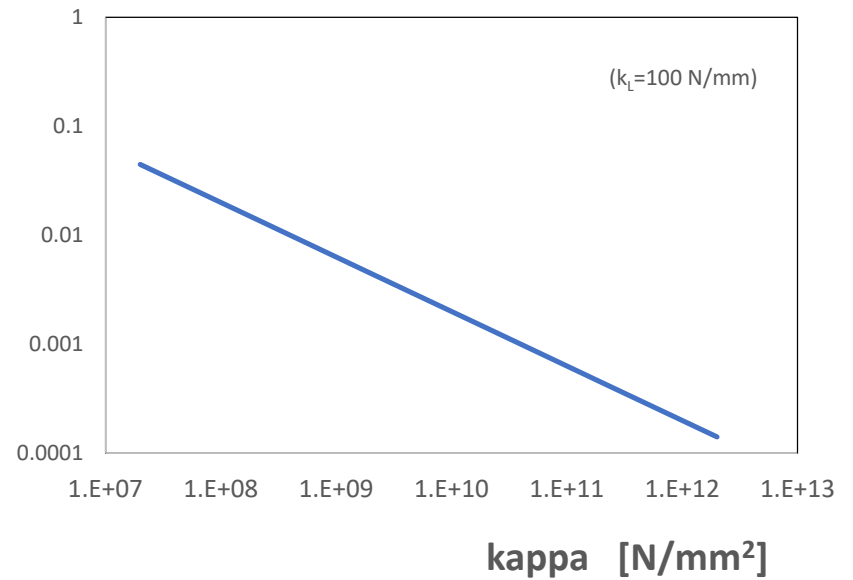
$$F_{c_i} = \left((k_L + H(q_2 - q_3)) \cdot (q_2 - q_3) \right)_i =$$
$$= (k_L + H \cdot \text{PENETRATION}) \cdot \text{PENETRATION}$$



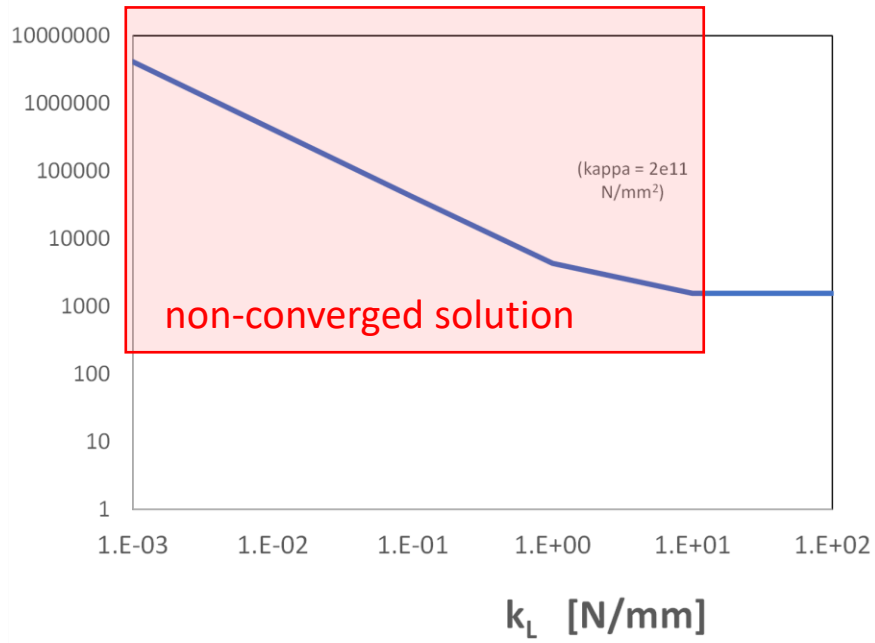
penetration [mm]



penetration [mm]



condition number



condition number

