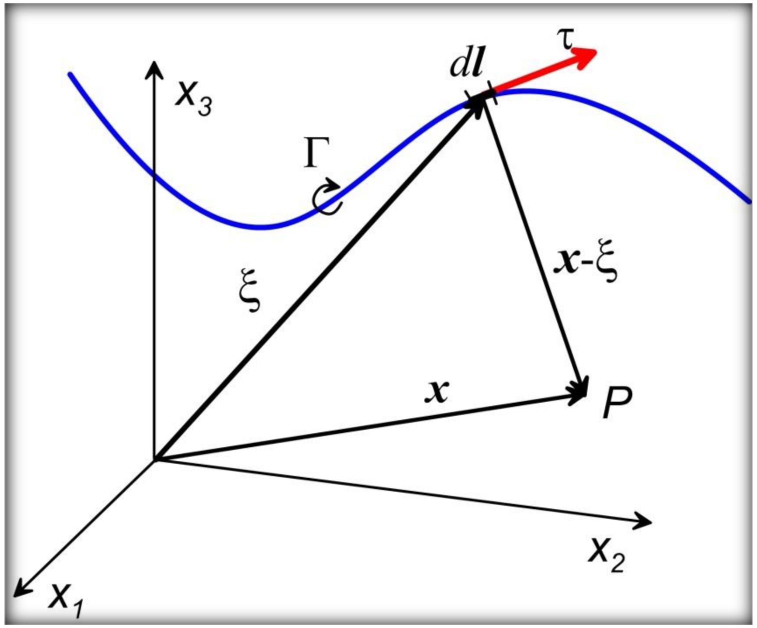
LECTURE 6

AERODYNAMICS OF A WING

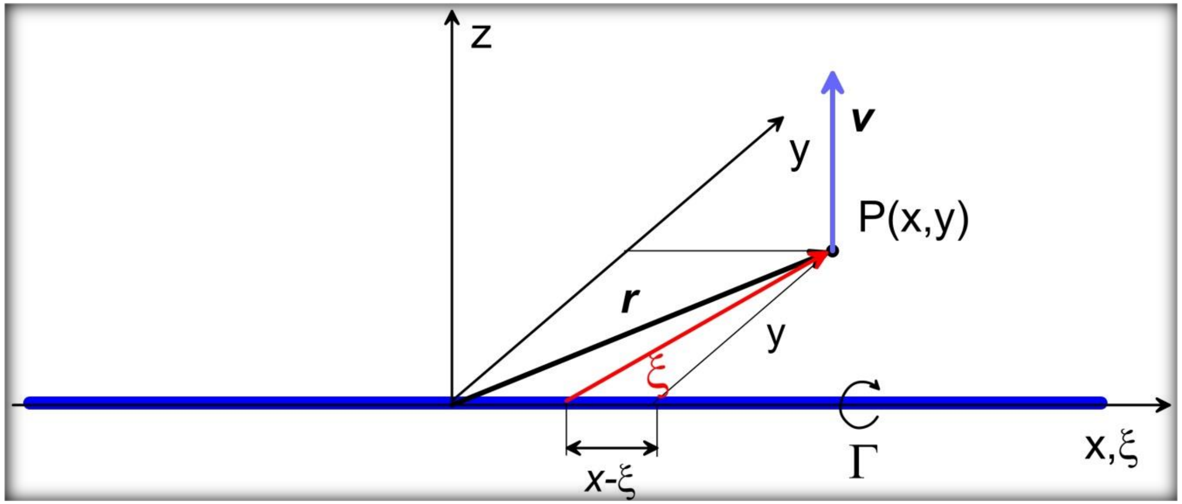
FUNDAMENTALS OF THE LIFTING-LINE THEORY

The Biot-Savart Law



The velocity induced by the singular vortex line with the circulation Γ can be determined by means of the Biot-Savart formula

$$\boldsymbol{v}(\boldsymbol{x}) = \frac{\Gamma}{4\pi}\int_{VL}\frac{d\boldsymbol{l}\times(\boldsymbol{x}-\boldsymbol{\xi})}{\left|\boldsymbol{x}-\boldsymbol{\xi}\right|^{3}}$$



Special case – induction of the straight vortex line:

$$d\boldsymbol{l} = d\xi\,\boldsymbol{e}_x \quad , \quad \boldsymbol{x}-\boldsymbol{\xi} = (x-\xi)\boldsymbol{e}_x + y\boldsymbol{e}_y$$

$$d\boldsymbol{l}\times(\boldsymbol{x}-\boldsymbol{\xi}) = y\,d\xi\,\boldsymbol{e}_x\times\boldsymbol{e}_x = y\,d\xi\,\boldsymbol{e}_z$$

$$\left|\boldsymbol{x}-\boldsymbol{\xi}\right|^{3} = \left[(x-\xi)^2 + y^2\right]^{3/2}$$

From the Biot-Savart formula one gets

$$\boldsymbol{v}(x,y,0)=\left[\frac{\Gamma}{4\pi}\int_{\xi_1}^{\xi_2}\frac{y}{\sqrt{[(x-\xi)^2+y^2]^3}}\,d\xi\right]\boldsymbol{e}_z$$

where

$$\int_{\xi_1}^{\xi_2}\frac{y}{\sqrt{[(x-\xi)^2+y^2]^3}}\,d\xi=\left|\begin{matrix}s=(\xi-x)/y\\ ds=d\xi/y\end{matrix}\right|=\frac{1}{y}\int_{\frac{\xi_1-x}{y}}^{\frac{\xi_2-x}{y}}\frac{1}{(1+s^2)^{3/2}}\,ds=\frac{1}{y}\frac{s}{\sqrt{1+s^2}}\Bigg|_{\frac{\xi_1-x}{y}}^{\frac{\xi_2-x}{y}}=$$

$$=\frac{1}{y}\left[\frac{x-\xi_1}{\sqrt{(x-\xi_1)^2+y^2}}-\frac{x-\xi_2}{\sqrt{(x-\xi_2)^2+y^2}}\right]$$

Case 1 – induction of the infinite vortex line (equivalent to the 2D point vortex!)

$$\boldsymbol{v}(x,y,0)=\frac{\Gamma}{4\pi y}\lim_{\xi\to\infty}\left[\frac{x+\xi}{\sqrt{(x+\xi)^2+y^2}}-\frac{x-\xi}{\sqrt{(x-\xi)^2+y^2}}\right]\boldsymbol{e}_z=\frac{\Gamma}{2\pi y}\boldsymbol{e}_z$$

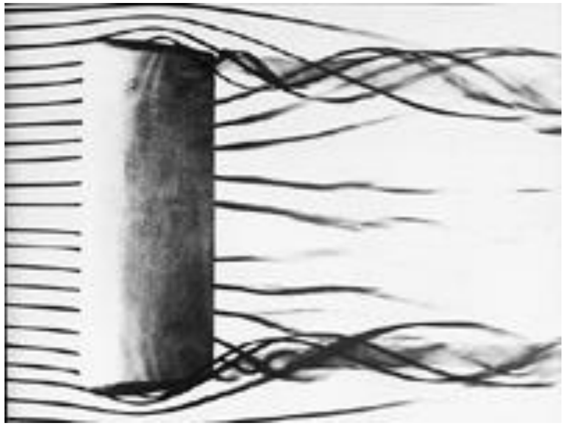
Case 2 – induction of the semi-infinite vortex line segment $\xi \in [0,\infty)$

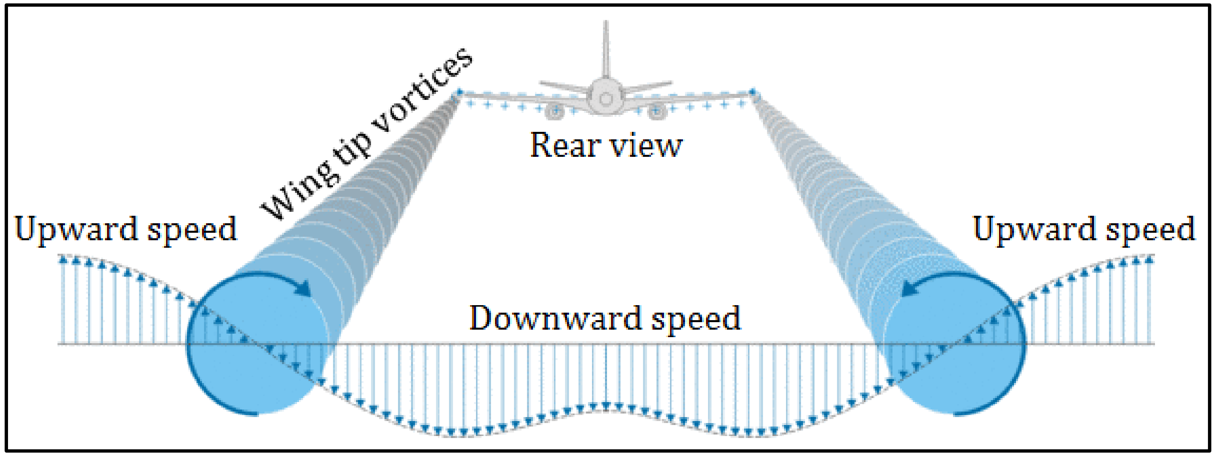
$$\boldsymbol{v}(x,y,0) = \frac{\Gamma}{4\pi y} \lim_{\xi \to \infty} \left[\frac{x}{\sqrt{x^2+y^2}} - \frac{x-\xi}{\sqrt{(x-\xi)^2+y^2}} \right] \boldsymbol{e}_z = \frac{\Gamma}{4\pi y} \left[\frac{x}{\sqrt{x^2+y^2}} + 1 \right] \boldsymbol{e}_z$$

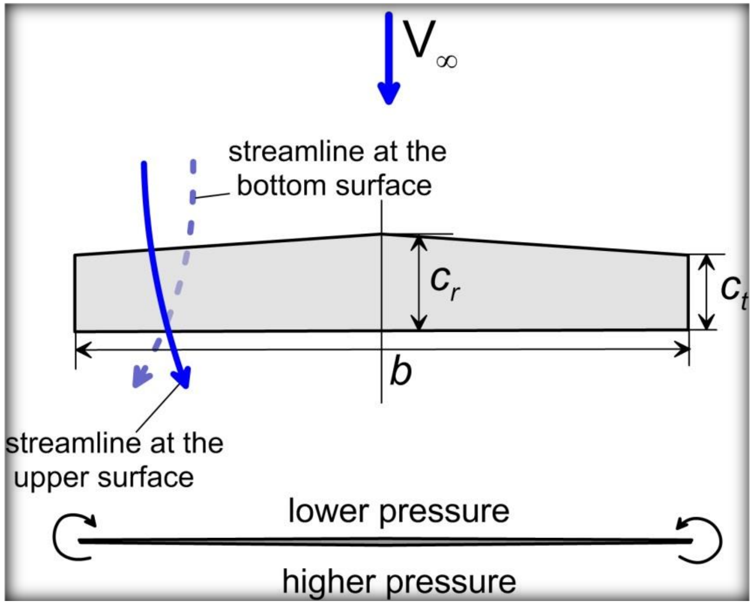
If $x = 0$ then

$$\boldsymbol{v}(0,y,0) = \frac{\Gamma}{4\pi y} \boldsymbol{e}_z$$

Flow past a finite-span wing – physical properties

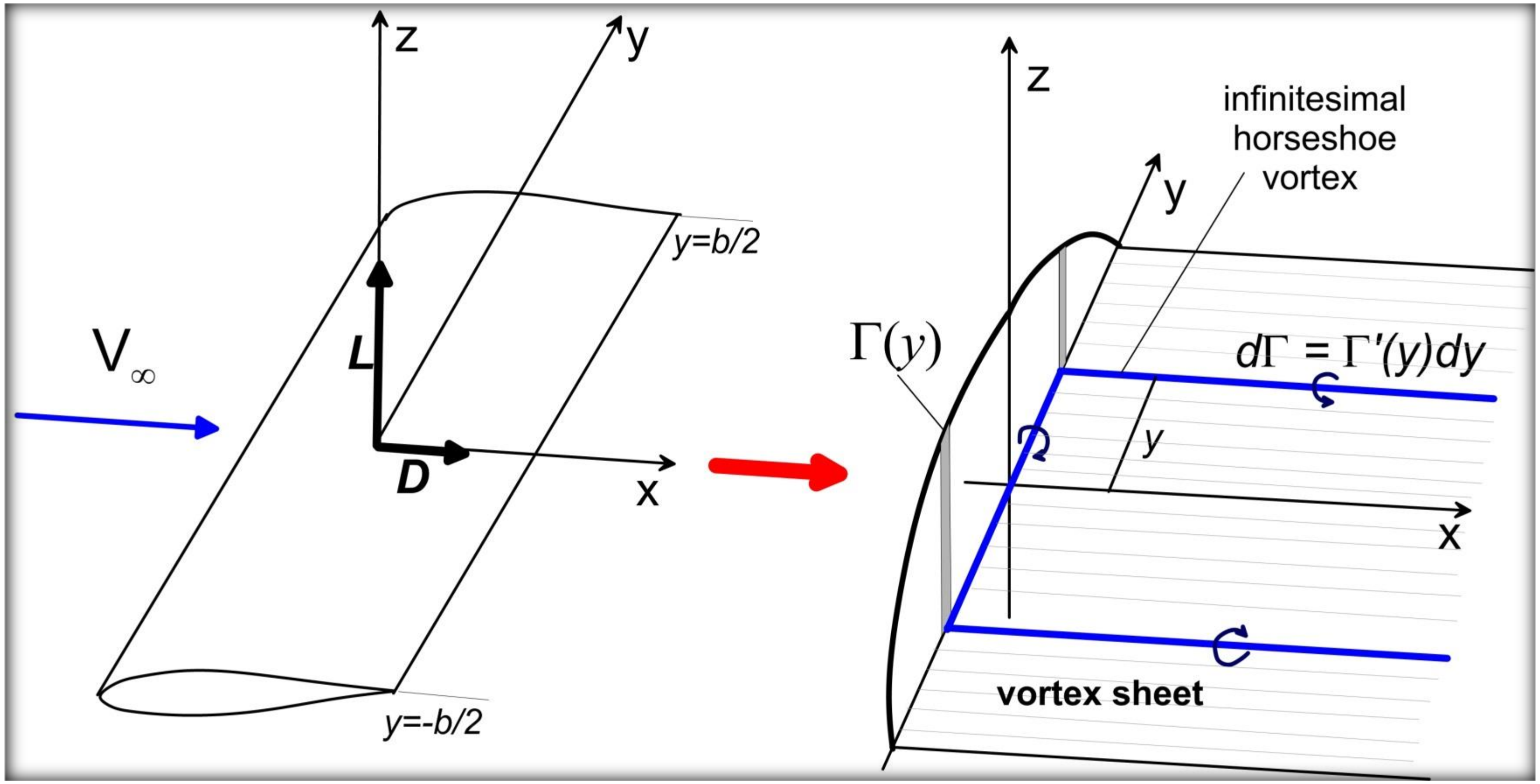






Lifting-line model of a finite-span wing

Flow past a wing is modeled by the superposition of the uniform free stream and the velocity induced by a plane vortex sheet "pretending" to be the cortex wave behind the wing.



The vortex sheet behind the wing is "woven" from continuum of infinitesimally weak horseshoe vortices. These vortices are "attached" to the lifting line leading to a continuous distribution of circulation along the wing span.

The vortex sheet induces vorticity all around. The idea is to calculate the calculate the velocity induced by this sheet on its front edge, i.e., along the **lifting line**. Next, it is assumed that **each infinitely thin slice of the wing generates the (differential) contribution to the total aerodynamic force as it were a two-dimensional airfoil**. Each slice "senses" its individual direction of "free stream", which results from the real free stream vector $\boldsymbol{V}_\infty$ and the vertical (normal to the vortex sheet) velocity induces at the lifting line in the point corresponding to the position of the wing slice.

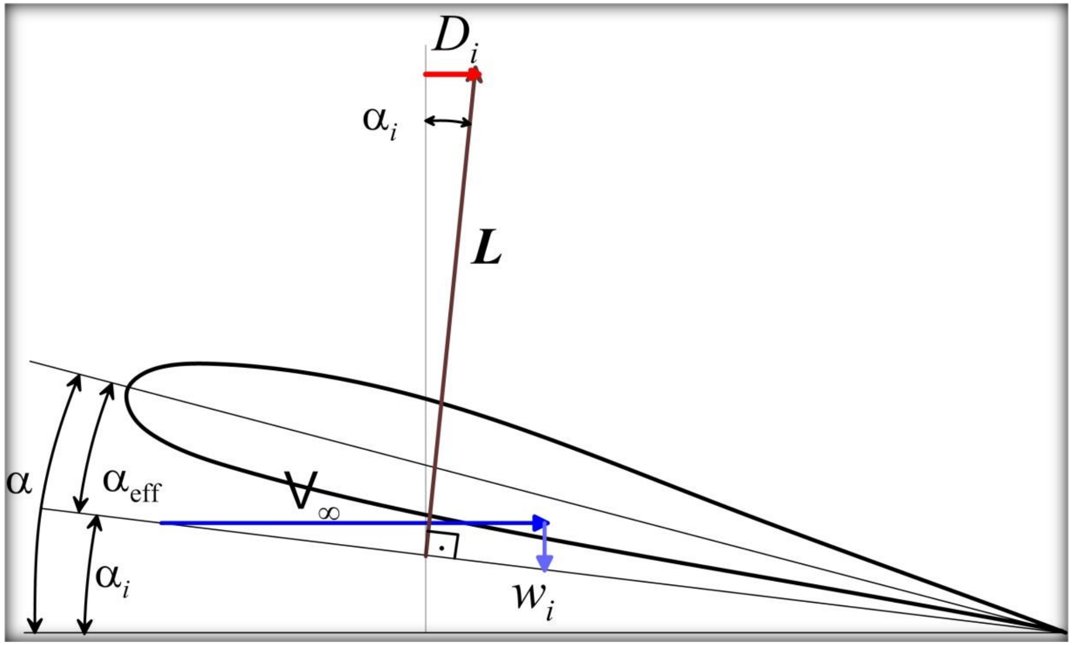
According to the Biot-Savart formula, the infinitesimal contribution to the velocity induces along the lifting line at the point $(0, y_0, 0)$ is

$$dw = -\frac{\Gamma'(y)dy}{4\pi(y_0 - y)}$$

The total velocity induces at this point is obtained by integration

$$w(y_0) = -\frac{1}{4\pi} \int_{-b/2}^{b/2} \frac{\Gamma'(y)dy}{y_0 - y}$$

Due to (generally) non-uniform distribution of the induced velocity along the wing span, the effective angle of attack has an individual value of each wing section – see figure below.



The direction of flow "sensed" by the wing section at $y = y_0$ is rotated clockwise by the induces angle

$$\alpha_i(y_0) = \operatorname{atan}[-w(y_0)/V_\infty]$$

For small angles …

$$\alpha_i(y_0) \approx -\frac{w(y_0)}{V_\infty} = \frac{1}{4\pi V_\infty} \int_{-b/2}^{b/2} \frac{\Gamma'(y)dy}{y_0 - y}$$

Clearly, an effective angle $\alpha_{eff} = \alpha_{eff}(y_0)$.

For small angles one can assume that the local lift coefficient changes linearly with the (local) angle. Hence

$$c_L(y_0) = a_\infty[\alpha_{eff}(y_0) - \alpha_0(y_0)]$$

Here, a_∞ denotes the slope of the lift characteristics for the wing section, α_0 is the angle of attack corresponding to the zero lift. Note that $a_\infty = 2\pi$ if the thin-airfoil theory is used. Note also that – in general – the angle $\alpha_0 = \alpha_0(y_0)$.

Next, we assume that the spanwise density of the lift force developed on the wing can be computed from the Kutta-Joukovski formula, namely

$$L'(y_0) = \tfrac{1}{2}\rho_\infty V_\infty^2 c_L(y_0) c(y_0) = \rho_\infty V_\infty \Gamma(y_0)$$

where $c(y_0)$ is local chord of the wing section

Hence, the local lift coefficient is $$c_L(y_0) = \frac{2\Gamma(y_0)}{V_\infty c(y_0)}$$

Assuming that $a_\infty = 2\pi$, the local effective angle of attack is

$$\alpha_{eff}(y_0) = \frac{\Gamma(y_0)}{\pi V_\infty c(y_0)} + \alpha_0(y_0)$$

Finally, the sum of the two local angles: $\alpha_{eff}(y_0)$ and $\alpha_i(y_0)$ is equal to the geometric angle of attack α. If the wing has geometric twist, this angle also depends of the spanwise location, i.e., $\alpha = \alpha(y_0)$.

Hence, we have obtained the following **integro-differential equation for the spanwise distribution of the circulation**

$$\boxed{\frac{\Gamma(y_0)}{\pi V_\infty c(y_0)} + \frac{1}{4\pi V_\infty} \int_{-b/2}^{b/2} \frac{\Gamma'(y)dy}{y_0 - y} = \alpha(y_0) - \alpha_0(y_0)}$$

One this equation is solved, then the spanwise distribution of the circulation is known. The lift force developed on the wing can be calculated as follows

$$L = \int_{-b/2}^{b/2} L'(y)dy = \rho_\infty V_\infty \int_{-b/2}^{b/2} \Gamma(y)dy$$

The (global) lift coefficient is

$$\boxed{C_L = \frac{L}{q_\infty S} = \frac{2}{V_\infty S} \int_{-b/2}^{b/2} \Gamma(y)dy}$$

The local contribution to the drag force is

$$D'_i = L' \sin\alpha_i \approx L'\alpha_i$$

The **induced drag** force is equal

$$D_i = \int_{-b/2}^{b/2} L'(y)\alpha_i(y)dy = \rho_\infty V_\infty \int_{-b/2}^{b/2} \Gamma(y)\alpha_i(y)dy$$

Thus, the **coefficient of the induced drag** is equal

$$C_{D_i} = \frac{D_i}{q_\infty S} = \frac{2}{V_\infty S} \int_{-b/2}^{b/2} \Gamma(y)\alpha_i(y)dy$$

Important case - elliptical distribution of the circulation

Assume
$$\Gamma(y) = \Gamma_0 \sqrt{1 - \left(\frac{2y}{b}\right)^2}$$

meaning that
$$L'(y) = \rho_\infty V_\infty \Gamma_0 \sqrt{1 - \left(\frac{2y}{b}\right)^2}$$

We have
$$\Gamma'(y) = -\frac{4\Gamma_0}{b^2} \frac{y}{\sqrt{1 - 4y^2 / b^2}}$$

Hence
$$w(y_0) = -\frac{1}{4\pi} \int_{-b/2}^{b/2} \frac{\Gamma'(y)dy}{y_0 - y} = \frac{\Gamma_0}{\pi b^2} \int_{-b/2}^{b/2} \frac{y\,dy}{(y_0 - y)\sqrt{1 - 4y^2 / b^2}}$$

Let us apply the following change of coordinates

$$y=\tfrac{1}{2}b\cos\theta \quad , \; dy=-\tfrac{1}{2}b\sin\theta\, d\theta$$

Thus

$$w(\theta_0)=-\frac{\Gamma_0}{2\pi b}\int_0^{\pi}\frac{\cos\theta}{\cos\theta-\cos\theta_0}d\theta=-\frac{\Gamma_0}{2b}$$

<u>Conclusion:</u> **for the elliptical distribution of the circulation, the downwash velocity is constant!**

The induced angle is $$\alpha_i=-\frac{w_i}{V_\infty}=\frac{\Gamma_0}{2bV_\infty}$$

The lift force

$$L=\rho_\infty V_\infty \Gamma_0\int_{-b/2}^{b/2}\sqrt{1-4y^2/b^2}\,dy=\rho_\infty V_\infty \Gamma_0\frac{b}{2}\int_0^{\pi}\sin^2\theta\, d\theta=\tfrac{1}{4}\pi\rho_\infty V_\infty \Gamma_0 b$$

Thus, the maximal circulation is $$\Gamma_0=\frac{4L}{\pi\rho_\infty V_\infty b}$$

On the other hand $$L = \tfrac{1}{2}\rho_\infty V_\infty^2 S C_L$$

Hence $$\Gamma_0 = \frac{2V_\infty S C_L}{\pi b}$$

and the induced angle is $$\alpha_i = \frac{\Gamma_0}{2bV_\infty} = \frac{2V_\infty S C_L}{\pi b}\frac{1}{2bV_\infty} = \frac{S C_L}{\pi b^2}$$

We define the **aspect ratio of the wing** $$\boxed{\Lambda = \frac{b^2}{S}}$$

Then, the alternative form of the formula for the induced angle for the elliptical distribution of vorticity is

$$\boxed{\alpha_i = \frac{C_L}{\pi \Lambda}}$$

The coefficient of the induced drag is calculated as follows

$$C_{D_i} = \frac{2\alpha_i}{V_\infty S}\int_{-b/2}^{b/2}\Gamma(y)dy = \frac{2\alpha_i\Gamma_0}{V_\infty S}\frac{b}{2}\int_0^\pi \sin^2\theta\, d\theta = \frac{\pi\alpha_i\Gamma_0 b}{2V_\infty S} = \frac{\pi b}{2V_\infty S}\frac{C_L}{\pi\Lambda}\frac{2V_\infty S C_L}{\pi b}$$

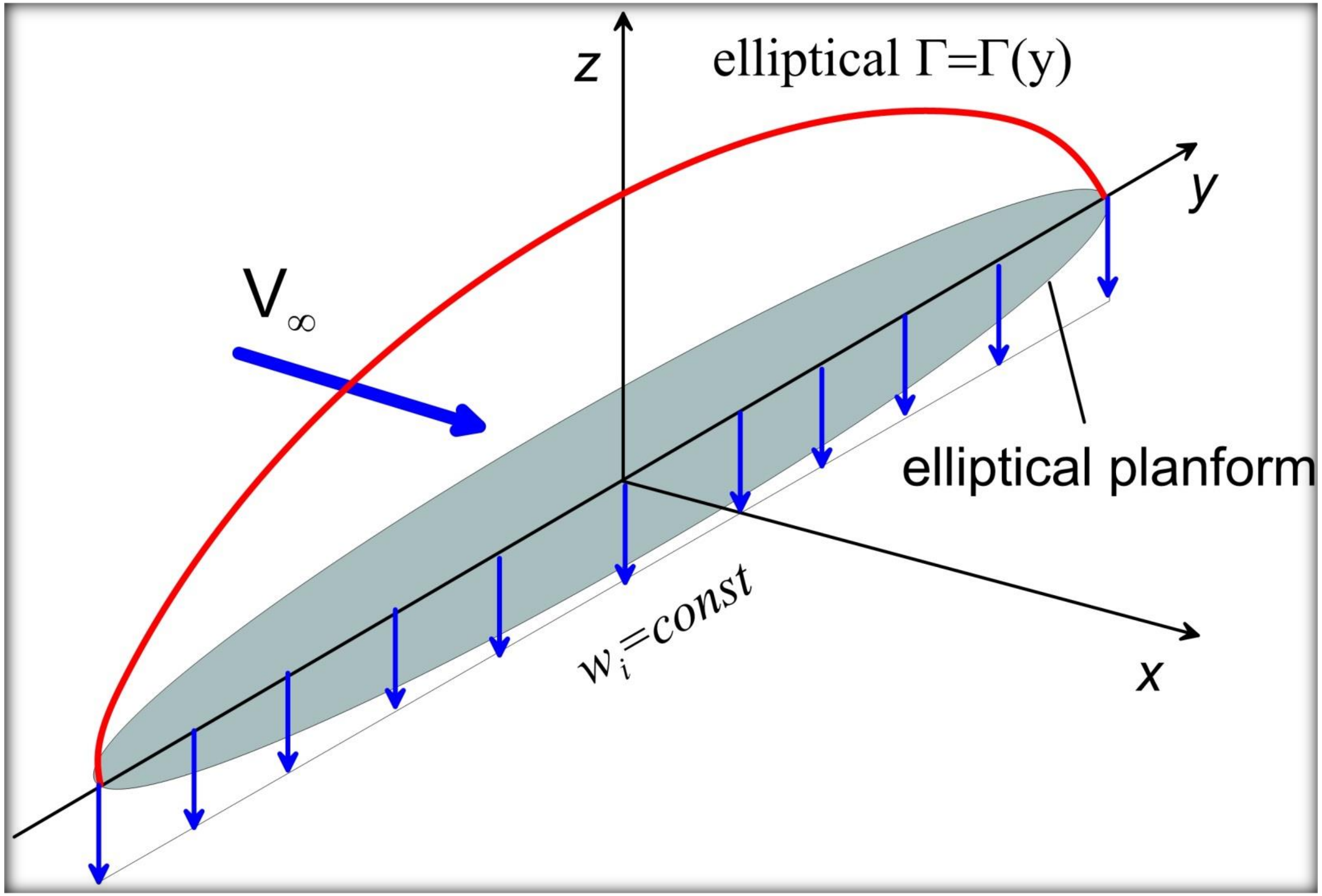
Thus, we have obtained the formula

$$\boxed{C_{D_i} = \frac{C_L^2}{\pi\Lambda}}$$

Consider the **wing with no geometrical or aerodynamic twist**.
Then, both α and α_0 are constant along the wing span. For the elliptical load distribution the angle α_i is also constant, hence the effective angle of attack α_{eff} and the lift coefficient $c_L = a_\infty(\alpha_{eff} - \alpha_0)$ are also constant along the wing span.

Since $L'(y) = c_L q_\infty c(y)$ then $c(y) = \dfrac{L'(y)}{c_L q_\infty}$.

<u>Conclusion:</u> the spanwise variation of the wing chord follows the variation of the aerodynamic load. Hence, **the planform of such wing is also elliptical**!



General lift distribution

Again, we use the transformation $y = -\tfrac{1}{2} b \cos\theta \quad , \quad \theta \in [0, \pi]$

The elliptic distribution is expressed now as $\Gamma(\theta) = \Gamma_0 \sqrt{1 - \cos^2\theta} = \Gamma_0 \sin\theta$

<u>Generalization:</u>

$$\Gamma(\theta) = 2 V_\infty b \sum_{n=1}^{\infty} A_n \sin n\theta$$

We will need the derivative …

$$\frac{d\Gamma}{dy} = \frac{d\Gamma}{d\theta}\frac{d\theta}{dy} = 2 V_\infty b \frac{d\theta}{dy} \sum_{n=1}^{\infty} n A_n \cos n\theta$$

The central equation of the lifting-line theory takes the form

$$\alpha(\theta_0) - \alpha_0(\theta_0) = \frac{2b}{\pi c(\theta_0)} \sum_{n=1}^{\infty} A_n \sin n\theta_0 + \frac{1}{\pi} \sum_{n=1}^{\infty} \left[n A_n \int_0^{\pi} \frac{\cos n\theta}{\cos\theta - \cos\theta_0} d\theta \right]$$

The Glauert integral appears again $$\int_0^\pi \frac{\cos n\theta}{\cos\theta - \cos\theta_0} d\theta = \frac{\pi \sin n\theta_0}{\sin\theta_0}$$

Hence, the main equation is transformed to the algebraic form

$$\alpha(\theta_0) - \alpha_0(\theta_0) = \frac{2b}{\pi c(\theta_0)} \sum_{n=1}^{\infty} A_n \sin n\theta_0 + \frac{1}{\pi} \sum_{n=1}^{\infty} n A_n \frac{\sin n\theta_0}{\sin\theta_0}$$

In order to find approximate solution, we first truncate the infinite series …

$$\alpha(\theta_0) - \alpha_{L=0}(\theta_0) = \frac{2b}{\pi c(\theta_0)} \sum_{n=1}^{N} A_n \sin n\theta_0 + \frac{1}{\pi} \sum_{n=1}^{N} n A_n \frac{\sin n\theta_0}{\sin\theta_0}$$

… and make use of the collocation method, i.e., require fulfillment of this equation at N different point $\eta_m \in [0,\pi],\ m = 1,..,N$.

This way, the linear algebraic system is obtained for the unknown coefficients $\{A_1, A_2, ..., A_N\}$, which can be solved, e.g., by the Gauss Elimination Method.

Once $\Gamma(\theta)$ is known, one can calculate all aerodynamic characteristics of the wing. We have

$$C_L = \frac{2}{V_\infty S}\int_{-b/2}^{b/2}\Gamma(y)dy = \frac{2b^2}{S}\sum_{n=1}^{N} A_n \int_0^{\pi}\sin n\theta \sin\theta d\theta$$

We use the orthogonality of the Fourier modes $\int_0^{\pi}\sin n\theta \sin\theta d\theta = \begin{cases}\pi/2 \ , & n=1 \\ 0 \ , & n\neq 1\end{cases}$

and obtain the formula

$$\boxed{C_L = \pi A_1 \frac{b^2}{S} = A_1 \pi \Lambda}$$

We see that only the first coefficient of the Fourier series is needed to calculate the lift force coefficient!

The calculation of the induced drag is more complicated …

We have

$$C_{D_i} = \frac{2}{V_\infty S}\int_{-b/2}^{b/2}\Gamma(y)\alpha_i(y)dy = \frac{2b^2}{S}\int_0^\pi \alpha_i(\theta)\sin\theta\left[\sum_{n=1}^{N} A_n \sin n\theta\right]d\theta$$

We need expression for the induced angle, namely

$$\alpha_i = \frac{1}{4\pi V_\infty}\int_{-b/2}^{b/2}\frac{\Gamma'(y)dy}{y_0 - y} = \frac{1}{\pi}\sum_{n=1}^{N} nA_n\left[\int_0^\pi \frac{\cos n\theta}{\cos\theta - \cos\theta_0}d\theta\right] = \sum_{n=1}^{N} nA_n\frac{\sin n\theta_0}{\sin\theta_0}$$

Hence, the formula for the C_{D_i} can be transformed as follows

$$C_{D_i} = \frac{2b^2}{S}\int_0^\pi \sin\theta\left[\sum_{k=1}^{N} kA_k\frac{\sin k\theta}{\sin\theta}\right]\left[\sum_{n=1}^{N} A_n\sin n\theta\right]d\theta =$$

$$= \frac{2b^2}{S}\sum_{k,n=1}^{N} kA_k A_n\int_0^\pi \sin k\theta \sin n\theta d\theta$$

Again, using the orthogonality property of the Fourier modes

$$\int_0^{\pi} \sin k\theta \sin n\theta d\theta = \begin{cases} 0 & , \; k \neq m \\ \frac{1}{2}\pi & , \; k = m \end{cases}$$

the formula for the induced drag coefficient simplifies to the form

$$C_{D_i} = \frac{2b^2}{S}\frac{\pi}{2}\sum_{n=1}^{N} nA_n^2 = \pi\Lambda\sum_{n=1}^{N} nA_n^2 = \pi\Lambda(A_1^2 + \sum_{n=2}^{N} nA_n^2) = \pi\Lambda A_1^2\left[1 + \sum_{n=2}^{N} n\frac{A_n^2}{A_1^2}\right]$$

We can write shortly

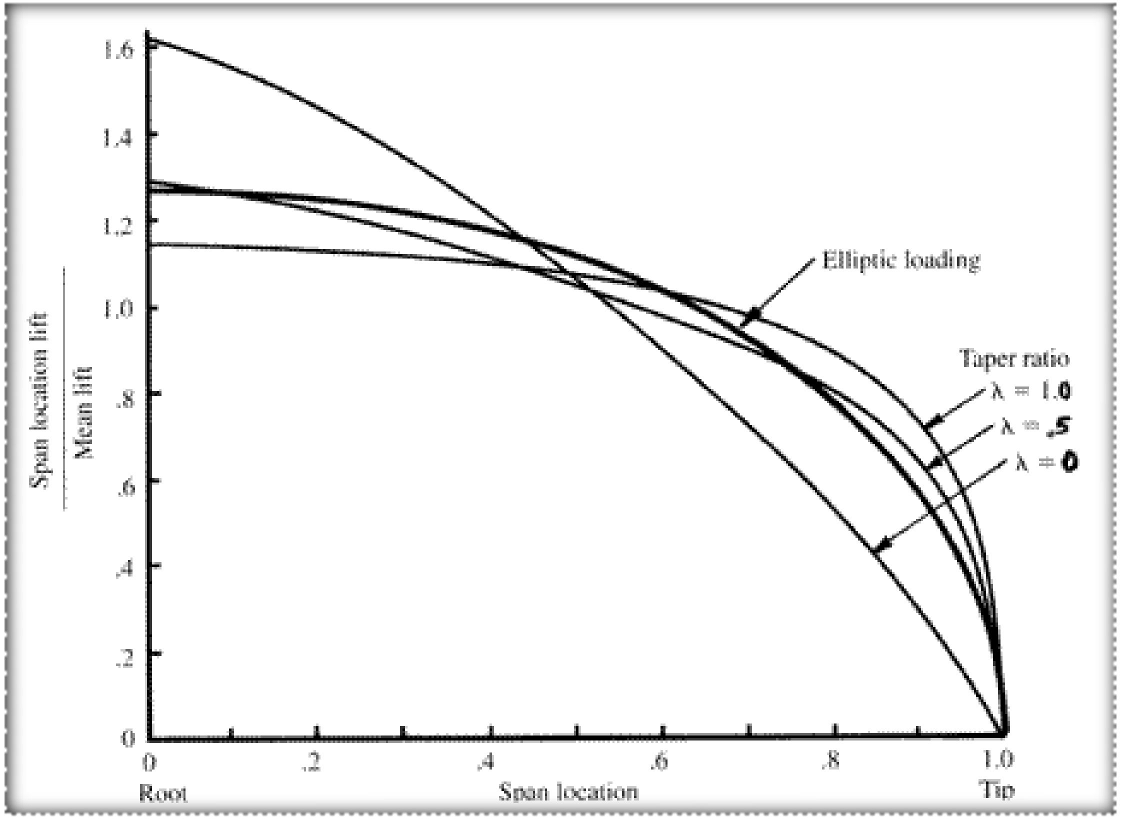
$$\boxed{C_{D_i} = \frac{C_L^2}{\pi\Lambda}(1+\delta) = \frac{C_L^2}{\pi\Lambda e}}$$

where $\delta = \sum_{n=2}^{N} n\frac{A_n^2}{A_1^2}$ and $e = (1+\delta)^{-1}$ (Oswald aerodynamic efficiency parameter).

Note that $\delta \geq 0$, hence $\left.C_{D_i}\right|_{elliptic\ wing} \leq \left.C_{D_i}\right|_{any\ wing}$ (optimality!)

The most famous airplane with the elliptical wing:

Trapezoidal wings are easier to construct and to build. Theoretically, they are nearly as good as elliptic ones if only the taper ratio (i.e., c_{tip} / c_{root}) is near the optimal value. In the wide range of aspect ratios, ($\lambda = 4 \div 10$), the smallest values of δ are achieved when the taper ratio is close to 0.3. Other factors, like the **stall pattern** also matters!



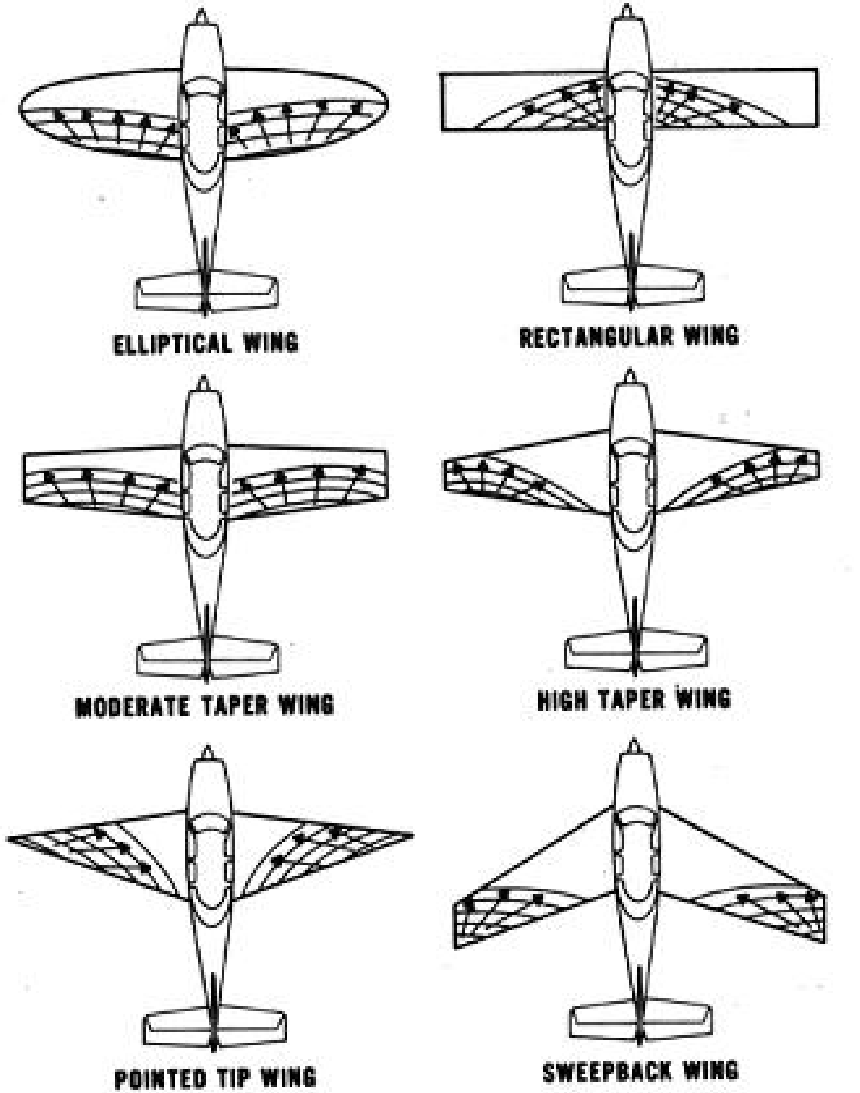


Figure 17–13 Wing Planforms (Exaggerated)

LEFT: Spanwise lift distribution for trapezoidal wing with different taper ratios.
RIGHT: Stall patterns for different planforms.

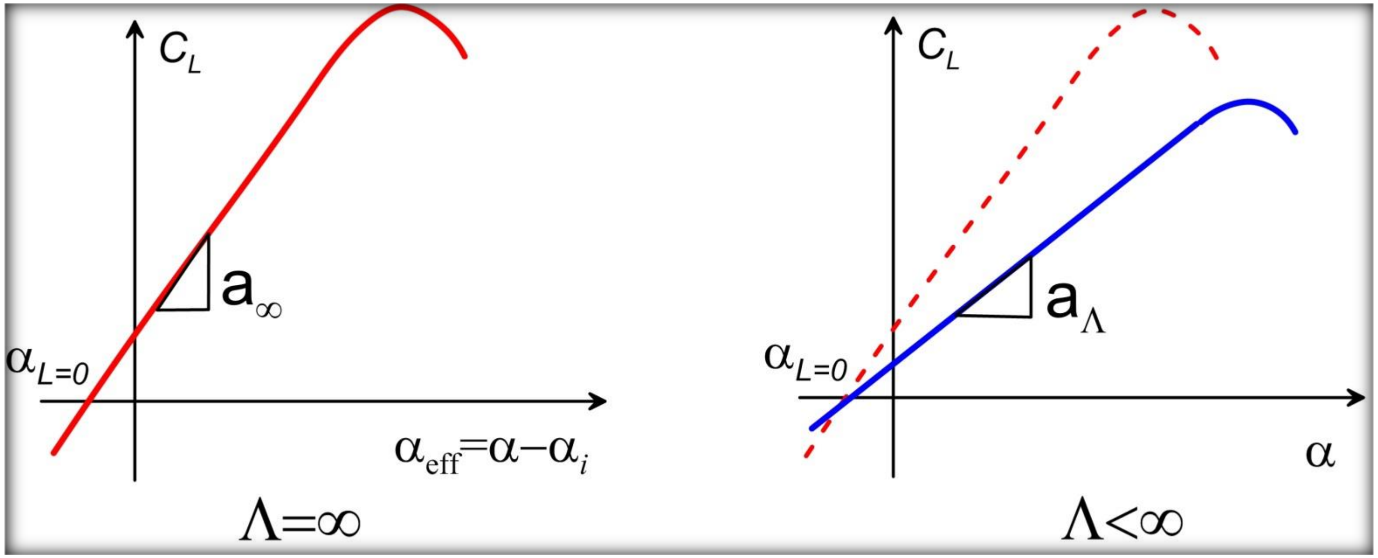
Reduction of the lift slope

The finite span not only leads to the appearance of the induced drag – it also changes (reduces) the slope of the "lift vs angle of attack" characteristic.

Denote:

$a_\infty = \frac{dc_L}{d\alpha}$ - slope of the lift characteristic for the 2D wing section (equivalent to $\Lambda = \infty$)

$a_\Lambda = \frac{dC_L}{d\alpha}$ - slope of the lift characteristic for the 3D wing.



Due to appearance of the induced angle, the 3D wing achieves the same value of the lift coefficient at larger geometric angle of attack – see figure.

We have …

$$C_L = a_\infty(\alpha - \alpha_i) + const$$

Hence, for the elliptic wing $$C_L = a_\infty(\alpha - \frac{C_L}{\pi\Lambda}) + const$$

Thus $$\frac{dC_L}{d\alpha} = a_\Lambda = \frac{a_\infty}{1 + a_\infty/\pi\Lambda}$$

For other planforms … $$a_\Lambda = \frac{a_\infty}{1 + (a_\infty/\pi\Lambda)(1+\tau)}$$

Correction factor τ typically ranges from 0.05 to 0.25. The value of this factor can be expressed by the coefficients $\{A_1, A_2, ..., A_N\}$