PROGRAM ROZWOJOWY
POLITECHNIKI WARSZAWSKIEJ

KINEMATICS OF FLUID – ADDITIONAL MATERIAL

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CIRCULATION, VORTICITY AND STREAMFUNCTIONS

CIRCULATION

Definition: Circulation of the vector field $\boldsymbol{w}$ along the (closed) contour $\mathfrak{L}$ is defined as

$$\Gamma = \oint_{\mathfrak{L}} \boldsymbol{w} \cdot d\boldsymbol{l}$$

Kelvin's Theorem:

Assume that:

- the volume force field $\boldsymbol{f}$ **potential**,
- the fluid is **inviscid** and **barotropic**
- the flow is **stationary**.

Then: the circulation of the velocity field $\boldsymbol{v}$ along any **closed material line** $\mathfrak{L}(t)$ is constant in time, i.e.

$$\tfrac{d}{dt}\Gamma(t) \equiv \tfrac{d}{dt} \oint_{\mathfrak{L}(t)} \boldsymbol{v}(t,\boldsymbol{x}) \cdot d\boldsymbol{l} = 0$$

Proof of the Kelvin Theorem:

Since the flow is barotropic and the volume force field is potential, we can write

$$\nabla P = \frac{1}{\rho}\nabla p \quad , \quad \boldsymbol{f} = \nabla \Phi$$

Thus, the acceleration (which consists of the **convective part** only) can be expressed as

$$\boldsymbol{a} \equiv (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\nabla(P-\Phi)$$

In order to evaluate the time derivative of the circulation along the material line, it is convenient to use Lagrange approach. Thus, the circulation can be expresses as

$$\Gamma(t) = \oint_{\mathfrak{L}(t)} \boldsymbol{v}(t,\boldsymbol{x})\cdot d\boldsymbol{l} = \oint_{\mathfrak{L}_0(t)} \boldsymbol{V}(t,\boldsymbol{\xi})\cdot \boldsymbol{J}(t,\boldsymbol{\xi})\,d\boldsymbol{l}_0$$

where $\boldsymbol{J}(t,\boldsymbol{\xi}) = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}}$ denotes the **Jacobi matrix** of the transformation between Eulerian and Lagrangian coordinates.

Then, the time derivative of the circulation is evaluated as follows

$$\frac{d}{dt}\oint_{\mathfrak{L}(t)} \boldsymbol{v}(t,\boldsymbol{x})\cdot d\boldsymbol{l} = \frac{d}{dt}\oint_{\mathfrak{L}_0(t)} \boldsymbol{V}(t,\boldsymbol{\xi})\cdot \boldsymbol{J}(t,\boldsymbol{\xi})d\boldsymbol{l}_0 = \oint_{\mathfrak{L}_0(t)} \boldsymbol{a}(t,\boldsymbol{\xi})\cdot \boldsymbol{J}(t,\boldsymbol{\xi})d\boldsymbol{l}_0 +$$

$$+\oint_{\mathfrak{L}_0(t)} \boldsymbol{V}(t,\boldsymbol{\xi})\cdot\nabla_{\xi}\boldsymbol{V}(t,\boldsymbol{\xi})d\boldsymbol{l}_0 = \oint_{\mathfrak{L}(t)} \boldsymbol{a}(t,\boldsymbol{x})\cdot d\boldsymbol{l} + \underbrace{\oint_{\mathfrak{L}_0(t)} \nabla_{\xi}(\tfrac{1}{2}\boldsymbol{V}\cdot\boldsymbol{V})(t,\boldsymbol{\xi})\cdot d\boldsymbol{l}_0}_{\substack{=0 \\ \text{int. of the grad. along the closed loop}}} =$$

$$= \underbrace{-\oint_{\mathfrak{L}(t)} \nabla(P+\Phi)\cdot d\boldsymbol{l}}_{\substack{=0 \\ \text{int. of the grad. along} \\ \text{the closed loop}}} = 0$$

where the relation $\partial_t \boldsymbol{J}(t,\boldsymbol{\xi}) = \nabla_{\xi}\boldsymbol{V}(t,\boldsymbol{\xi})$ has been used.

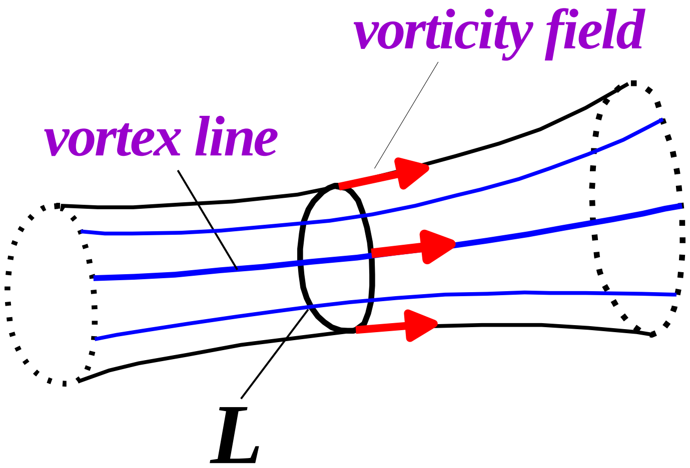
VORTICITY

As we already know, the vorticity is defined as the rotation of the velocity:

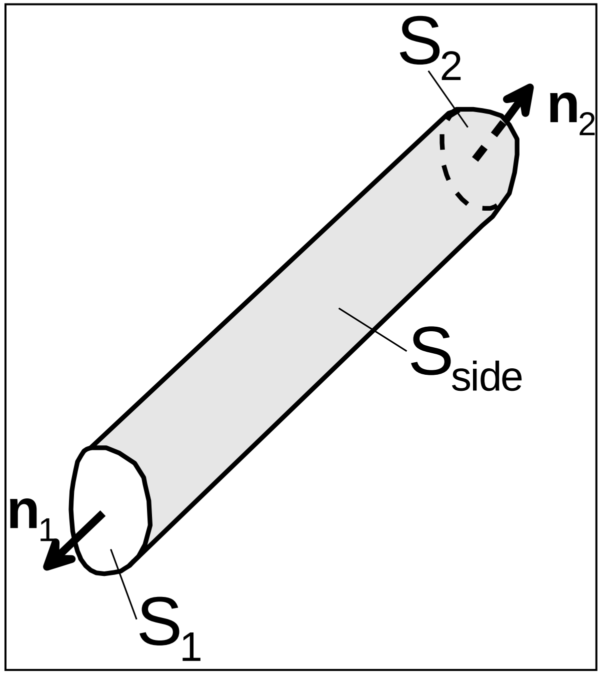
$$\boldsymbol{\omega} = rot\,\boldsymbol{v} \equiv \nabla \times \boldsymbol{v}$$

Definitions:

- A **vortex line** is the line of the vorticity vector field. At each point of such line, the vorticity vector is **tangent** to this line.
- The **vortex tube** is the subset of the flow domain bounded by the surface made of the vortex lines passing through all point of a given closed contour (the contour L on the picture below)



STRENGTH OF THE VORTEX TUBE



It is defined as the **flux of vorticity** through a cross-section of the tube. Using the Stokes' Theorem we can write:

$$\int_S \boldsymbol{\omega} \cdot \boldsymbol{n}\, d\sigma = \oint_l \boldsymbol{v} \cdot d\boldsymbol{x} = \Gamma$$

We see that the **strength of the vortex tube is equal to the circulation of the velocity along a closed contour wrapped around the tube**.

The above definition does not depend on the choice of a particular contour. Indeed, since the **vorticity field is divergence-free**, the flux of the vorticity is fixed along the vortex tube. To see this, consider the tube segment Ω located between two cross-section S_1 and S_2 .

From the **GGO theorem** we have

$$0 = \int_\Omega \nabla \cdot \boldsymbol{\omega}\, d\boldsymbol{x} = \int_{S_1} \boldsymbol{\omega} \cdot \boldsymbol{n}\, ds + \int_{S_2} \boldsymbol{\omega} \cdot \boldsymbol{n}\, ds + \underbrace{\int_{S_{side}} \boldsymbol{\omega} \cdot \boldsymbol{n}\, ds}_{=0} = 0$$

Note that the last integral vanishes because the **surface** S_{side} **is made of the vortex lines** and thus at each point of S_{side} the normal versor $\boldsymbol{n}$ is **perpendicular to the vorticity vector**.

Note also that the orientations of the normal versors at S_1 and S_2 are opposite (in order to apply the **GGO Theorem**, the **normal versor must point outwards** at all components of the boundary $\partial\Omega$).

Reversing the orientation of $\boldsymbol{n}$ at S_2, we conclude that

$$\int_{S_1} \boldsymbol{\omega}\cdot\boldsymbol{n}\,ds = \int_{S_2} \boldsymbol{\omega}\cdot\boldsymbol{n}\,ds$$

HELMHOLTZ (3^{RD}) THEOREM

Assume that:

- the flow is **inviscid** and **barotropic**,
- the volume force field is **potential**.

Then: the vortex lines consist of the same fluid elements, i.e. the **lines of the vorticity field are material lines**.

Proof:

We need the **transformation rule for the vectors tangent to a material line**.

Let at initial time t = 0 the material line be described parametrically as $l_0 : \boldsymbol{a} = \boldsymbol{a}(s)$.
At some later time instant t > 0, the shape of the material line follows from the flow mapping $\mathfrak{F}_t : \boldsymbol{R}^3 \ni a \mapsto \boldsymbol{x} \in \boldsymbol{R}^3$, i.e., $l : \boldsymbol{x} = \boldsymbol{x}(s) = \mathfrak{F}_t[\boldsymbol{a}(s)]$.

The corresponding transformation of the tangent vector can be evaluated as follows

$$\boldsymbol{\tau}(s) = \tfrac{d}{ds}\boldsymbol{x}(s) = \tfrac{d}{ds}\mathfrak{F}_t(\boldsymbol{a}(s)) = \underbrace{[\tfrac{\partial x}{\partial \xi}](\boldsymbol{a}(s))}_{Jacobi\ matrix} \tfrac{d}{ds}\boldsymbol{a}(s) = [\tfrac{\partial x}{\partial \xi}](\boldsymbol{a}(s))\,\boldsymbol{\tau}_0(s)$$

Let's now write the acceleration in the Lamb-Gromeko form:

$$\boldsymbol{a} = \frac{D}{Dt}\boldsymbol{v} = \partial_t \boldsymbol{v} + \nabla\left(\tfrac{1}{2}\upsilon^2\right) + \boldsymbol{\omega}\times\boldsymbol{v}$$

The rotation of $\boldsymbol{a}$ can be expressed as

$$\nabla\times\boldsymbol{a} = \partial_t(\nabla\times\boldsymbol{v}) + \nabla\times(\boldsymbol{\omega}\times\boldsymbol{v}) = \frac{D}{Dt}\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\nabla)\boldsymbol{v} + (\nabla\cdot\boldsymbol{v})\boldsymbol{\omega}$$

In the above, the following vector identity, written for $\boldsymbol{p} = \boldsymbol{\omega}$ and $\boldsymbol{q} = \boldsymbol{v}$, is used

$$\nabla\times(\boldsymbol{p}\times\boldsymbol{q}) = (\boldsymbol{q}\cdot\nabla)\,\boldsymbol{p} - (\boldsymbol{p}\cdot\nabla)\boldsymbol{q} + (\nabla\cdot\boldsymbol{q})\,\boldsymbol{p} - (\nabla\cdot\boldsymbol{p})\boldsymbol{q}$$

Next, one can calculate the Lagrangian derivative of the vector field $\boldsymbol{\omega}/\rho$ as follows

$$\frac{D}{Dt}\left(\frac{1}{\rho}\boldsymbol{\omega}\right) = \frac{1}{\rho}\frac{D}{Dt}\boldsymbol{\omega} - \frac{1}{\rho^2}\boldsymbol{\omega}\underbrace{\frac{D}{Dt}\rho}_{=-\rho\nabla\cdot\boldsymbol{v}} = \frac{1}{\rho}\left[\nabla\times\boldsymbol{a} + (\boldsymbol{\omega}\cdot\nabla)\boldsymbol{v} - (\nabla\cdot\boldsymbol{v})\boldsymbol{\omega}\right] +$$

$$+\frac{1}{\rho}\boldsymbol{\omega}\nabla\cdot\boldsymbol{v} = \frac{1}{\rho}\nabla\times\boldsymbol{a} + \left(\frac{1}{\rho}\boldsymbol{\omega}\cdot\nabla\right)\boldsymbol{v}$$

From the **equation of motion** and assumed flow properties that the acceleration field is **potential** and thus

$$\nabla \times \boldsymbol{a} = \boldsymbol{0}$$

Then, the equation for the vector field $\boldsymbol{\omega} / \rho$ reduces to

$$\frac{D}{Dt}\left(\frac{1}{\rho}\boldsymbol{\omega}\right) = \left(\frac{1}{\rho}\boldsymbol{\omega}\cdot\nabla\right)\boldsymbol{v}$$

Define the vector field $\boldsymbol{c}$ such that $\omega_i = \rho \frac{\partial x_i}{\partial \xi_j} c_j$, or equivalently, $\boldsymbol{\omega} = \rho \left[\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}}\right] \boldsymbol{c}$.

Jacobi matrix

In the above, the symbol $\boldsymbol{\xi}$ denotes the Lagrange variables.

The left-hand side of the above equation can be transformed as follows

$$L = \frac{D}{Dt}\left(\frac{1}{\rho}\boldsymbol{\omega}\right) = \frac{d}{dt}\left[\left[\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}}\right]\boldsymbol{c}\right] = \left[\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}}\right]\frac{d}{dt}c + \left[\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{\xi}}\right]\boldsymbol{c} = \left[\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}}\right]\frac{d}{dt}c + \left[\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}}\right]\left[\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}}\right]\boldsymbol{c}$$

The right-hand side can be written as

$$R = \left(\frac{1}{\rho}\boldsymbol{\omega}\cdot\nabla\right)\boldsymbol{v} = \left[\left[\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}}\right]\boldsymbol{c}\cdot\nabla\right]\boldsymbol{v} = \left[\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}}\right]\left[\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}}\right]\cdot\boldsymbol{c}$$

Since $L = R$, we conclude that

$$\frac{d}{dt}\boldsymbol{c} = 0$$

Thus, **c is constant along trajectories of the fluid elements.**

Using the Lagrange description, we can write $\boldsymbol{c}(t,\xi) = \boldsymbol{c}(0,\xi) = \boldsymbol{c}_0$

Note that for the initial time $t = 0$ the transformation between Lagrange and Euler descriptions **reduces to identity**.

$$\left[\frac{\partial \boldsymbol{x}}{\partial \xi}\right]\Bigg|_{t=0} = \boldsymbol{I}$$

Therefore $\boldsymbol{c}_0 = \frac{1}{\rho_0}\boldsymbol{\omega}_0$ and since $\boldsymbol{c}(t) \equiv \boldsymbol{c}_0$ we get

$$\frac{1}{\rho}\boldsymbol{\omega} = \left[\frac{\partial \boldsymbol{x}}{\partial \xi}\right]\frac{1}{\rho_0}\boldsymbol{\omega}_0.$$

The last equality has the form of the **transformation rule for the vectors tangent to material lines**. Since the vector $\boldsymbol{\omega}_0 / \rho_0$ is tangent to the vortex line passing through the point ξ at $t = 0$, it follows that the vector $\boldsymbol{\omega} / \rho$ is tangent to image of this line at some later time t. But $\boldsymbol{\omega} / \rho$ is also tangent to the vortex line passing through the point **x**, which means that **the vortex lines must be material**.

Since the vortex lines are material, so are the **vortex tubes**. If we define a closed, material contour lying on the vortex tube's surface (and wrapped around it), then such a contour remains on this surface for any time. It follows from the Kelvin Theorem that the circulation along such contour remains constant. Consequently, **the strength of any vortex tube also remains constant in time**. It is important conclusion showing that the vortex motion of the inviscid, barotropic fluid exposed to a potential force field cannot be created or destroyed.

EQUATION OF THE VORTICITY TRANSPORT

In fluid mechanics the **vorticity** plays a very important role, in particular in understanding of the phenomenon of **turbulence**. In this section we derive the differential equation governing spatial/temporal evolution of this field.

Recall that the equation of motion of an inviscid fluid can be written in the following form

$$\partial_t \boldsymbol{v} + \nabla(\tfrac{1}{2}\upsilon^2) + \boldsymbol{\omega}\times\boldsymbol{v} = -\tfrac{1}{\rho}\nabla p + \boldsymbol{f}$$

Thus, the application of the rotation operator yields

$$\partial_t \boldsymbol{\omega} + \nabla\times(\boldsymbol{\omega}\times\boldsymbol{v}) = -\nabla\times(\tfrac{1}{\rho}\nabla p) + \nabla\times\boldsymbol{f}$$

The pressure term can be transformed as follows

$$\nabla\times(\tfrac{1}{\rho}\nabla p) = \nabla(\tfrac{1}{\rho})\times\nabla p + \tfrac{1}{\rho}\underbrace{\nabla\times\nabla p}_{0} = -\tfrac{1}{\rho^2}\nabla\rho\times\nabla p$$

Note: the above term **vanishes** identically when the **fluid is barotropic** since the **gradients of pressure and density are in such case parallel**.

The **equation of the vorticity transport** can be written in the form

$$\partial_t \boldsymbol{\omega} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\nabla)\boldsymbol{v} = -\frac{1}{\rho^2}\nabla\rho\times\nabla p + \nabla\times\boldsymbol{f}$$

or, using the full derivative

$$\frac{D}{Dt}\boldsymbol{\omega} = \underbrace{(\boldsymbol{\omega}\cdot\nabla)\boldsymbol{v}}_{\text{vortex stretching term}} \underbrace{-\frac{1}{\rho^2}\nabla\rho\times\nabla p}_{\text{baroclinic term}} + \underbrace{\nabla\times\boldsymbol{f}}_{\text{nonpotential volume force term}}$$

The change of the vorticity appears due to the following factors:

- Local deformation of the pattern of vortex lines (or vortex tubes) known as the **"vortex stretching"** effect. This mechanism is believed to be crucial for generating spatial/temporal complexity of turbulent flows. **The vortex stretching term vanishes identically for 2D flows**.
- Presence of **baroclinic** effects. If the flow is not barotropic then the gradients of pressure and density field are nonparallel. It can be shown that in such situation a **torque** is developed which perpetuates rotation of fluid elements (generates vorticity).
- Presence of nonpotential volume forces. This factor is important e.g. for electricity-conducting fluids.

For the **barotropic** (in particular – incompressible) motion of **inviscid fluid**, the vorticity equation reduces to

$$\frac{D}{Dt}\boldsymbol{\omega} = (\boldsymbol{\omega}\cdot\nabla)\boldsymbol{v}$$

In the **2D case** it reduces further to $\frac{D}{Dt}\boldsymbol{\omega} = 0$

We conclude that in any 2D flow the vorticity is conserved along trajectories of fluid elements.

If the fluid is **viscous**, the vorticity equation contains the **diffusion term**. We will derive this equation assuming that the fluid is incompressible. Again, we begin with the Navier-Stokes equation in the Lamb-Gromeko form

$$\partial_t\boldsymbol{v} + \nabla\left(\tfrac{1}{2}\upsilon^2\right) + \boldsymbol{\omega}\times\boldsymbol{v} = -\tfrac{1}{\rho}\nabla p + \nu\,\Delta\boldsymbol{v} + \boldsymbol{f}$$

If the rotation operator is applied, we get the equation

$$\partial_t\boldsymbol{\omega} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\nabla)\boldsymbol{v} = \nu\,\Delta\boldsymbol{\omega} + \nabla\times\boldsymbol{f}$$

which reduces to $\partial_t\boldsymbol{\omega} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\nabla)\boldsymbol{v} = \nu\,\Delta\boldsymbol{\omega}$

when the field of the volume forces $\boldsymbol{f}$ is potential.

In the above, the following operator identity has been used

$$rot\, \Delta \mathbf{v} = rot\,(grad\, div\mathbf{v} - rot\, rot\mathbf{v}) = -rot\, rot\boldsymbol{\omega} = grad\, div\boldsymbol{\omega} - rot\, rot\, \boldsymbol{\omega} = \Delta\boldsymbol{\omega}$$

showing that the **vector Laplace and rotation operators commute**.

The vorticity equation can be also written equivalently as

$$\frac{D}{Dt}\boldsymbol{\omega} = (\boldsymbol{\omega}\cdot\nabla)\boldsymbol{\upsilon} + \nu\,\Delta\boldsymbol{\omega}$$

The viscous term describes the **diffusion of vorticity** due to **fluid viscosity**. This effect smears the vorticity over the whole flow domain. Thus, in the viscous case the vortex lines are not material lines anymore.

There exists a **relation between the streamfunction and vorticity**. Since the flow is 2D, the vorticity field is perpendicular to the flow's plane and can be expressed as

$$\boldsymbol{\omega} \equiv \nabla\times\boldsymbol{\upsilon} = (\partial_1\upsilon_2 - \partial_2\upsilon_1)\boldsymbol{e}_3 \equiv \omega\boldsymbol{e}_3$$

Then, the **streamfunction satisfies the Poisson equation**

$$\Delta\psi \equiv \partial_{11}\psi + \partial_{22}\psi = -(\partial_1\upsilon_2 - \partial_2\upsilon_1) = -\omega$$

Two dimensional motion of an incompressible viscous fluid can be described in terms of the purely kinematical quantities: velocity, vorticity and streamfunction. The pressure field is eliminated and the continuity equation $div\,\boldsymbol{\upsilon}=0$ is automatically satisfied. The complete description consists of the following equations:

- Equation of the vorticity transport (2D) $\partial_t\omega+\upsilon_1\partial_1\omega+\upsilon_2\partial_2\omega=\nu\,\Delta\omega$
- Equation for the streamfunction $\Delta\psi=-\omega$
- Relation between the streamfunction and velocity $\upsilon_1=\partial_2\psi\ ,\ \upsilon_2=-\partial_1\psi$
- Definition of vorticity (2D) $\omega=\partial_1\upsilon_2-\partial_2\upsilon_1$

accompanied by appropriately formulated **boundary and initial conditions**.
