PROGRAM ROZWOJOWY
POLITECHNIKI WARSZAWSKIEJ

ADDITIONAL TOPICS

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PROOF OF THE LAMB-GROMEKO FORMULA

We have $\boldsymbol{\omega} = \in_{ijk} \frac{\partial}{\partial x_j} \upsilon_k \, \boldsymbol{e}_i$ and $\upsilon^2 = \upsilon_i \upsilon_i$.

Then

$$\boldsymbol{\omega} \times \boldsymbol{\upsilon} = \in_{k\beta\gamma} \omega_k \upsilon_\beta \boldsymbol{e}_\gamma = \in_{k\beta\gamma} \in_{ijk} \frac{\partial \upsilon_j}{\partial x_i} \upsilon_\beta \, \boldsymbol{e}_\gamma = (\delta_{i\beta}\delta_{j\gamma} - \delta_{i\gamma}\delta_{j\beta}) \frac{\partial \upsilon_j}{\partial x_i} \upsilon_\beta \, \boldsymbol{e}_\gamma =$$

$$= \left(\delta_{i\beta}\delta_{j\gamma} \frac{\partial \upsilon_j}{\partial x_i} \upsilon_\beta - \delta_{i\gamma}\delta_{j\beta} \frac{\partial \upsilon_j}{\partial x_i} \upsilon_\beta \right) \boldsymbol{e}_\gamma = \left(\frac{\partial \upsilon}{\partial x_\beta} \upsilon_\beta - \frac{\partial \upsilon_\beta}{\partial x_\gamma} \upsilon_\beta \right) \boldsymbol{e}_\gamma =$$

$$= \left(\frac{\partial \upsilon_\gamma}{\partial x_\beta} \upsilon_\beta - \frac{\partial}{\partial x_\gamma} \left(\tfrac{1}{2} \upsilon_\beta \upsilon_\beta \right) \right) \boldsymbol{e}_\gamma = (\boldsymbol{\upsilon} \cdot \nabla) \boldsymbol{\upsilon} - \nabla \left(\tfrac{1}{2} \upsilon^2 \right)$$

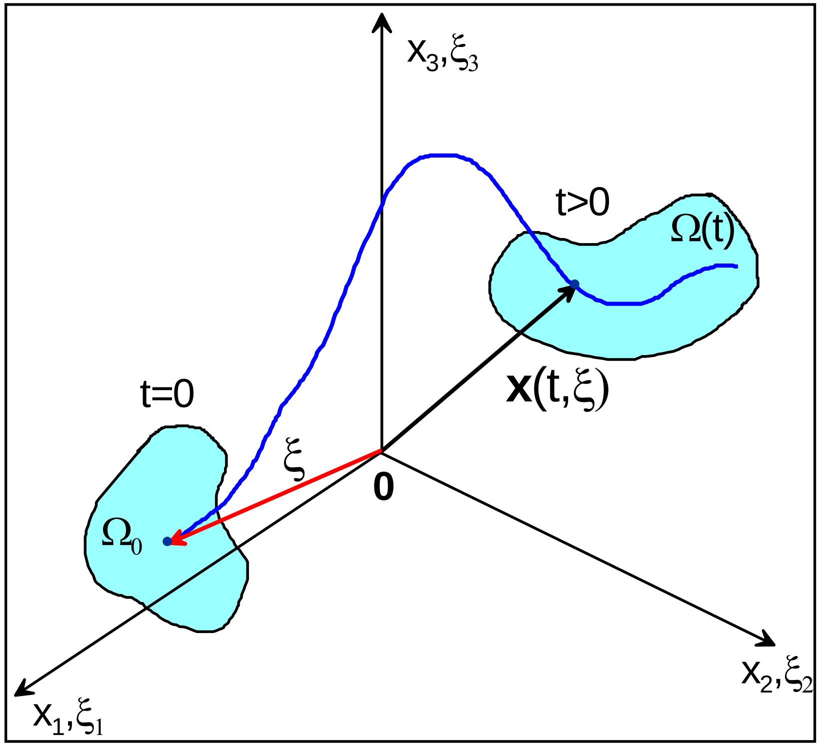
Thus

$$(\boldsymbol{\upsilon} \cdot \nabla) \boldsymbol{\upsilon} = \nabla \left(\tfrac{1}{2} \upsilon^2 \right) + \boldsymbol{\omega} \times \boldsymbol{\upsilon}$$

and the Lamb-Gromeko formula follows immediately. ♣

REYNOLDS TRANSPORT THEOREM (1)

We will prove the mathematical result known as the **Reynolds' Transport Theorem**, which plays the fundamental role in derivation of **differential forms of the conservation principles** in **Continuum Mechanics**.



Consider any sufficiently regular scalar field $f = f(t, \boldsymbol{x})$. Consider the integral of f calculated over an arbitrary material volume $\Omega(t)$.

$$C(t) = \int_{\Omega(t)} f(t, \boldsymbol{x})\, d\boldsymbol{x}.$$

We need to compute the time derivative $C'(t) = \dfrac{d}{dt} \displaystyle\int_{\Omega(t)} f(t, \boldsymbol{x})\, d\boldsymbol{x}.$

NOTE: This task is nontrivial since **the integration domain is itself time dependent**!

Reynolds Transport Theorem (2)

To calculate the derivative, we will switch from Euler variables $\boldsymbol{x}=[x_1,x_2,x_3]$ to Lagrangian variables $\boldsymbol{\xi}=[\xi_1,\xi_2,\xi_3]$. The integral $C(t)$ can be written as

$$C(t)=\int_{\Omega(t)} f(t,\boldsymbol{x})\,d\boldsymbol{x}=\int_{\Omega_0} f[t,\boldsymbol{x}(t,\boldsymbol{\xi})]\,J(t,\boldsymbol{\xi})\,d\boldsymbol{\xi}=\int_{\Omega_0} f_0(t,\boldsymbol{\xi})\,J(t,\boldsymbol{\xi})\,d\boldsymbol{\xi}\,.$$

In the above formula we have used the composite function f_0

$$f_0=f_0(t,\boldsymbol{\xi})=f[t,\boldsymbol{x}(t,\boldsymbol{\xi})],$$

and also the **Jacobi determinant** (Jacobian) defined as

$$J(t,\boldsymbol{\xi})=\det\begin{bmatrix}\frac{\partial x_1}{\partial \xi_1} & \frac{\partial x_1}{\partial \xi_2} & \frac{\partial x_1}{\partial \xi_3}\\ \frac{\partial x_2}{\partial \xi_1} & \frac{\partial x_2}{\partial \xi_2} & \frac{\partial x_2}{\partial \xi_3}\\ \frac{\partial x_3}{\partial \xi_1} & \frac{\partial x_3}{\partial \xi_2} & \frac{\partial x_3}{\partial \xi_3}\end{bmatrix}(t,\boldsymbol{\xi}).$$

REYNOLDS TRANSPORT THEOREM (3)

Since the domain Ω_0 is time-independent (it is actually the initial form of the material volume $\Omega(t)$ at the time $t = 0$), we can move the differentiation operator under the sign of the integral and get

$$C'(t) = \frac{d}{dt} \int_{\Omega_0} f_0(t,\xi)\, J(t,\xi)\, d\xi = \int_{\Omega_0} \frac{\partial f_0}{\partial t}(t,\xi)\, J(t,\xi)\, d\xi + \int_{\Omega_0} f_0(t,\xi) \frac{\partial J}{\partial t}(t,\xi)\, d\xi$$

Note that time differentiation of the composite function f_0 yields

$$\tfrac{\partial}{\partial t} f_0(t,\xi) = \tfrac{d}{dt} f[t,\boldsymbol{x}(t,\xi)] = \tfrac{\partial}{\partial t} f[t,\boldsymbol{x}(t,\xi)] + \tfrac{\partial}{\partial x_i} f[t,\boldsymbol{x}(t,\xi)] \cdot \underbrace{\tfrac{\partial}{\partial t} x_i(t,\xi)}_{=V_i(t,\xi)=\upsilon_i[t,\boldsymbol{x}(t,\xi)]} =$$

$$= \left(\tfrac{\partial}{\partial t} f + \boldsymbol{\upsilon} \cdot \nabla f \right)[t,\boldsymbol{x}(t,\xi)]$$

This part was easy! **We need to calculate the time derivative of the Jacobian** which has appeared in the second integral in the formula for $C'(t)$. This is much more complicated … .

Basically, we have two methods.

Method A

We write the Jacobian using the alternating symbol $J(t,\xi) = \in_{ijk} \frac{\partial x_1}{\partial \xi_i} \frac{\partial x_2}{\partial \xi_j} \frac{\partial x_3}{\partial \xi_k}$

Note that partial derivatives with respect to time and Lagrangian variables commute, hence

$$\frac{\partial}{\partial t}\frac{\partial x_1}{\partial \xi_i} = \frac{\partial}{\partial \xi_i}\frac{\partial x_1}{\partial t} = \frac{\partial V_1}{\partial \xi_i} \quad , \quad \frac{\partial}{\partial t}\frac{\partial x_2}{\partial \xi_j} = \frac{\partial}{\partial \xi_j}\frac{\partial x_2}{\partial t} = \frac{\partial V_2}{\partial \xi_j} \quad , \quad \frac{\partial}{\partial t}\frac{\partial x_3}{\partial \xi_k} = \frac{\partial}{\partial \xi_k}\frac{\partial x_3}{\partial t} = \frac{\partial V_3}{\partial \xi_k}$$

The time derivative

$$\frac{\partial}{\partial t} J = \in_{ijk} \frac{\partial V_1}{\partial \xi_i}\frac{\partial x_2}{\partial \xi_j}\frac{\partial x_3}{\partial \xi_k} + \in_{ijk} \frac{\partial x_1}{\partial \xi_i}\frac{\partial V_2}{\partial \xi_j}\frac{\partial x_3}{\partial \xi_k} + \in_{ijk} \frac{\partial x_1}{\partial \xi_i}\frac{\partial x_2}{\partial \xi_j}\frac{\partial V_3}{\partial \xi_k} =$$

$$= \begin{vmatrix} \frac{\partial V_1}{\partial \xi_1} & \frac{\partial V_1}{\partial \xi_2} & \frac{\partial V_1}{\partial \xi_3} \\ \frac{\partial x_2}{\partial \xi_1} & \frac{\partial x_2}{\partial \xi_2} & \frac{\partial x_2}{\partial \xi_3} \\ \frac{\partial x_3}{\partial \xi_1} & \frac{\partial x_3}{\partial \xi_2} & \frac{\partial x_3}{\partial \xi_3} \end{vmatrix} + \begin{vmatrix} \frac{\partial x_1}{\partial \xi_1} & \frac{\partial x_1}{\partial \xi_2} & \frac{\partial x_1}{\partial \xi_3} \\ \frac{\partial V_2}{\partial \xi_1} & \frac{\partial V_2}{\partial \xi_2} & \frac{\partial V_2}{\partial \xi_3} \\ \frac{\partial x_3}{\partial \xi_1} & \frac{\partial x_3}{\partial \xi_2} & \frac{\partial x_3}{\partial \xi_3} \end{vmatrix} + \begin{vmatrix} \frac{\partial x_1}{\partial \xi_1} & \frac{\partial x_1}{\partial \xi_2} & \frac{\partial x_1}{\partial \xi_3} \\ \frac{\partial x_2}{\partial \xi_1} & \frac{\partial x_2}{\partial \xi_2} & \frac{\partial x_2}{\partial \xi_3} \\ \frac{\partial V_3}{\partial \xi_1} & \frac{\partial V_3}{\partial \xi_2} & \frac{\partial V_3}{\partial \xi_3} \end{vmatrix} =$$

$$= \sum_{i=1}^{3}\sum_{j=1}^{3} \underbrace{\frac{\partial}{\partial \xi_j} V_i}_{(\nabla_\xi \boldsymbol{V})_{ij}} \quad \underbrace{[cof\ \boldsymbol{J}]_{ij}}_{cofactor\ (i,j)\ of\ \boldsymbol{J}}$$

Consider two square matrices $\boldsymbol{A}$ and $\boldsymbol{B}$**,** and also the product $\boldsymbol{C} = \boldsymbol{A}\boldsymbol{B}^T$. It means that

$$c_{ik} = \sum_j a_{ij} b_{kj} \equiv a_{ij} b_{kj},$$

so we conclude that

$$tr\,\boldsymbol{C} \equiv c_{ii} = a_{ij} b_{ij}$$ (**trace** of the matrix $\boldsymbol{C}$)

Moreover, from the construction of the inverse Jacobi matrix we have

$$\boldsymbol{J}^{-1} = \frac{1}{\det \boldsymbol{J}} (cof\ \boldsymbol{J})^T \quad \Rightarrow \quad (cof\ \boldsymbol{J})^T = \det\boldsymbol{J}\ \boldsymbol{J}^{-1} = J\ \boldsymbol{J}^{-1}$$

Hence, the formula for the time derivative of the Jacobi determinant can be written as follows

$$\tfrac{\partial}{\partial t} J(t,\xi) = tr\left[\nabla_\xi \boldsymbol{V} \cdot (cof\ \boldsymbol{J})^T\right](t,\xi) = J(t,\xi)\, tr\left[\nabla_\xi \boldsymbol{V} \cdot \boldsymbol{J}^{-1}\right](t,\xi)$$

Finally, **we need to get back to the Euler variables**. To this end, we use the relation between Lagrange and Euler definitions of the fluid velocity

$$\underbrace{\boldsymbol{V}(t,\xi)}_{Lagrange} = \underbrace{\boldsymbol{v}[t, \boldsymbol{x}(t,\xi)]}_{Euler}$$

Next, we calculate the gradient operator with respect to the Lagrange variables

$$\left[\nabla_{\xi}\boldsymbol{V}\right]_{ij}(t,\boldsymbol{\xi}) = \frac{\partial}{\partial \xi_j} V_i(t,\boldsymbol{\xi}) = \sum_{k=1}^{3} \frac{\partial}{\partial x_k} \upsilon_i[t,\boldsymbol{x}(t,\boldsymbol{\xi})] \frac{\partial x_k}{\partial \xi_j}(t,\boldsymbol{\xi}).$$

The above formula can be written shortly as

$$\nabla_{\xi}\boldsymbol{V}(t,\boldsymbol{\xi}) = \nabla\boldsymbol{\upsilon}[t,\boldsymbol{x}(t,\boldsymbol{\xi})]\cdot\boldsymbol{J}(t,\boldsymbol{\xi})$$

Thus, the time derivative of the Jacobian can be re-written in the following form

$$\frac{\partial}{\partial t} J(t,\boldsymbol{\xi}) = J(t,\boldsymbol{\xi})\left(tr\,\nabla\boldsymbol{\upsilon}\right)[t,\boldsymbol{x}(t,\boldsymbol{\xi})].$$

Taking into account that $tr\,\nabla\boldsymbol{\upsilon} = \frac{\partial}{\partial x_i}\upsilon_i = div\,\boldsymbol{\upsilon} \equiv \nabla\cdot\boldsymbol{\upsilon}$

we finally get the formula

$$\frac{\partial}{\partial t} J(t,\boldsymbol{\xi}) = J(t,\boldsymbol{\xi})\,\nabla\cdot\boldsymbol{\upsilon}[t,\boldsymbol{x}(t,\boldsymbol{\xi})]$$

Method B

This method is based upon the **group property** of the transformation of the **material volume** at initial time $t = 0$ to the volume (consisting of the same fluid particles) at some later time $t > 0$. We can write $\boldsymbol{x}(t+s,\boldsymbol{\xi}) = \boldsymbol{x}[t,\boldsymbol{x}(s,\boldsymbol{\xi})]$ or ($i = 1,2,3$).

$$x_i(t+s,\xi_1,\xi_2,\xi_3) = x_i[t,x_1(s,\xi_1,\xi_2,\xi_3),x_2(s,\xi_1,\xi_2,\xi_3),x_3(s,\xi_1,\xi_2,\xi_3)],$$

Let's differentiate the above formula with respect to the Lagrange coordinate ξ_j:

$$\frac{\partial x_i}{\partial \xi_j}(t+s,\boldsymbol{\xi}) = \frac{\partial x_i}{\partial \xi_k}[t,\boldsymbol{x}(s,\boldsymbol{\xi})]\,\frac{\partial x_k}{\partial \xi_j}(s,\boldsymbol{\xi}),$$

which can also be written as $[\boldsymbol{J}]_{ij}(t+s,\boldsymbol{\xi}) = [\boldsymbol{J}]_{ik}[t,\boldsymbol{x}(s,\boldsymbol{\xi})]\,[\boldsymbol{J}]_{kj}(s,\boldsymbol{\xi}),$

which is equivalent to $\boldsymbol{J}(t+s,\boldsymbol{\xi}) = \boldsymbol{J}[t,\boldsymbol{x}(s,\boldsymbol{\xi})]\,\boldsymbol{J}(s,\boldsymbol{\xi}).$

From the fundamental property of determinant

$$J(t+s,\boldsymbol{\xi}) = J[t,\boldsymbol{x}(s,\boldsymbol{\xi})]\,J(s,\boldsymbol{\xi}).$$

We need to calculate the derivative

$$\frac{\partial}{\partial t}J(t,\xi) := \lim_{\Delta t \to 0}\frac{J(t+\Delta t,\xi)-J(t,\xi)}{\Delta t} = \lim_{\Delta t \to 0}\frac{J(t,\xi)\,J[\Delta t,\boldsymbol{x}(t,\xi)]-J(t,\xi)}{\Delta t} =$$

$$= J(t,\xi)\lim_{\Delta t \to 0}\frac{J[\Delta t,\boldsymbol{x}(t,\xi)]-1}{\Delta t}$$

Note that $J[\Delta t,\boldsymbol{x}(t,\xi)]$ is the Jacobian of the **"nearly identical" transformation** $\boldsymbol{x}(t,\xi) \mapsto \boldsymbol{x}(t+\Delta t,\xi)$, which can be written shortly as $\boldsymbol{x} \mapsto \Psi_{\Delta t}(\boldsymbol{x})$.

The explicit form of this transformation is ($i = 1, 2, 3$),

$$[\Psi_{\Delta t}(\boldsymbol{x})]_i = x_i + \upsilon_i(t,x_1,x_2,x_3)\,\Delta t + O(\Delta t^2)$$

This, the Jacobi matrix can be calculated as follows

$$[\boldsymbol{J}]_{ij}(\Delta t,\boldsymbol{x}) = \frac{\partial}{\partial x_j}[\Psi_{\Delta t}(\boldsymbol{x})]_i = \delta_{ij} + \frac{\partial \upsilon_i}{\partial x_j}(t,\boldsymbol{x})\,\Delta t + O(\Delta t^2)$$

or simply

$$\boldsymbol{J}(\Delta t,\boldsymbol{x}) = \boldsymbol{I} + \nabla\boldsymbol{\upsilon}(t,\boldsymbol{x})\,\Delta t + O(\Delta t^2).$$

Now, it is not difficult to show (do it!) that

$$J(\Delta t, \boldsymbol{x}) = 1 + \underbrace{\left(\frac{\partial \upsilon_1}{\partial x_1} + \frac{\partial \upsilon_2}{\partial x_2} + \frac{\partial \upsilon_3}{\partial x_3} \right)}_{div\,\boldsymbol{\upsilon}} (t, \boldsymbol{x}) \Delta t + O(\Delta t^2) = 1 + \nabla \cdot \boldsymbol{\upsilon}(t, \boldsymbol{x}) \Delta t + O(\Delta t^2)$$

Thus, we get

$$\lim_{\Delta t \to 0} \frac{J(\Delta t, \boldsymbol{x}) - 1}{\Delta t} = \nabla \cdot \boldsymbol{\upsilon}(t, \boldsymbol{x})$$

and – after returning back to the Lagrange variables - the formula for the time derivative of the Jacobian is obtained

$$\frac{\partial}{\partial t} J(t, \boldsymbol{\xi}) := J(t, \boldsymbol{\xi}) \left(\nabla \cdot \boldsymbol{\upsilon} \right) [t, \boldsymbol{x}(t, \boldsymbol{\xi})].$$

REYNOLDS TRANSPORT THEOREM (4)

The time derivative $C'(t)$ can be now evaluated as follows

$$C'(t) = \int_{\Omega_0} \left(\tfrac{\partial}{\partial t} f + \boldsymbol{v}\cdot\nabla f + f\,\nabla\cdot\boldsymbol{v} \right)[t, \boldsymbol{x}(t,\boldsymbol{\xi})]\, J(t,\boldsymbol{\xi})\, d\boldsymbol{\xi} =$$

$$= \int_{\Omega(t)} \left(\tfrac{\partial}{\partial t} f + \boldsymbol{v}\cdot\nabla f + f\,\nabla\cdot\boldsymbol{v} \right)(t,\boldsymbol{x})\, d\boldsymbol{x} = \int_{\Omega(t)} \left[\tfrac{\partial}{\partial t} f + \nabla\cdot(f\,\boldsymbol{v}) \right](t,\boldsymbol{x})\, d\boldsymbol{x} =$$

$$= \int_{\Omega(t)} \tfrac{\partial}{\partial t} f\, d\boldsymbol{x} + \int_{\Omega(t)} \nabla\cdot(f\,\boldsymbol{v})\, d\boldsymbol{x} \underset{\substack{\Uparrow \\ \text{GGO} \\ \text{Theorem}}}{=} \int_{\Omega(t)} \tfrac{\partial}{\partial t} f\, d\boldsymbol{x} + \int_{\partial\Omega(t)} f \underbrace{\upsilon_n}_{\substack{\boldsymbol{v}\cdot\boldsymbol{n} \\ \text{normal} \\ \text{velocity}}} ds$$

Note that the last equality has been obtained by the use of the **Green-Gauss-Ostrogradsky (GGO) Theorem**. We see that the rate of change of $C(t)$ is the sum of **two components**. The **first component** appears due to the **local time variation** of the integrated function f and it appears even if the fluid is in rest (no motion). In contrast, the **second term is entirely due to the fluid motion** and it assumes nonzero value even if the field f is stationary (i.e. $\frac{\partial}{\partial t} f \equiv 0$).

TIME RATE OF CHANGE OF AN EXTENSIVE QUANTITY

Consider an **extensive** physical quantity, characterized by its **mass-specific density** $H = H(t, \boldsymbol{x})$. The amount of this quantity contained in the **material volume** $\Omega(t)$ is expressed by the following volume integral

$$h(t) = \int_{\Omega(t)} \rho H \, d\boldsymbol{x}$$

The examples are: the Cartesian components of the linear momentum, kinetic and internal energy. We need to know how to evaluate **the time derivative of such integrals**.

Using the **Reynolds' theorem** and the **differential equation of mass conservation** we can write

$$\frac{d}{dt} h(t) = \frac{d}{dt} \int_{\Omega(t)} \rho H \, d\boldsymbol{x} \underset{\substack{\Uparrow \\ \textit{Reynolds} \\ \textit{Trans.Th.}}}{=} \int_{\Omega(t)} \left[\frac{\partial}{\partial t}(\rho H) + \nabla \cdot (\rho H \boldsymbol{v}) \right] d\boldsymbol{x} =$$

$$= \int_{\Omega(t)} H \underbrace{\left[\frac{\partial}{\partial t} \rho + \nabla \cdot (\rho \boldsymbol{v}) \right]}_{=0!} dx + \int_{\Omega(t)} \rho \underbrace{\left(\frac{\partial}{\partial t} H + \boldsymbol{v} \cdot \nabla H \right)}_{= \frac{DH}{Dt}} d\boldsymbol{x} = \int_{\Omega(t)} \rho \frac{D}{Dt} H \, d\boldsymbol{x}$$