PROGRAM ROZWOJOWY POLITECHNIKI WARSZAWSKIEJ

MATHEMATICAL PRELIMINARIES

KAPITAŁ LUDZKI
NARODOWA STRATEGIA SPÓJNOŚCI

UNIA EUROPEJSKA
EUROPEJSKI FUNDUSZ SPOŁECZNY

ALGEBRA OF VECTORS AND TENSORS

Orthogonal basic unary vectors (versors) : $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$

$$(\boldsymbol{e}_i, \boldsymbol{e}_j) = \delta_{ij} \equiv \begin{cases} 1 & if \quad i = j \\ 0 & if \quad i \neq j \end{cases}$$

Any vector in E^3 is expressed as unique linear combinations of the basic versors

$$\boldsymbol{a} = a_1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2 + a_3\boldsymbol{e}_3 \equiv a_i\boldsymbol{e}_i$$ - summation (Einstein) convention

$$\boldsymbol{a} = [a_1, a_2, a_3]$$ - canonical equivalence of E^3 and R^3

INNER (SCALAR) PRODUCT

Let $\boldsymbol{a} = a_i\boldsymbol{e}_i$ and $\boldsymbol{b} = b_j\boldsymbol{e}_j$. We define the inner product of $\boldsymbol{a}$ and $\boldsymbol{b}$:

$$\boldsymbol{a} \cdot \boldsymbol{b} \equiv (\boldsymbol{a}, \boldsymbol{b}) = a_i b_j (\boldsymbol{e}_i, \boldsymbol{e}_j) = a_i b_j \delta_{ij} = a_i b_i$$

Note that $(\boldsymbol{a}, \boldsymbol{e}_i) = a_i$ hence we can write $\boldsymbol{a} = (\boldsymbol{a}, \boldsymbol{e}_i)\boldsymbol{e}_i$

VECTOR (CROSS) PRODUCT

We define the operation $\times$ on the basic vectors:

$$\boldsymbol{e}_1 \times \boldsymbol{e}_2 = \boldsymbol{e}_3 \ , \ \boldsymbol{e}_2 \times \boldsymbol{e}_3 = \boldsymbol{e}_1 \ , \ \boldsymbol{e}_3 \times \boldsymbol{e}_1 = \boldsymbol{e}_2,$$

$$\underbrace{\boldsymbol{e}_i \times \boldsymbol{e}_i}_{no\ summation!} = \boldsymbol{0} \ , \ \boldsymbol{e}_i \times \boldsymbol{e}_j = -\boldsymbol{e}_j \times \boldsymbol{e}_i$$

Assuming linearity with respect to both arguments, we extend this operation to all vectors in the space E^3

$$\boldsymbol{a} \times \boldsymbol{b} = a_i \boldsymbol{e}_i \times b_j \boldsymbol{e}_j = a_i b_j \, \boldsymbol{e}_i \times \boldsymbol{e}_j = (a_2 b_3 - a_3 b_2)\boldsymbol{e}_1 + (a_3 b_1 - a_1 b_3)\boldsymbol{e}_2 + (a_1 b_2 - a_2 b_1)\boldsymbol{e}_3$$

Practical way of computing the vector product

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \boldsymbol{e}_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \boldsymbol{e}_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \boldsymbol{e}_3$$

ALTERNATING SYMBOL

$$\in_{ijk} = \begin{cases} 0 \ \ if \ \ \ i=j \ \ \ or \ \ \ i=k \ \ \ or \ \ \ j=k \\ 1 \ \ if \ \ \{i,j,k\} \ \ is \ \ an \ \ even \ \ permutation \ of \ \{1,2,3\} \\ -1 \ \ if \ \ \{i,j,k\} \ \ is \ \ an \ \ odd \ \ permutation \ of \ \{1,2,3\} \end{cases}$$

For instance we have $\in_{213} = -1 \quad , \quad \in_{311} = 0 \quad , \quad \in_{231} = 1.$

The **vector product** of **a** and **b** can be nicely written as follows

$$\boldsymbol{a} \times \boldsymbol{b} = \in_{ijk} a_j b_k \boldsymbol{e}_i$$

Another useful operation is the **mixed product** of three vectors

$$\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \in_{ijk} a_i b_j c_k$$

Determinant of the matrix **A** (dim **A** = 3): $\det \boldsymbol{A} = \in_{ijk} a_{1,i} a_{2,j} a_{3,k}$

2^{ND}-RANK TENSORS IN E^3

Tensors as bilinear transformations (functionals) $E^3 \times E^3 \to R$

Bi-linearity means that

$$T(\alpha_1 \boldsymbol{x}_1 + \alpha_2 \boldsymbol{x}_2, \boldsymbol{y}) = \alpha_1 T(\boldsymbol{x}_1, \boldsymbol{y}) + \alpha_2 T(\boldsymbol{x}_2, \boldsymbol{y}),$$
$$T(\boldsymbol{x}, \alpha_1 \boldsymbol{y}_1 + \boldsymbol{\alpha}_2 \boldsymbol{y}_2) = \alpha_1 T(\boldsymbol{x}, \boldsymbol{y}_1) + \alpha_2 T(\boldsymbol{x}, \boldsymbol{y}_2).$$

For two arbitrary vectors **x** and **y** we can write

$$T(\boldsymbol{x}, \boldsymbol{y}) = T(x_i \boldsymbol{e}_i, y_j \boldsymbol{e}_j) = x_i y_j T(\boldsymbol{e}_i, \boldsymbol{e}_j) = t_{ij} x_i y_j$$

The **matrix** $\boldsymbol{T}$ such that $[\boldsymbol{T}]_{ij} = t_{ij}$ **represents the tensor T in the assumed reference frame** (or with respect to assumed basic versors)

Some operations on tensors:

Addition: $T = T_1 + T_2 \Rightarrow \boldsymbol{T} = \boldsymbol{T}_1 + \boldsymbol{T}_2 \Rightarrow t_{ij} = t_{ij}^1 + t_{ij}^2$

Multiplication by a scalar $T = \beta T_1 \Rightarrow \boldsymbol{T} = \beta \boldsymbol{T}_1 \Rightarrow t_{ij} = \beta t_{ij}^1$

Multiplication of two tensors $T = T_1 T_2 \Rightarrow \boldsymbol{T} = \boldsymbol{T}_1 \boldsymbol{T}_2 \Rightarrow t_{ij} = t_{ik}^1 t_{kj}^2$

Scalar (Frobenius) product of two tensors $s = T_1 : T_2 := t_{ij}^1 t_{ij}^2 \quad (double\ \ summation\,!)$

Basic linear functionals $E^3 \rightarrow R$:

$$f_i(\boldsymbol{e}_j) \coloneqq \delta_{ij}$$

In the case of the orthogonal base, the basic functionals (covectors) can be identified with the "normal" base. The "canonical" identity between base and co-base follows from the following formula

$$f_i(\boldsymbol{w}) = (\boldsymbol{e}_i, \boldsymbol{w}) \;\;,\;\; i = 1,2,3 \;,\;\; \boldsymbol{w} \in E^3$$

Tensor product of the basic functionals:

$$(f_i \otimes f_j)(\boldsymbol{x}, \boldsymbol{y}) \coloneqq f_i(\boldsymbol{x})\, f_j(\boldsymbol{y}) = f_i(x_k \boldsymbol{e}_k)\, f_j(y_m \boldsymbol{e}_m) =$$
$$= x_k y_m f_i(\boldsymbol{e}_k)\, f_j(\boldsymbol{e}_m) = x_k y_m (\boldsymbol{e}_i, \boldsymbol{e}_k)(\boldsymbol{e}_j, \boldsymbol{e}_m) = x_k y_m \delta_{ik} \delta_{jm} = x_i y_j$$

Thus we can write $T(\boldsymbol{x}, \boldsymbol{y}) = t_{ij} x_i y_j = t_{ij}\,(f_i \otimes f_j)(\boldsymbol{x}, \boldsymbol{y})$ or $T = t_{ij}\, f_i \otimes f_j$.

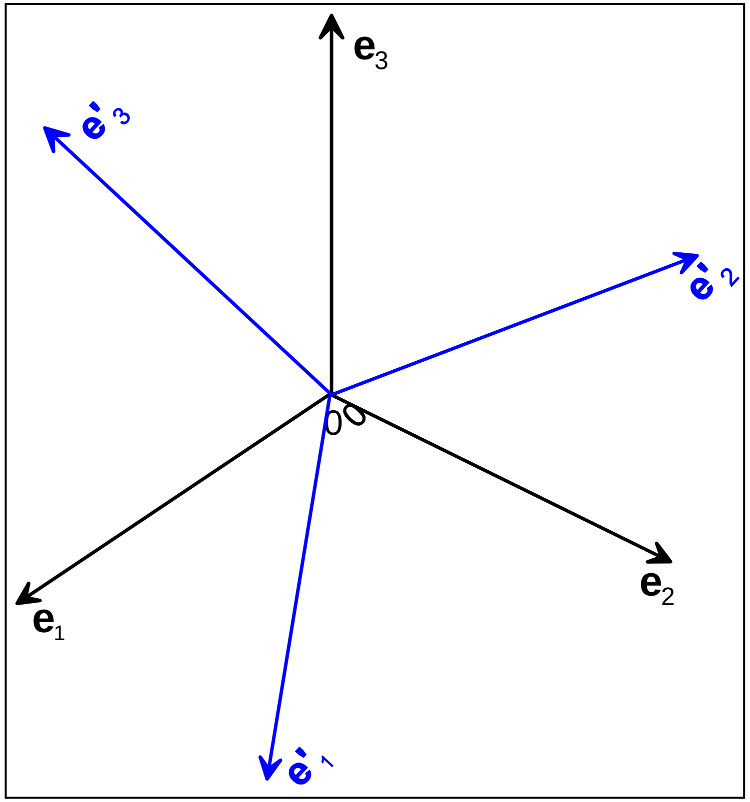
Due to the above identification between base and co-base we may equally well write

$$T = t_{ij}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j$$

The linear space of the 2nd-rank tensors is 9-dimensional.

ORTHOGONAL TRANSFORMATIONS OF COORDINATE SYSTEMS

Assume that **different basic vectors** are introduced $\boldsymbol{e}'_1, \boldsymbol{e}'_2, \boldsymbol{e}'_3$ (see figure). These vectors **can be expressed by means of the "old" basic vectors**.



Consider $\boldsymbol{e}'_i = z_{ik}\boldsymbol{e}_k$, $\boldsymbol{e}'_j = z_{jm}\boldsymbol{e}_m$.

The orthogonality condition for the new base yields

$$(\boldsymbol{I})_{ij} = \delta_{ij} = (\boldsymbol{e}'_i, \boldsymbol{e}'_j) = z_{ik} z_{jm} (\boldsymbol{e}_k, \boldsymbol{e}_m) = z_{ik} z_{jm} \delta_{km} =$$
$$= z_{ik} z_{jk} = (\boldsymbol{Z}\boldsymbol{Z}^T)_{ij} = (\boldsymbol{Z}^T\boldsymbol{Z})_{ij}.$$

We conclude that the transformation of the basis **preserves orthonormality** of the basic vectors if and only if **the transformation matrix Z satisfies the relation** $\boldsymbol{Z}^{-1} = \boldsymbol{Z}^T$**, i.e., it is the orthogonal matrix**.

Each vector **x** from E^3 **can be expressed with respect to both basis**, namely

$$\boldsymbol{x} = x_i \boldsymbol{e}_i = x'_i \boldsymbol{e}'_i .$$

Thus $\boldsymbol{x} = x_i \boldsymbol{e}_i = x'_i z_{ij} \boldsymbol{e}_j = x'_j z_{ji} \boldsymbol{e}_i$,

meaning that

$$x_i = z_{ji} x'_j = (\boldsymbol{Z}^T)_{ij} x'_j = (\boldsymbol{Z}^{-1})_{ij} x'_j$$

and

$$x'_i = (\boldsymbol{Z})_{ij} x_j .$$

These are the transformation rules for the vectors!

Consider the **tensor** T and its representation with respect to both basis (reference frames)

$$T(\boldsymbol{x}, \boldsymbol{y}) = t_{ij} x_i y_j = t'_{ij} x'_i y'_j .$$

We can write

$$T(\boldsymbol{x}, \boldsymbol{y}) = t_{ij} x_i y_j = t_{ij} z_{ki} x'_k z_{mj} y'_m = x'_k \underbrace{z_{ki} t_{ij}}_{(\boldsymbol{ZT})_{kj}} z_{mj} y'_m =$$

$$= x'_k \underbrace{(\boldsymbol{ZT})_{kj} (\boldsymbol{Z}^T)_{jm}}_{(\boldsymbol{ZTZ}^T)_{km}} y'_m = x'_k \underbrace{(\boldsymbol{ZTZ}^T)_{km}}_{t'_{km}} y'_m = x'_k t'_{km} y'_m$$

The matrix representing the tensor T in the new base is given as

$$\boldsymbol{T}' = \boldsymbol{Z}\ \boldsymbol{T}\ \boldsymbol{Z}^{T} = \boldsymbol{Z}\ \boldsymbol{T}\ \boldsymbol{Z}^{-1}$$

Thus, we have obtained the transformation rule for the 2nd – rank tensors!

DIFFERENT VIEW: 2ND –RANK TENSORS AS LINEAR MAPPINGS $E^3 \to E^3$

Consider the 2nd-rank tensor T and two vectors **x** and **y**.

We have
$$T(\boldsymbol{x},\boldsymbol{y}) = x_i \underbrace{t_{ij} y_j}_{w_i} = x_i w_i = \underbrace{(\boldsymbol{x},\boldsymbol{w})}_{\text{inner product}}.$$

The vector **w** can be defined as $\boldsymbol{w} = \mathfrak{T}\boldsymbol{y}$.

The **linear transformation** $\mathfrak{T} : E^3 \to E^3$ **is defined by its action on the basic versors** as

$$\mathfrak{T}\boldsymbol{e}_j = t_{ij}\boldsymbol{e}_i$$

Indeed, for **any vector w** we get

$$\boldsymbol{w} = \mathfrak{T}\boldsymbol{y} = \mathfrak{T}(y_j \boldsymbol{e}_j) = y_j \mathfrak{T}\boldsymbol{e}_j = t_{ij} y_j \boldsymbol{e}_i = w_i \boldsymbol{e}_i.$$

Equivalence between 2-rank tensors and linear mappings can be established as follows

$$\mathfrak{T} \to T:\; T(\boldsymbol{x},\boldsymbol{y}) := (\boldsymbol{x},\mathfrak{T}\boldsymbol{y}) \quad , \quad T \to \mathfrak{T}:\; \mathfrak{T}\boldsymbol{y} := T(\boldsymbol{e}_i,\boldsymbol{y})\boldsymbol{e}_i.$$

EIGENVECTORS, EIGENVALUES AND TENSOR INVARIANTS

The eigenvalue problem:

1st formulation: find $\lambda \in C$ and nonzero **w** such that $\mathfrak{T}\boldsymbol{w} = \lambda \boldsymbol{w}$, or
2nd formulation: find $\lambda \in C$ and nonzero **w** such that $T(\boldsymbol{x},\boldsymbol{v}) = \lambda(\boldsymbol{x},\boldsymbol{v})$ for each vector **x** from the space E^3.

Equivalently, we have

$$(t_{ij}v_j - \lambda v_i)\boldsymbol{e}_i = \boldsymbol{0} \;\Rightarrow\; p_T(\lambda) = \det(\boldsymbol{T} - \lambda \boldsymbol{I}) = 0.$$

Thus **eigenvalues are the roots of the characteristic polynomial** $p_T(\lambda)$.

Tensor T is **symmetric** when $T(\boldsymbol{x},\boldsymbol{y}) = T(\boldsymbol{y},\boldsymbol{x})$, i.e. when $t_{ij} = t_{ji}$ (check!) or $\boldsymbol{T} = \boldsymbol{T}^T$.

If the tensor T is **symmetric** then its **all eigenvalues are real** and the **eigenvectors corresponding to different eigenvalues are orthogonal** (the proof can be found in standard algebra textbooks).

The **characteristic polynomial is invariant**, i.e. it is the same **in all orthogonal reference frames**. Indeed, according to the transformation rule we have

$$p_T(\lambda) = \det(\boldsymbol{T}' - \lambda \boldsymbol{I}) = \det(\boldsymbol{Z}\boldsymbol{T}\boldsymbol{Z}^{-1} - \lambda \boldsymbol{I}) = \det[\boldsymbol{Z}(T - \lambda \boldsymbol{I})\boldsymbol{Z}^{-1}] =$$

$$= \det \boldsymbol{Z} \cdot \det(\boldsymbol{T} - \lambda \boldsymbol{I}) \cdot \det \boldsymbol{Z}^{-1} = \det \boldsymbol{Z} \cdot \det(\boldsymbol{T} - \lambda \boldsymbol{I}) \cdot (\det \boldsymbol{Z})^{-1} = \det(\boldsymbol{T} - \lambda \boldsymbol{I})$$

We are mostly interested in **3D case**. Then, we can write

$$p_T(\lambda) = -\lambda^3 + J_1 \lambda^2 - J_2 \lambda + J_3$$

where

$$J_1 = tr\, T := t_{ii} \equiv t_{11} + t_{22} + t_{33}$$ (“tr” means trace),

$$J_2 = \tfrac{1}{2}[(tr\, T)^2 - tr\, T^2]$$ (calculate for 2D case!),

$$J_3 = \det T .$$

The following **relations** hold **between invariants and the eigenvalues** (Viete formulas for 3rd-order polynomial)

$$J_1 = \lambda_1 + \lambda_2 + \lambda_3 \quad , \quad J_2 = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 \quad , \quad J_3 = \lambda_1\lambda_2\lambda_3 .$$

CAYLEY-HAMILTON THEOREM

Any square matrix **A** satisfies its own characteristic polynomial $p_A(\lambda) = \det(\boldsymbol{A} - \lambda \boldsymbol{I})$, i.e. we have $p_A(\boldsymbol{A}) = \boldsymbol{0}$.

<u>Proof:</u>

For invertible square matrix **M** we have $\boldsymbol{M}^{-1} = \left(\det \boldsymbol{M}\right)^{-1} (cof\, \boldsymbol{M})^T$. Thus $\boldsymbol{M}\,(cof\, \boldsymbol{M})^T = \det \boldsymbol{M} \cdot \boldsymbol{I}$. Let $\boldsymbol{M} = \boldsymbol{A} - \lambda \boldsymbol{I}$. Then $\boldsymbol{B}(\lambda) := [cof\,(\boldsymbol{A} - \lambda \boldsymbol{I})]^T$ is the matrix polynomial of the order not larger than n -1 (n – dimension of **A**)

$$\boldsymbol{B}(\lambda) = \lambda^{n-1} \boldsymbol{B}_{n-1} + \lambda^{n-1} \boldsymbol{B}_{n-1} + \ldots + \lambda \boldsymbol{B}_1 + \boldsymbol{B}_0$$

and we have

$$(\boldsymbol{A} - \lambda \boldsymbol{I})\left(\lambda^{n-1} \boldsymbol{B}_{n-1} + \lambda^{n-1} \boldsymbol{B}_{n-1} + \ldots + \lambda \boldsymbol{B}_1 + \boldsymbol{B}_0\right) =$$

$$= \det(\boldsymbol{A} - \lambda \boldsymbol{I}) \cdot \boldsymbol{I} = (\lambda^n + c_{n-1} \lambda^{n-1} + \ldots + c_1 \lambda + c_0)\, \boldsymbol{I}$$

The above equality **is satisfied for any number λ** so the corresponding matrix coefficients at both sides should be **the same**.

Thus

$$
\begin{aligned}
&-\boldsymbol{B}_{n-1} = \boldsymbol{I} \\
&-\boldsymbol{B}_{k-1} + \boldsymbol{A}\boldsymbol{B}_k = c_k \boldsymbol{I} \quad , \quad k = n-1, n-2, .., 1 \\
&\boldsymbol{A}\boldsymbol{B}_0 = c_0 \boldsymbol{I}
\end{aligned}
$$

Let's multiply (from the left side) the first equation by $\boldsymbol{A}^n$, the second one by $\boldsymbol{A}^{n-1}$ and so on (then the last equation remains unchanged) and sum up all equations. The left-hand side of the obtained equation is zero since all terms will cancel out in pairs! Thus we get

$$
0 = \boldsymbol{A}^n + c_{n-1}\boldsymbol{A}^{n-1} + \ldots + c_1 \boldsymbol{A} + c_0 \boldsymbol{I} \equiv p_A(\boldsymbol{A})
$$

as stated.

For the matrices with the dimension equal 3 we have $-\boldsymbol{T}^3 + J_1 \boldsymbol{T}^2 - J_2 \boldsymbol{T} + J_3 \boldsymbol{I} = \boldsymbol{0}$.

This relation will be used in the section devoted to the constitutive relations in fluid mechanics. In particular, note that the **third** power of such matrix can be expressed as the linear combination of $\boldsymbol{I}$, $\boldsymbol{A}$ and $\boldsymbol{A}^2$. Using recursion one can show that this conclusion holds true for any natural power of the matrix $\boldsymbol{A}$.

PRODUCT OF ALTERNATING SYMBOLS

Important identity $\in_{ijk}\in_{k\beta\gamma}=\delta_{i\beta}\delta_{j\gamma}-\delta_{i\gamma}\delta_{j\beta}$

Proof

Consider $\begin{vmatrix}\delta_{11} & \delta_{12} & \delta_{13}\\ \delta_{21} & \delta_{22} & \delta_{23}\\ \delta_{31} & \delta_{32} & \delta_{33}\end{vmatrix}=\begin{vmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{vmatrix}=1.$

After row's permutation one gets $\begin{vmatrix}\delta_{i1} & \delta_{i2} & \delta_{i3}\\ \delta_{j1} & \delta_{j2} & \delta_{j3}\\ \delta_{k1} & \delta_{k2} & \delta_{k3}\end{vmatrix}=\in_{ijk}.$

Then, after column's permutation we obtain $\begin{vmatrix}\delta_{i\alpha} & \delta_{i\beta} & \delta_{i\gamma}\\ \delta_{j\alpha} & \delta_{j\beta} & \delta_{j\gamma}\\ \delta_{k\alpha} & \delta_{k\beta} & \delta_{k\gamma}\end{vmatrix}=\in_{ijk}\in_{\alpha\beta\gamma}.$

Now, put k = α and apply summation.

The result is as follows $\begin{vmatrix} \delta_{ik} & \delta_{i\beta} & \delta_{i\gamma} \\ \delta_{jk} & \delta_{j\beta} & \delta_{j\gamma} \\ \delta_{kk} & \delta_{k\beta} & \delta_{k\gamma} \end{vmatrix} = \in_{ijk} \in_{k\beta\gamma}$, or

$$\in_{ijk} \in_{k\beta\gamma} = \delta_{ik}(\delta_{j\beta}\delta_{k\gamma} - \delta_{k\beta}\delta_{j\gamma}) - \delta_{i\beta}(\delta_{jk}\delta_{k\gamma} - \underbrace{\delta_{kk}}_{3}\,\delta_{j\gamma}) + \delta_{i\gamma}(\delta_{jk}\delta_{k\beta} - \underbrace{\delta_{kk}}_{3}\,\delta_{j\beta}) =$$

$$= \delta_{j\beta}\delta_{i\gamma} - \delta_{i\beta}\delta_{j\gamma} - \delta_{i\beta}\delta_{j\gamma} + 3\delta_{i\beta}\delta_{j\gamma} + \delta_{j\beta}\delta_{i\gamma} - 3\delta_{j\beta}\delta_{i\gamma} = \delta_{i\beta}\delta_{j\gamma} - \delta_{j\beta}\delta_{i\gamma}$$

Exercise: Using index formalism derive the following vector identity

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a}, \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a}, \boldsymbol{b})\boldsymbol{c}$$

Basic differential operators (in Cartesian c.s.)

Gradient of a scalar field $f = f(t, \boldsymbol{r})$

$$\nabla f = \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3} \right] = \frac{\partial f}{\partial x_i} \boldsymbol{e}_i \qquad \text{(vector)}$$

∇ - nabla operator

Divergence of the vector field $\boldsymbol{w} = w_i(t, \boldsymbol{r}) \boldsymbol{e}_i$

$$div\, \boldsymbol{w} \equiv \underbrace{\nabla \cdot \boldsymbol{w}}_{\text{formal inner product}} = \frac{\partial w_1}{\partial x_1} + \frac{\partial w_2}{\partial x_2} + \frac{\partial w_3}{\partial x_3} = \frac{\partial w_j}{\partial x_j} \qquad \text{(scalar)}$$

Rotation (curl) of the vector field $\boldsymbol{w} = w_i(t, \boldsymbol{r}) \boldsymbol{e}_i$

$$rot\, \boldsymbol{w} \equiv \underbrace{\nabla \times \boldsymbol{w}}_{\text{formal vector product}} = \left[\frac{\partial w_3}{\partial x_2} - \frac{\partial w_2}{\partial x_3} \right] \boldsymbol{e}_1 + \left[\frac{\partial w_1}{\partial x_3} - \frac{\partial w_3}{\partial x_1} \right] \boldsymbol{e}_2 + \left[\frac{\partial w_2}{\partial x_1} - \frac{\partial w_1}{\partial x_2} \right] \boldsymbol{e}_3 = \qquad \text{(vector)}$$

$$= \in_{ijk} \frac{\partial w_k}{\partial x_j} \boldsymbol{e}_i = \in_{ijk} \frac{\partial w_j}{\partial x_i} \boldsymbol{e}_k$$

Gradient of the vector field $\boldsymbol{w} = w_i(t, \boldsymbol{r})\boldsymbol{e}_i$

$$Grad\,\boldsymbol{w} \equiv \underbrace{\nabla \boldsymbol{w}}_{\text{formal dyadic product}} = \frac{\partial w_i}{\partial x_j} e_i \otimes e_j \quad \text{(2}^{\text{nd}}\text{–rank tensor)}$$

Divergence of the tensor field $T = t_{ij}(t, \boldsymbol{r}) e_i \otimes e_j$

$$Div\,T \equiv \underbrace{\nabla \cdot T}_{\text{formal matrix–vector product}} = \frac{\partial t_{ij}}{\partial x_j} \boldsymbol{e}_i \quad \text{(vector)}$$

Scalar Laplace operator

$$\Delta f \equiv \nabla \cdot (\nabla f) \equiv \nabla^2 f = \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \frac{\partial^2 f}{\partial x_3^2} \equiv \frac{\partial^2 f}{\partial x_k \partial x_k}$$

Vector Laplace operator

$$\boldsymbol{\Delta w} \equiv \underbrace{\nabla \cdot (\nabla \boldsymbol{w})}_{\text{Divergence of the tensor } \nabla \boldsymbol{w}} \equiv \nabla(\nabla \cdot \boldsymbol{w}) - \nabla \times (\nabla \times \boldsymbol{w}) = \underbrace{\Delta w_j}_{\text{Scalar Laplacian of the component } w_j} \boldsymbol{e}_j = \frac{\partial^2 w_j}{\partial x_k \partial x_k} \boldsymbol{e}_j$$

NOTE: only in the Cartesian coordinate system the components of the vector Laplacian are equal to scalar Laplacians of the vector field components!

USEFUL DIFFERENTIAL FORMULAE

1) $\nabla(\varphi\psi) = \psi\nabla\varphi + \varphi\nabla\psi$

2) $\nabla\cdot(\varphi\boldsymbol{w}) = \nabla\varphi\cdot\boldsymbol{w} + \varphi\nabla\cdot\boldsymbol{w}$

3) $\nabla\times(\varphi\boldsymbol{w}) = \nabla\varphi\times\boldsymbol{w} + \varphi\nabla\times\boldsymbol{w}$

4) $\nabla\cdot(\boldsymbol{u}\times\boldsymbol{w}) = \boldsymbol{w}\cdot(\nabla\times\boldsymbol{u}) - \boldsymbol{u}\cdot(\nabla\times\boldsymbol{w})$

5) $\nabla\times(\boldsymbol{u}\times\boldsymbol{w}) = \nabla\boldsymbol{u}\cdot\boldsymbol{w} - \nabla\boldsymbol{w}\cdot\boldsymbol{u} + (\nabla\cdot\boldsymbol{w})\boldsymbol{u} - (\nabla\cdot\boldsymbol{u})\boldsymbol{w}$

6) $\nabla(\boldsymbol{u}\cdot\boldsymbol{w}) = \nabla\boldsymbol{u}\cdot\boldsymbol{w} + \nabla\boldsymbol{w}\cdot\boldsymbol{u} + \boldsymbol{u}\times(\nabla\times\boldsymbol{w}) + \boldsymbol{w}\times(\nabla\times\boldsymbol{u})$

7) $\nabla(\tfrac{1}{2}u^2) \equiv \tfrac{1}{2}\nabla(\boldsymbol{u}\cdot\boldsymbol{u}) = \nabla\boldsymbol{u}\cdot\boldsymbol{u} + \boldsymbol{u}\times(\nabla\times\boldsymbol{u})$

8) $\nabla\cdot\nabla\varphi = \nabla^2\varphi \equiv \Delta\varphi$

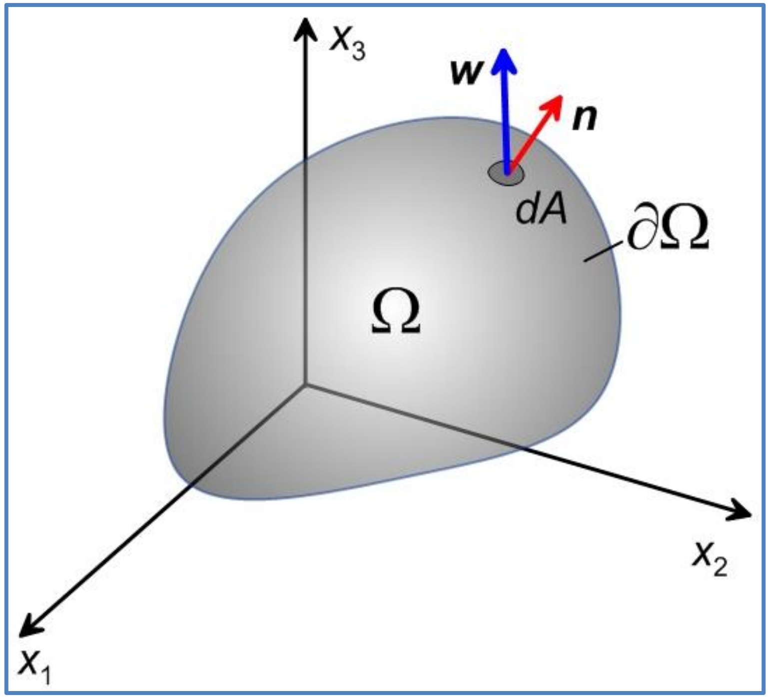
9) $\nabla\times\nabla\varphi \equiv \boldsymbol{0} \quad , \quad \nabla\cdot(\nabla\times\boldsymbol{w}) = 0$

10) $\boldsymbol{\Delta}\boldsymbol{w} = \nabla(\nabla\cdot\boldsymbol{w}) - \nabla\times(\nabla\times\boldsymbol{w})$

Exercise: Derive all the formulae using the index calculus.

INTEGRAL THEOREMS

GREEN-GAUSS-OSTROGRADSKY (GGO) THEOREM



Consider the vector field $\boldsymbol{w} = \boldsymbol{w}(\boldsymbol{x})$ defined in a 3D volume Ω bounded by sufficiently regular surface $\partial\,\Omega$. Then

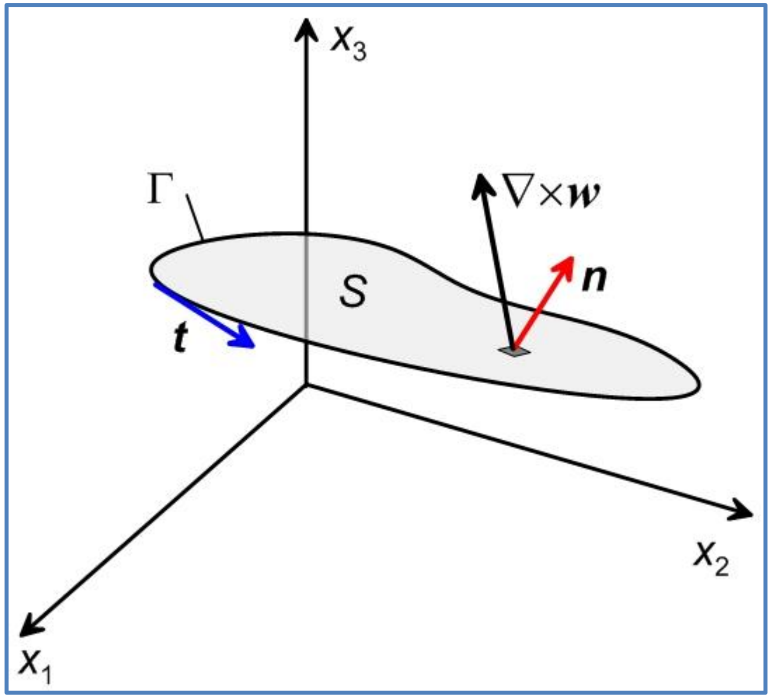
$$\int_{\partial\Omega} \underbrace{(\boldsymbol{w},\boldsymbol{n})}_{\boldsymbol{w}\cdot\boldsymbol{n}=w_n} \, dS = \int_{\Omega} \nabla\cdot\boldsymbol{w} \; d\boldsymbol{x}$$

*component of **w** normal to S* — *divergence of **w***

We have analogous (dual) theorem with vector products, namely

$$\int_{\partial\Omega} \boldsymbol{n}\times\boldsymbol{w}\, dS = \int_{\Omega} \underbrace{\nabla\times\boldsymbol{w}}_{\text{rotation of } \boldsymbol{w}} d\boldsymbol{x}$$

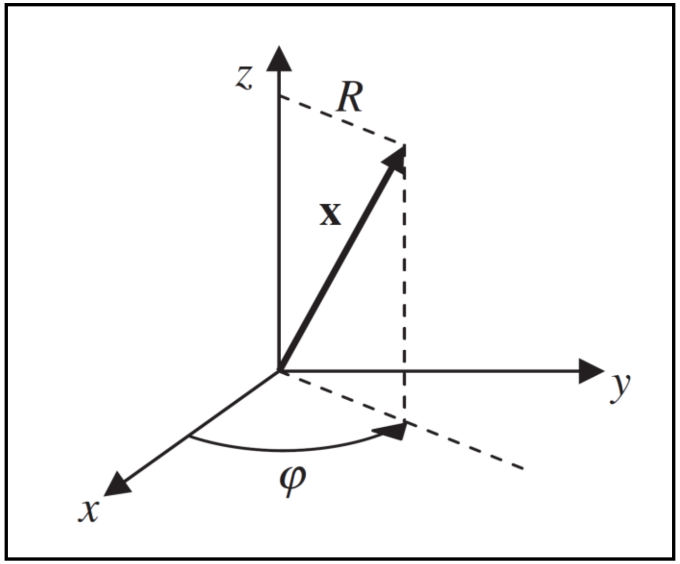
STOKES THEOREM



Consider the vector field $\boldsymbol{w}=\boldsymbol{w}(\boldsymbol{x})$, the closed line (loop) γ and sufficiently regular (yet arbitrary) surface S spanned (like a soap bubble) by this line. Then

$$\oint_{\gamma} \underbrace{(\boldsymbol{w},\boldsymbol{\tau})}_{\substack{\boldsymbol{w}\cdot\boldsymbol{\tau}\equiv w_{\tau} \\ \text{component of} \\ \boldsymbol{w}\ \text{tangent to}\ \gamma}} dl = \int_{S} \underbrace{(\nabla\times\boldsymbol{w},\boldsymbol{n})}_{\substack{\text{component of} \\ \text{rot}\ \boldsymbol{w}\ \text{normal to}\ S}} dS$$

POLAR AND CYLINDRICAL SYSTEMS OF COORDINATES



$$x = R\cos\varphi \;,\quad y = R\sin\varphi \;,\quad z \equiv z$$

$$R = \sqrt{x^2 + y^2} \;,\quad \varphi = \arctan(y/x)$$

Basic vectors: $\begin{cases} e_R = e_x \cos\varphi + e_y \sin\varphi \\ e_\varphi = -e_x \sin\varphi + e_y \cos\varphi \\ e_z = e_z \end{cases}$

Gradient of f : $\nabla f = \frac{\partial}{\partial R} f\, \boldsymbol{e}_R + \frac{1}{R}\frac{\partial}{\partial \varphi} f\, \boldsymbol{e}_\varphi + \frac{\partial}{\partial z} f\, \boldsymbol{e}_z$

Scalar Laplacian of f : $\Delta f = \frac{1}{R}\frac{\partial}{\partial R}(R\frac{\partial}{\partial R} f) + \frac{1}{R^2}\frac{\partial^2}{\partial \varphi^2} f + \frac{\partial^2}{\partial z^2} f$

Divergence of $\boldsymbol{u} = u_R \boldsymbol{e}_R + u_\varphi \boldsymbol{e}_\varphi + u_z \boldsymbol{e}_z$: $\nabla \cdot \boldsymbol{u} = \frac{1}{R}\frac{\partial}{\partial R}(R u_R) + \frac{1}{R}\frac{\partial}{\partial \varphi} u_\varphi + \frac{\partial}{\partial z} u_z$

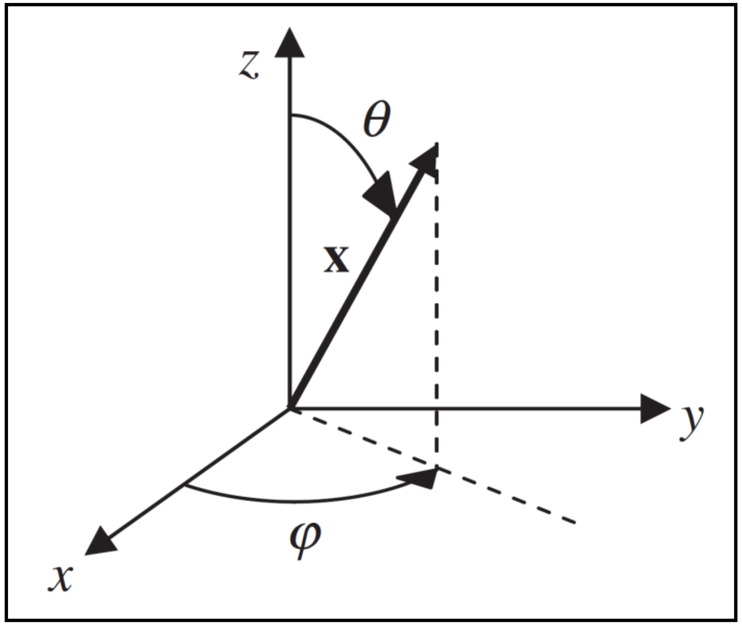
Rotation of $\boldsymbol{u} = u_R \boldsymbol{e}_R + u_\varphi \boldsymbol{e}_\varphi + u_z \boldsymbol{e}_z$:

$$\nabla \times \boldsymbol{u} = (\frac{1}{R}\frac{\partial}{\partial \varphi} u_z - \frac{\partial}{\partial z} u_\varphi)\boldsymbol{e}_R + (\frac{\partial}{\partial z} u_R - \frac{\partial}{\partial R} u_z)\boldsymbol{e}_\varphi + \frac{1}{R}[\frac{\partial}{\partial R}(R u_\varphi) - \frac{\partial}{\partial \varphi} u_R]\boldsymbol{e}_z$$

Vector Laplacian of $\boldsymbol{u} = u_R \boldsymbol{e}_R + u_\varphi \boldsymbol{e}_\varphi + u_z \boldsymbol{e}_z$:

$$\boldsymbol{\Delta u} = (\Delta u_R - \frac{1}{R^2} u_R - \frac{2}{R^2}\frac{\partial}{\partial \varphi} u_\varphi)\boldsymbol{e}_R + (\Delta u_\varphi + \frac{2}{R^2}\frac{\partial}{\partial \varphi} u_R - \frac{1}{R^2} u_\varphi)\boldsymbol{e}_\varphi + \Delta u_z \boldsymbol{e}_z$$

SPHERICAL COORDINATE SYSTEM



$$x = r\cos\varphi\sin\theta \ , \ y = r\sin\varphi\sin\theta \ , \ z = r\cos\theta$$

$$r = \sqrt{x^2 + y^2 + z^2}, \varphi = arctg\left(\tfrac{y}{x}\right), \theta = arctg\left(\tfrac{\sqrt{x^2+y^2}}{z}\right)$$

Basic vectors:

$$\begin{cases} \boldsymbol{e}_r = \boldsymbol{e}_x \sin\theta\cos\varphi + \boldsymbol{e}_y \sin\theta\sin\varphi + \boldsymbol{e}_z \cos\theta \\ \boldsymbol{e}_\varphi = -\boldsymbol{e}_x \sin\varphi + \boldsymbol{e}_y \cos\varphi \\ \boldsymbol{e}_\theta = \boldsymbol{e}_x \cos\theta\cos\varphi + \boldsymbol{e}_y \cos\theta\sin\varphi - \boldsymbol{e}_z \sin\theta \end{cases}$$

Gradient of f :

$$\nabla f = \tfrac{\partial}{\partial r} f\, \boldsymbol{e}_r + \tfrac{1}{R}\tfrac{\partial}{\partial \theta} f\, \boldsymbol{e}_\theta + \tfrac{1}{r\sin\theta}\tfrac{\partial}{\partial \varphi} f\, \boldsymbol{e}_\varphi$$

Scalar Laplasjan of f :

$$\Delta f = \tfrac{1}{r^2}[\tfrac{\partial}{\partial r}(r^2 \tfrac{\partial}{\partial r} f) + \tfrac{1}{\sin\theta}\tfrac{\partial}{\partial\theta}(\sin\theta \tfrac{\partial}{\partial\theta} f) + \tfrac{1}{\sin^2\theta}\tfrac{\partial^2}{\partial\varphi^2} f]$$

Divergence of $\boldsymbol{u} = u_r \boldsymbol{e}_r + u_\theta \boldsymbol{e}_\theta + u_\varphi \boldsymbol{e}_\varphi$:

$$\nabla\cdot\boldsymbol{u} = \tfrac{1}{r^2}\tfrac{\partial}{\partial r}(r^2 u_r) + \tfrac{1}{r\sin\theta}[\tfrac{\partial}{\partial\theta}(u_\theta \sin\theta) + \tfrac{\partial}{\partial\varphi} u_\varphi]$$

Rotation of $\boldsymbol{u} = u_r \boldsymbol{e}_r + u_\theta \boldsymbol{e}_\theta + u_\varphi \boldsymbol{e}_\varphi$:

$$\nabla \times \boldsymbol{u} = \frac{1}{r\sin\theta}[\frac{\partial}{\partial\theta}(u_\varphi \sin\theta) - \frac{\partial}{\partial\varphi}u_\theta)\boldsymbol{e}_r + \frac{1}{r}[\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}u_r - \frac{\partial}{\partial r}(ru_\varphi)]\boldsymbol{e}_\theta + \frac{1}{r}[\frac{\partial}{\partial r}(ru_\theta) - \frac{\partial}{\partial\theta}u_r]\boldsymbol{e}_\varphi$$

Vector Laplacian of $\boldsymbol{u} = u_r \boldsymbol{e}_r + u_\theta \boldsymbol{e}_\theta + u_\varphi \boldsymbol{e}_\varphi$:

$$\boldsymbol{\Delta u} = [\Delta u_r - \frac{2}{r^2}u_r - \frac{2}{r^2\sin\theta}\frac{\partial}{\partial\theta}(u_\theta \sin\theta) - \frac{2}{r^2\sin\theta}\frac{\partial}{\partial\varphi}u_\varphi]\boldsymbol{e}_r +$$
$$+ (\Delta u_\theta + \frac{2}{r^2}\frac{\partial}{\partial\theta}u_r - \frac{1}{r^2\sin^2\theta}u_\theta - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial}{\partial\varphi}u_\varphi)\boldsymbol{e}_\theta +$$
$$+ (\Delta u_\varphi + \frac{2}{r^2\sin^2\theta}\frac{\partial}{\partial\varphi}u_\varphi + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial}{\partial\varphi}u_\theta - \frac{1}{r^2\sin^2\theta}u_\varphi)\boldsymbol{e}_\theta$$