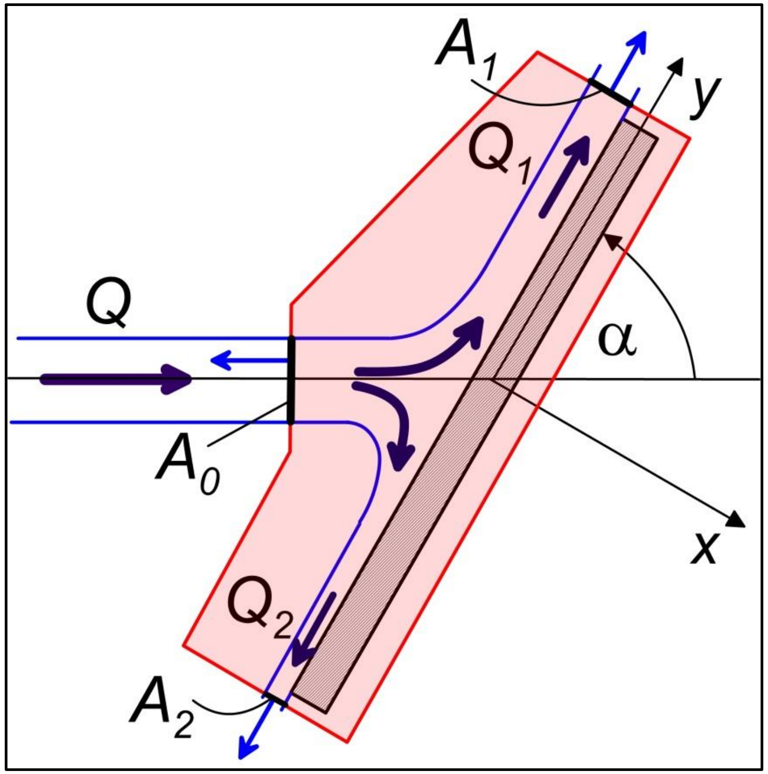
DYNAMICAL REACTION FORCES – FURTHER EXAMPLES

1. REACTION OF THE WALL-IMPINGING STREAM



The boundary: $\partial\Omega = A_0 \cup A_1 \cup A_2 \cup other$

In the whole domain $p \equiv p_a$

Formula for the reaction force:

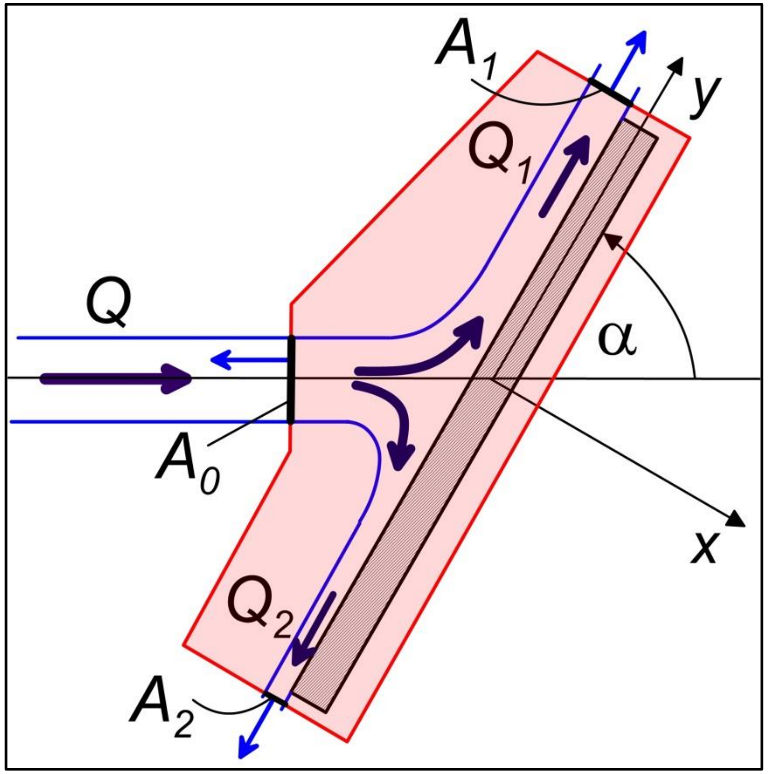
$$\boldsymbol{R} = -\int_{A_0} \rho \upsilon_n \boldsymbol{\upsilon}\, dS - \int_{A_1} \rho \upsilon_n \boldsymbol{\upsilon}\, dS - \int_{A_2} \rho \upsilon_n \boldsymbol{\upsilon}\, dS$$

Next … $\upsilon_0 = Q_0 / A_0$ and $\upsilon_0 = \upsilon_1 = \upsilon_2.$

At the inlet A_0**:**

$$\boldsymbol{\upsilon} = \upsilon_0 \sin\alpha\, \boldsymbol{e}_x + \upsilon_0 \cos\alpha\, \boldsymbol{e}_y \quad , \quad \boldsymbol{n} = -\sin\alpha\, \boldsymbol{e}_x - \cos\alpha\, \boldsymbol{e}_y \,, \; \upsilon_n \equiv \boldsymbol{\upsilon}\cdot\boldsymbol{n} = -\upsilon_0$$

$$\int_{A_0} \rho \upsilon_n \boldsymbol{\upsilon}\, dS = -\rho A_0 \upsilon_0^2 \sin\alpha\, \boldsymbol{e}_x - \rho A_0 \upsilon_0^2 \cos\alpha\, \boldsymbol{e}_y$$



At the outlet A_1:

$$\boldsymbol{\upsilon} = \upsilon_1 \boldsymbol{e}_y \equiv \upsilon_0 \boldsymbol{e}_y \quad , \quad \boldsymbol{n} = \boldsymbol{e}_y \; , \; \upsilon_n = \upsilon_1 \equiv \upsilon_0 \quad ,$$

$$\int_{A_1} \rho \upsilon_n \boldsymbol{\upsilon} \, dS = \rho A_1 \upsilon_0^2 \, \boldsymbol{e}_y$$

At the outlet A_2:

$$\boldsymbol{\upsilon} = -\upsilon_2 \boldsymbol{e}_y \equiv -\upsilon_0 \boldsymbol{e}_y \quad , \quad \boldsymbol{n} = -\boldsymbol{e}_y \; , \; \upsilon_n = \upsilon_2 \equiv \upsilon_0 \quad ,$$

$$\int_{A_2} \rho \upsilon_n \boldsymbol{\upsilon} \, dS = -\rho A_2 \upsilon_0^2 \, \boldsymbol{e}_y$$

After insertion we obtain

$$\boldsymbol{R} = \rho A_0 \upsilon_0^2 \sin\alpha \, \boldsymbol{e}_x + \rho \upsilon_0^2 (A_0 \cos\alpha - A_1 + A_2) \boldsymbol{e}_y =$$

$$= \rho Q_0 \upsilon_0 \sin\alpha \, \boldsymbol{e}_x + \rho \upsilon_0 (Q_0 \cos\alpha - Q_1 + Q_2) \boldsymbol{e}_y$$

But what is the flux distribution ?!

Assume that $R_y \equiv 0$ **(only normal reaction exists).**

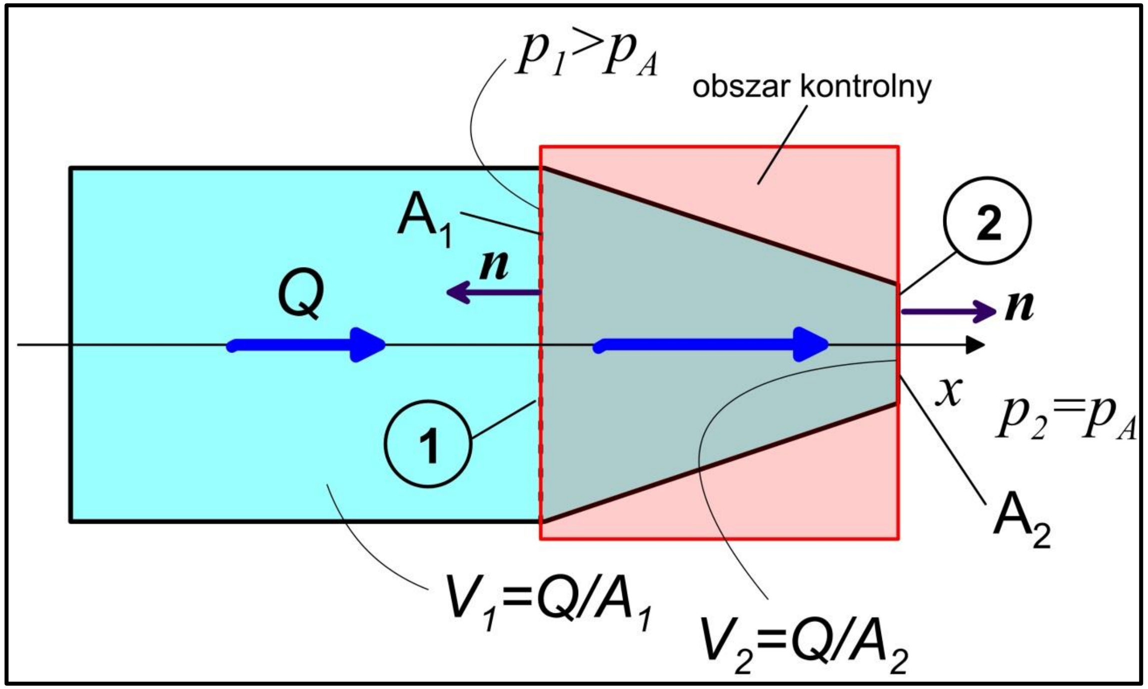
Thus:

$$\begin{cases} Q_1 + Q_2 = Q_0 \\ Q_1 - Q_2 = Q_0 \cos\alpha \end{cases} \Rightarrow \begin{cases} Q_1 = \frac{1+\cos\alpha}{2} Q_0 \\ Q_2 = \frac{1-\cos\alpha}{2} Q_0 \end{cases}$$

Finally, the reaction force is equal

$$\boldsymbol{R} = \rho Q_0 \upsilon_0 \sin\alpha\, \boldsymbol{e}_x$$

2. REACTION FORCE ON THE CONVERGENT TIP OF THE FIRE (OR GARDEN) HOSE



$$Q = \upsilon_1 A_1 = \upsilon_2 A_2 \quad \Rightarrow \quad \upsilon_1 = Q / A_1 \ , \ \upsilon_2 = Q / A_2$$

The Bernoulli Eq. 1-2:

$$p_1 + \tfrac{1}{2}\rho\upsilon_1^2 = p_a + \tfrac{1}{2}\rho\upsilon_2^2 \quad \Rightarrow \quad p_1 - p_a = \tfrac{1}{2}\rho\upsilon_2^2 - \tfrac{1}{2}\rho\upsilon_1^2$$

The reaction is calculated as follows …

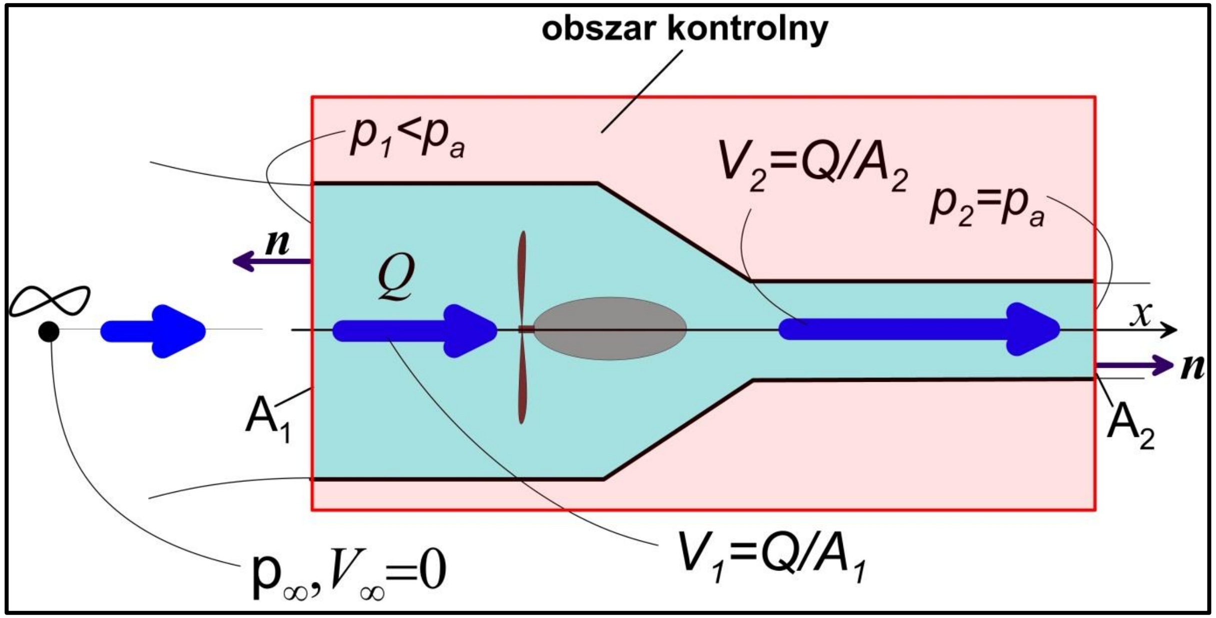
$$F_x = -\int_{A_1} \rho \upsilon_n \upsilon_x dS - \int_{A_1} (p - p_a) \underset{-1}{n_x} dS - \int_{A_2} \rho \upsilon_n \upsilon_x dS - \int_{A_2} \underbrace{(p - p_a)}_{0} n_x dS =$$

$$= -\rho A_1(-\upsilon_1)\upsilon_1 - A_1(p_1 - p_a)(-1) - \rho A_2 \upsilon_2 \upsilon_2 = \rho A_1 \upsilon_1^2 - \rho A_2 \upsilon_2^2 + A_1(\tfrac{1}{2}\rho \upsilon_2^2 - \tfrac{1}{2}\rho \upsilon_1^2) =$$

$$= \tfrac{1}{2}\rho A_1 \upsilon_1^2 - \rho A_2 \upsilon_2^2 + \tfrac{1}{2}\rho A_1 \upsilon_2^2 = \tfrac{1}{2}\rho \frac{Q^2}{A_1} - \rho \frac{Q^2}{A_2} + \tfrac{1}{2}\rho A_1 \frac{Q^2}{A_2^2} = \tfrac{1}{2}\frac{\rho Q^2}{A_1 A_2^2}(A_2^2 - 2A_1 A_2 + A_1^2) =$$

$$= \tfrac{1}{2}\rho Q^2 \frac{(A_1 - A_2)^2}{A_1 A_2^2}$$

3. THRUST FORCE OF THE JET PROPULSION

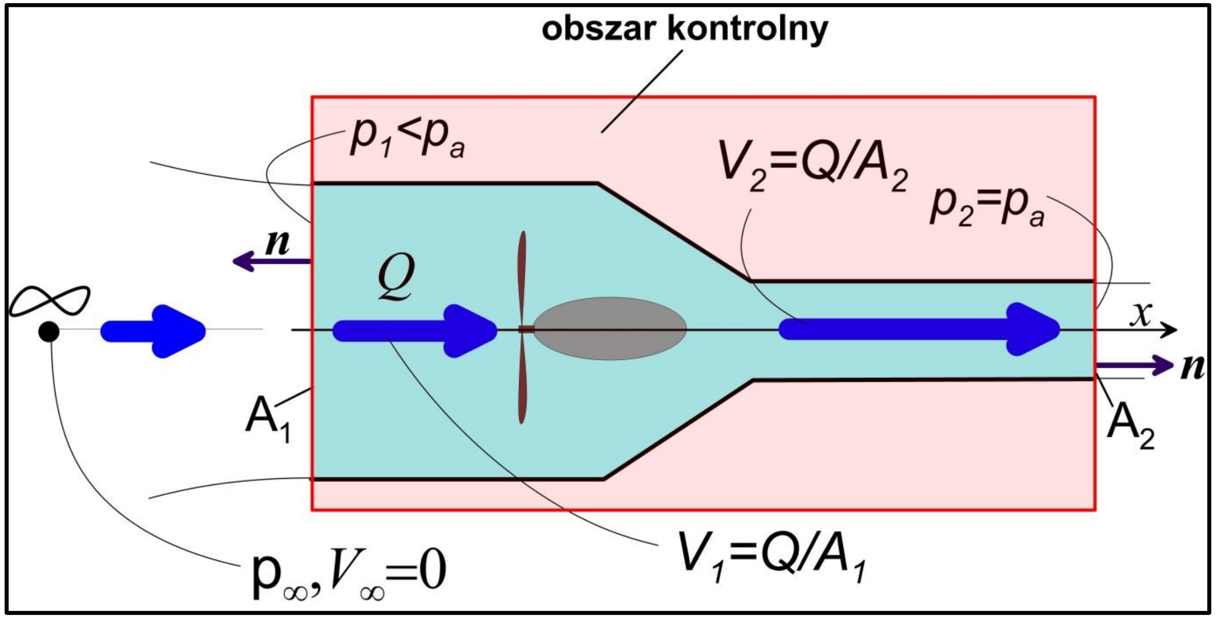


$$F_x = -\int_{A_1} \rho \upsilon_n \upsilon_x dS - \int_{A_1} (p - p_a)\underbrace{n_x}_{-1} dS - \int_{A_2} \rho \upsilon_n \upsilon_x dS - \int_{A_2} (\underbrace{p - p_a}_{0}) n_x dS =$$

$$= -\rho A_1 \upsilon_1(-\upsilon_1) - A_1(p_1 - p_a)(-1) - \rho A_2 \upsilon_2 \upsilon_2 = \rho A_1 \upsilon_1^2 - \rho A_2 \upsilon_2^2 + A_1(p_1 - p_a)$$

The Bernoulli Eq. from ∞ the inlet section …

$$p_a = p_1 + \tfrac{1}{2}\rho \upsilon_1^2 \quad \Rightarrow \quad p_1 - p_a = -\tfrac{1}{2}\rho \upsilon_1^2$$

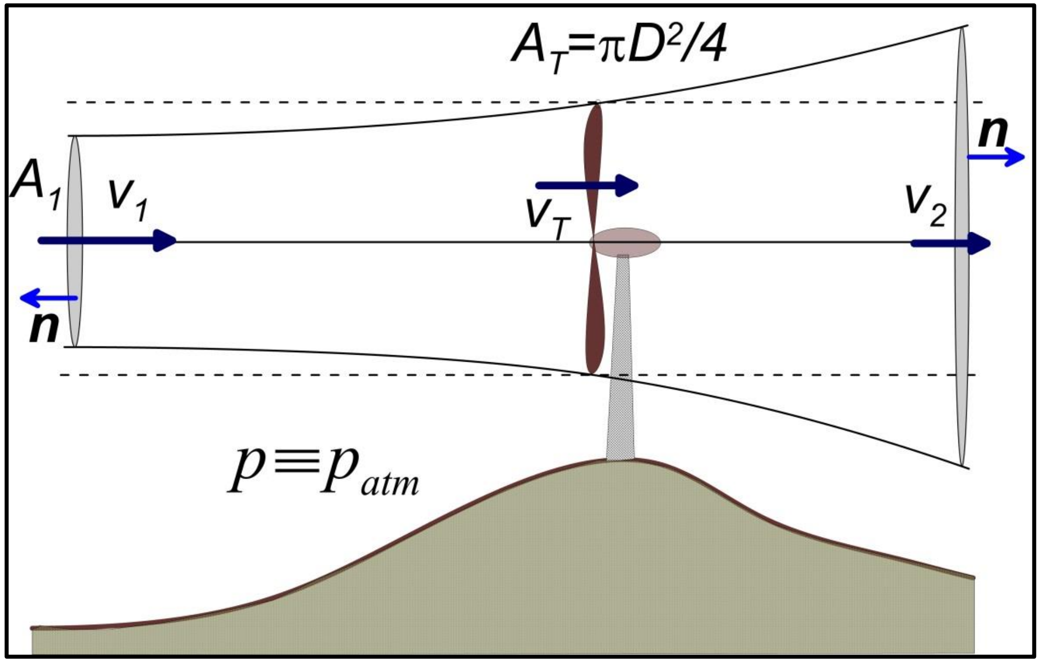


$$F_x = \rho A_1 \upsilon_1^2 - \rho A_2 \upsilon_2^2 - \tfrac{1}{2}\rho A_1 u_1^2 = \tfrac{1}{2}\rho A_1 \upsilon_1^2 - \rho A_2 \upsilon_2^2 = \tfrac{1}{2}\rho \frac{Q^2}{A_1} - \rho \frac{Q^2}{A_2} = \tfrac{1}{2}\rho \frac{Q^2}{A_1 A_2}(A_2 - 2A_1)$$

If $A_1 > \tfrac{1}{2} A_2$ **then** $F_x < 0$**, hence the thrust force is obtained.**

Exercise: Solve the above problem in the case when the jet engine moves with the velocity w **(in the left direction!)**

4. Simple model of a wind turbine. The Betz's limit.



We assume that in the whole domain

$$p \equiv p_a$$

Force exerted on the air stream …

$$F_x = \int_{A_1} \rho \upsilon_n \upsilon_x dS + \int_{A_2} \rho \upsilon_n \upsilon_x dS =$$

$$= -\rho A_1 \upsilon_1^2 + \rho A_2 \upsilon_2^2 < 0$$

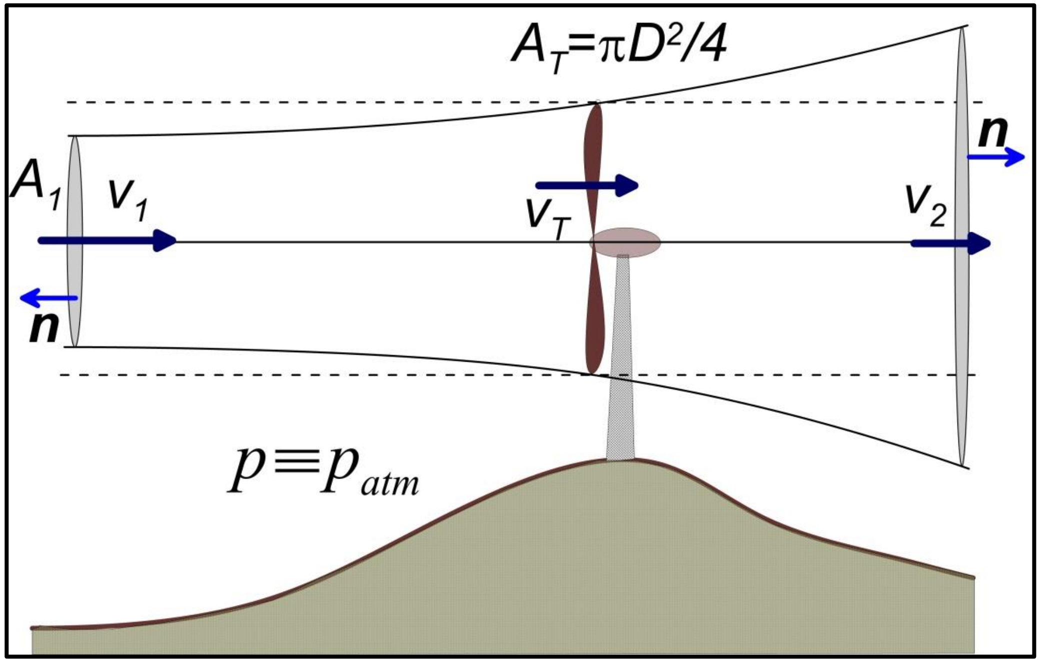
Kinetic energy (de facto – power) of the air stream

$$\Delta E_k = F_x \upsilon_T \quad \Rightarrow \quad \tfrac{1}{2} Q_m \upsilon_2^2 - \tfrac{1}{2} Q_m \upsilon_1^2 = \upsilon_T (-\rho A_1 \upsilon_1^2 + \rho A_2 \upsilon_2^2)$$

From the continuity condition: $Q_m = \rho Q = \rho A_1 \upsilon_1 = \rho A_2 \upsilon_2 = \rho A_T \upsilon_T$

Thus, we get: $\tfrac{1}{2} Q_m \upsilon_2^2 - \tfrac{1}{2} Q_m \upsilon_1^2 = \upsilon_T Q_m (\upsilon_2 - \upsilon_1)$

Conclusion: $\upsilon_T = \tfrac{1}{2}(\upsilon_1 + \upsilon_2)$



Assume:

$$\upsilon_1 \equiv V_\infty \quad \textbf{i} \quad \upsilon_T = \alpha V_\infty \quad , \quad \alpha \in (\tfrac{1}{2}, 1)$$

Then:

$$\upsilon_2 = 2\upsilon_T - \upsilon_1 = (2\alpha - 1)V_\infty$$

$$\upsilon_1 - \upsilon_2 = 2(1-\alpha)V_\infty$$

Formula for the force

$$|F_x| = \rho Q \upsilon_1 - \rho Q \upsilon_2 = \rho \upsilon_T A_T (\upsilon_1 - \upsilon_2) =$$

$$= 2\alpha(1-\alpha)\rho V_\infty^2 A_T$$

Power produced by the turbine $$P_T = |F_x| \upsilon_T = 2\alpha^2 (1-\alpha) \rho V_\infty^3 A_T$$

Calculation of the maximal power

$$q(\alpha) = 2\alpha^2(1-\alpha) \quad , \quad q'(\alpha) = 4\alpha - 6\alpha^2$$

$$q'(\alpha) = 0 \quad \Leftrightarrow \quad \alpha = 0 \quad \text{lub} \quad \alpha = \tfrac{2}{3} \quad \text{(max!)}$$

Hence $q_{\max} = q(\tfrac{2}{3}) = \tfrac{8}{27}$ **and** $$P_T^{\max} = \tfrac{8}{27} \rho V_\infty^3 A_T$$

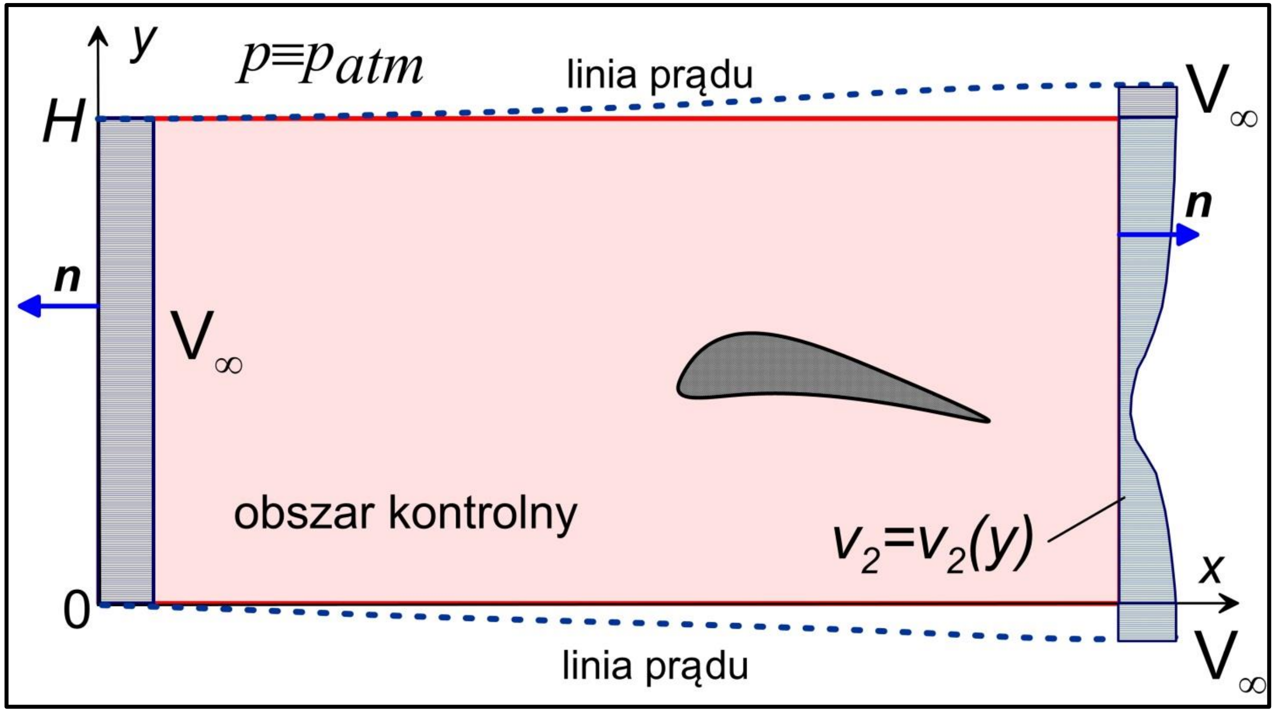
Efficiency – ratio between the produces power and the stream of kinetic energy carried by an undisturbed air stream with the cross-section area equal A_T

We have

$$\eta = \frac{P_T}{\frac{1}{2}\rho A_T V_\infty^3} = 2q(\alpha) = 4\alpha^2(1-\alpha)$$

Note that $\eta \le \eta_{\max} = \frac{16}{27} \approx\sim 59\%$ – ***Betz's limit***

5. DETERMINATION OF THE AERODYNAMIC DRAG BY MEASUREMENT OF THE LINEAR MOMENTUM DEFICIT IN THE WAKE BEHIND THE BODY

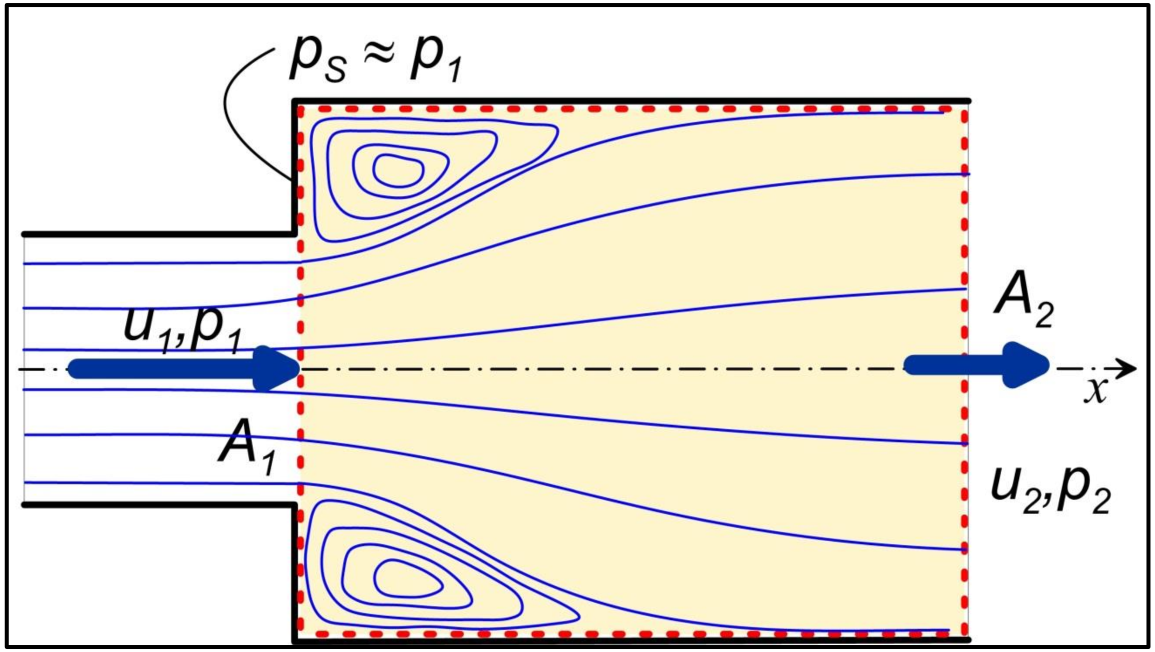


$$D = -\int_{in} \rho \upsilon_n \upsilon_x dS - \int_{out} \rho \upsilon_n \upsilon_x dS - \int_{side} \rho \upsilon_n \upsilon_x dS =$$

$$= -\rho H(-V_\infty)V_\infty - \rho \int_0^H \upsilon_2^2(y)dy - \Delta Q_m V_\infty =$$

$$= \rho \int_0^H [V_\infty^2 - \upsilon_2^2(y)]dy - \rho V_\infty \int_0^H [V_\infty - \upsilon_2(y)]dy = \rho \int_0^H \upsilon_2(y)[V_\infty - \upsilon_2(y)]dy$$

6. USING LINEAR MOMENTUM BALANCE TO ESTIMATE THE LOCAL PRESSURE LOSS.



Assumption: pressure at the vertical part of the wall is equal p_1

Increment of the linear momentum in the control volume is

$$\Delta P_x = \rho A_2 u_2^2 - \rho A_1 u_1^2 = \rho A_2 \frac{A_1^2}{A_2^2} u_1^2 - \rho A_1 u_1^2 = \rho u_1^2 A_2 \left(\frac{A_1^2}{A_2^2} - \frac{A_1}{A_2} \right)$$

The force acting on the fluid in the control volume is

$$F_x = p_1 A_1 + \underbrace{p_s}_{\approx p_1} (A_2 - A_1) - p_2 A_2 = (p_1 - p_2) A_2$$

Accordingly to the 2nd Principle of Dynamics

$$\Delta P_x = F_x$$

Hence

$$\Delta p_{ZZP} \equiv p_2 - p_1 = \tfrac{1}{2}\rho u_1^2 \left(\frac{2A_1}{A_2} - \frac{2A_1^2}{A_2^2} \right)$$

From "naively" applied Bernoulli Equation we obtain

$$\tfrac{1}{2}\rho u_1^2 + p_1 = \tfrac{1}{2}\rho u_2^2 + p_2$$

$$\Downarrow$$

$$\Delta p_{RB} \equiv p_2 - p_1 = \tfrac{1}{2}\rho u_1^2 - \tfrac{1}{2}\rho u_2^2 = \tfrac{1}{2}\rho u_1^2 \left(1 - \frac{u_2^2}{u_1^2} \right) = \tfrac{1}{2}\rho u_1^2 \left(1 - \frac{A_1^2}{A_2^2} \right)$$

We can see that

$$\Delta p_{ZZP} \neq \Delta p_{RB}$$

We introduce the correction term to the BE accounting for the pressure lost due to sudden expansion of the duct

$$\tfrac{1}{2}\rho u_1^2 + p_1 = \tfrac{1}{2}\rho u_1^2 + p_2 + \Delta p_{str} \;\Rightarrow\; p_2 - p_1 = \tfrac{1}{2}\rho u_1^2 - \tfrac{1}{2}\rho u_1^2 - \Delta p_{str}$$

Hence, we obtain the formula

$$\Delta p_{ZZP} = \Delta p_{RB} - \Delta p_{str} \;\Rightarrow\; \Delta p_{str} = \Delta p_{RB} - \Delta p_{ZZP}$$

The local loss of pressure can be expressed as

$$\Delta p_{str} = \tfrac{1}{2}\rho u_1^2 \left(1 - \frac{A_1^2}{A_2^2} - \frac{2A_1}{A_2} + \frac{2A_1^2}{A_2^2}\right) = \underbrace{\left(1 - \frac{A_1}{A_2}\right)^2}_{\zeta_1} \tfrac{1}{2}\rho u_1^2 = \zeta_1 \tfrac{1}{2}\rho u_1^2$$

WE have introduces the local pressure loss coefficient $\zeta_1 = \left(1 - A_1 / A_2\right)^2$. Here, the **reference velocity** is the average velocity in the duct in the front of the expansion.

If for some reason we prefer the reference velocity to be the **velocity behind the expansion** then we can easily transform our formula as follows

$$\Delta p_{str} = \left(1 - \frac{A_1}{A_2}\right)^2 \tfrac{1}{2}\rho u_1^2 = \left(1 - \frac{A_1}{A_2}\right)^2 \tfrac{1}{2}\rho \frac{A_2^2}{A_1^2} u_2^2 = \underbrace{\left(\frac{A_2}{A_1} - 1\right)^2}_{\zeta_2} \tfrac{1}{2}\rho u_2^2 = \zeta_2 \tfrac{1}{2}\rho u_2^2$$

Hence, this time the local pressure loss coefficient is $\zeta_2 = (A_2/A_1 - 1)^2$.

Note: In the limit case $\frac{A_2}{A_1} \to \infty$ (which corresponds to the outflow from the duct to large container) one gets

$$\lim_{\frac{A_2}{A_1} \to \infty} \zeta_1 = 1$$

Clearly the coefficient ζ_2 becomes unbounded.