



Institute of Aeronautics and Applied Mechanics

Finite element method 2 (FEM 2)

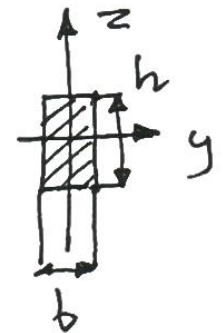
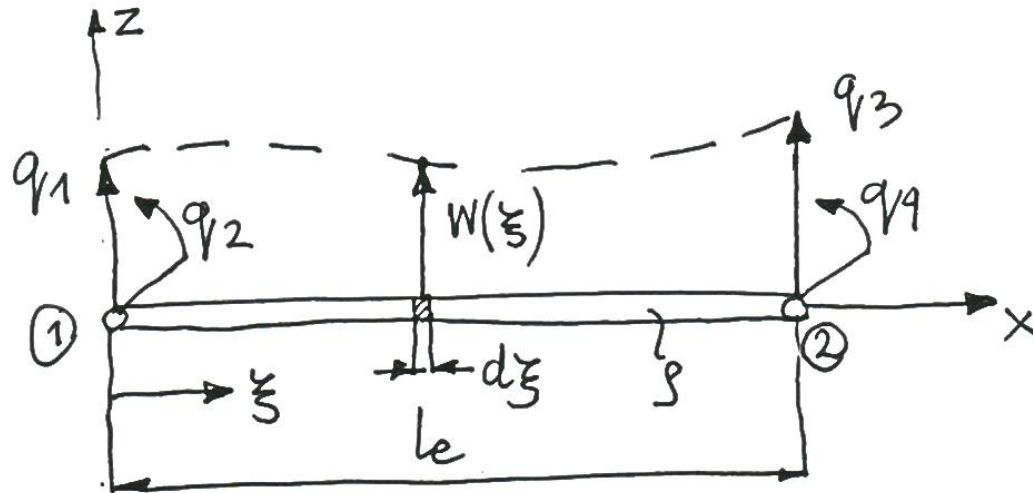
Mass matrix of a beam

MASS MATRIX OF A BEAM ELEMENT

$$n=2$$

$$n_p=2$$

$$n_e = n \cdot n_p = 4$$



$$\underset{1 \times 4}{[q]} = [q_1, q_2, q_3, q_4]$$

Kinetic energy of a small part:

$$d\bar{T}_e = \frac{1}{2} dm \dot{w}^2 = \frac{1}{2} \rho A d\xi \dot{w}^2, \quad \dot{w} = \frac{dw}{dt}$$

$$A = b \cdot h$$

$$J_y = \frac{bh^3}{12}$$

(rectangular cross-section)

Kinetic energy of a finite element:

$$T_e = \int dT_e = \frac{1}{2} \int_0^{l_e} \dot{w} \cdot \rho A \cdot \dot{w} d\xi =$$

$$\begin{matrix} \uparrow & \uparrow \\ \dot{w} & \dot{w} \\ L \dot{q}_e & [N_1, N_2, N_3, N_4] \cdot \left\{ \dot{q}_e \right\}_e \\ 1 \times 4 & \begin{Bmatrix} N_1(\xi) \\ N_2(\xi) \\ N_3(\xi) \\ N_4(\xi) \end{Bmatrix} \\ & 4 \times 1 \end{matrix}$$

$$= \frac{1}{2} L \dot{q}_e \cdot \int_0^{l_e} \rho A \begin{bmatrix} N_1 \cdot N_1 & N_1 \cdot N_2 & N_1 \cdot N_3 & N_1 \cdot N_4 \\ N_2 \cdot N_1 & N_2 \cdot N_2 & N_2 \cdot N_3 & N_2 \cdot N_4 \\ N_3 \cdot N_1 & N_3 \cdot N_2 & N_3 \cdot N_3 & N_3 \cdot N_4 \\ N_4 \cdot N_1 & N_4 \cdot N_2 & N_4 \cdot N_3 & N_4 \cdot N_4 \end{bmatrix} d\xi \cdot \left\{ \dot{q}_e \right\}_e \Rightarrow$$

where:

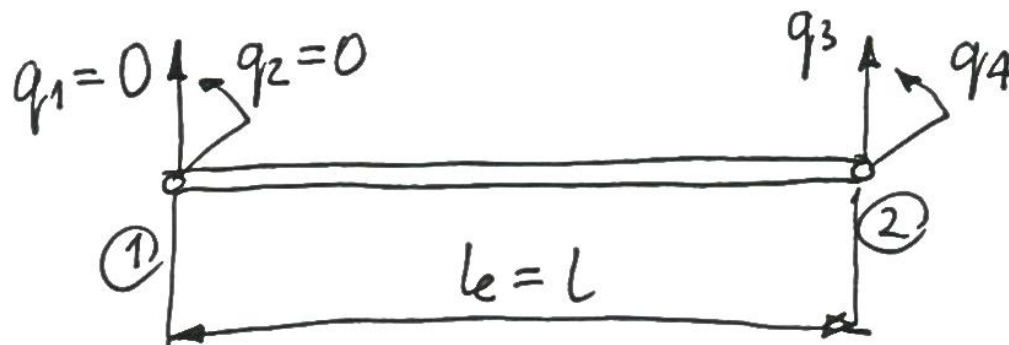
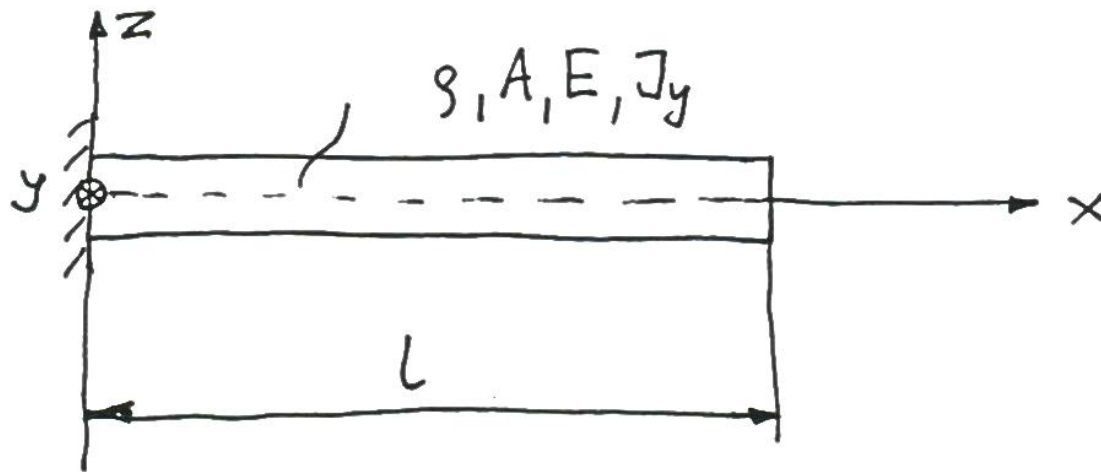
$$\left. \begin{aligned} N_1(\xi) &= 1 - \frac{3}{l^2} \xi^2 + \frac{2}{l^3} \xi^3 \\ N_2(\xi) &= \xi - \frac{2}{l} \xi^2 + \frac{1}{l^2} \xi^3 \\ N_3(\xi) &= \frac{3}{l^2} \xi^2 - \frac{2}{l^3} \xi^3 \\ N_4(\xi) &= -\frac{1}{l} \xi^2 + \frac{1}{l^2} \xi^3 \end{aligned} \right\} \begin{array}{l} \text{shape functions} \\ \text{of a beam F.E.} \end{array}$$

$$\Rightarrow T_e = \underbrace{\frac{1}{2} L \dot{q}}_{1 \times 4} \cdot \underbrace{[m]}_{4 \times 4} \cdot \underbrace{\{ \dot{q} \}}_{4 \times 1}_e$$

↑
consistent mass matrix of a beam FE.

$$[m]_e = \frac{\rho A l}{420} \begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix}$$

EXAMPLE : FIND NATURAL FREQUENCIES AND VIBRATION MODES
IN PLANE XZ FOR A CANTILEVER BEAM



$$\left(\underset{4 \times 4}{[K]} - \omega^2 \underset{4 \times 4}{[M]} \right) \underset{4 \times 1}{\{q\}} = \underset{4 \times 1}{\{0\}}$$

$$\left(\frac{2EI_y}{L^3} \begin{bmatrix} 6 & 3L & -6 & 3L \\ 3L & 2L^2 & -3L & L^2 \\ -6 & -3L & 6 & -3L \\ 3L & L^2 & -3L & 2L^2 \end{bmatrix} - \frac{\omega^2 \rho AL}{420} \begin{bmatrix} 156 & 22L & 54 & -13L \\ 22L & 4L^2 & 13L & -3L^2 \\ 54 & 13L & 156 & -22L \\ -13L & -3L^2 & -22L & 4L^2 \end{bmatrix} \right) \cdot \underset{4 \times 1}{\{q\}} = \underset{4 \times 1}{\{0\}}$$

a new constant: $\lambda = \frac{\rho AL^4}{840EI_y} \omega^2$, $q_1=0$, $q_2=0$

$$\left(\begin{bmatrix} 6 & -3L \\ -3L & 2L^2 \end{bmatrix} - \lambda \begin{bmatrix} 156 & -22L \\ -22L & 4L^2 \end{bmatrix} \right) \cdot \begin{Bmatrix} q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\det \left(\begin{bmatrix} 6-156\lambda & (22\lambda-3)\cdot l \\ (22\lambda-3)\cdot l & (2-4\lambda)\cdot l^2 \end{bmatrix} \right) = 0$$

$$(6-156\lambda)(2-4\lambda)l^2 - (22\lambda-3)^2 l^2 = 0$$

$$(12 - 24\lambda - 312\lambda + 624\lambda^2) l^2 - 484\lambda^2 l^2 + 132\lambda l^2 - 9l^2 = 0$$

$$140\lambda^2 - 204\lambda + 3 = 0 \quad \Rightarrow$$

$$\lambda_1 = 1.4857 \cdot 10^{-2}, \quad \lambda_2 = 1.4423$$

VIBRATION MODE	<u>FREQUENCY</u>		RELATIVE ERROR $\frac{ f_i - \bar{f}_i }{f_i}$
	FE MODEL (1FE) $f_i / \left(\sqrt{\frac{EJ_y}{\rho A L^4}} \right)$	ANALYTICAL $\bar{f}_i / \left(\sqrt{\frac{EJ_y}{\rho A L^4}} \right)$	
1st	0.5626	0.5598	0.5 %
2nd	5.543	3.5087	58 %

$$\begin{bmatrix} 6 - 156\lambda_i & (22\lambda_i - 3)L \\ (22\lambda_i - 3)L & (2 - 4\lambda_i)L^2 \end{bmatrix} \cdot \begin{Bmatrix} q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

two linearly dependent equations

→ 1st:

$$(6 - 156\lambda_i)q_3 + (22\lambda_i - 3)L \cdot q_4 = 0$$

$$q_4 = \frac{156\lambda_i - 6}{22\lambda_i - 3} \cdot \frac{q_3}{L}$$

if $q_3 = \Delta$ then: $q_4(\lambda_1) = 1.38 \cdot \frac{\Delta}{L}$

$$q_4(\lambda_2) = 7.62 \cdot \frac{\Delta}{L}$$

1st
mode



2nd
mode

