Warsaw University of Technology

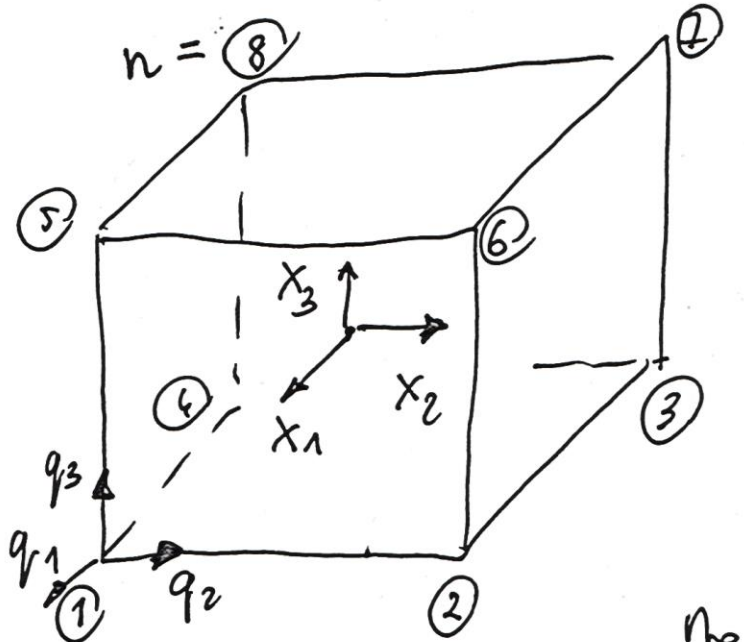
Faculty of Power and Aeronautical Engineering
WARSAW UNIVERSITY OF TECHNOLOGY

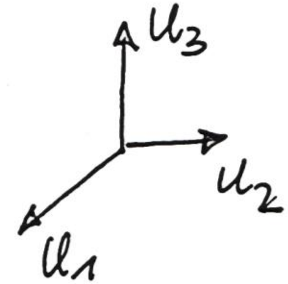
Institute of Aeronautics and Applied Mechanics

Finite element method 2 (FEM 2)

Nonlinear analysis

NONLINEAR STRAIN IN A FINITE ELEMENT.





$n_p = 3$

$n_e = n \cdot n_p = 24.$

DISPLACEMENT VECTOR :

$$\underset{3\times 1}{\{u\}} = \begin{Bmatrix} u_1(x_1, x_2, x_3) \\ u_2(x_1, x_2, x_3) \\ u_3(x_1, x_2, x_3) \end{Bmatrix} = \underset{3\times n_e}{[N(x_1, x_2, x_3)]} \cdot \underset{n_e \times 1}{\{q\}_e}$$

(shape functions)

LOCAL VECTOR OF NODAL PARAMETERS:

$$\underset{1\times ne}{\lfloor q \rfloor_e} = \lfloor q_1, q_2, \ldots, q_{ne} \rfloor$$

STRAIN VECTOR

$$\underset{1\times 6}{\lfloor \varepsilon \rfloor} = \lfloor \varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, \gamma_{12}, \gamma_{23}, \gamma_{31} \rfloor$$

STRESS VECTOR

$$\underset{1\times 6}{\lfloor \sigma \rfloor} = \lfloor \sigma_{11}, \sigma_{22}, \sigma_{33}, \tau_{12}, \tau_{23}, \tau_{31} \rfloor$$

KINEMATIC EQUATIONS

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i} + u_{\alpha,i} \cdot u_{\alpha,j}\right)$$

$i, j = 1, 2, 3$; $\alpha = 1, 2, 3$ (dummy index)

$$u_{i,j} = \frac{\partial u_i}{\partial x_j}$$

for $i = 1$, $j = 1$:

$$\varepsilon_{11} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_1}{\partial x_1} + \frac{\partial u_1}{\partial x_1} \cdot \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_1} \cdot \frac{\partial u_2}{\partial x_1} + \frac{\partial u_3}{\partial x_1} \cdot \frac{\partial u_3}{\partial x_1}\right) =$$

normal strain

$$= \underbrace{\frac{\partial u_1}{\partial x_1}}_{\text{linear part}} + \underbrace{\frac{1}{2}\left(\left(\frac{\partial u_1}{\partial x_1}\right)^2 + \left(\frac{\partial u_2}{\partial x_2}\right)^2 + \left(\frac{\partial u_3}{\partial x_1}\right)^2\right)}_{\text{nonlinear part}}$$

for $i=1,\ j=2$:

$$\underset{\uparrow \atop \text{shear strain}}{\varepsilon_{12}} = \frac{1}{2}\Bigg(\underbrace{\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}}_{\text{linear part}} + \underbrace{\frac{\partial u_1}{\partial x_1}\cdot\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\cdot\frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_1}\cdot\frac{\partial u_3}{\partial x_2}}_{\text{nonlinear part}} \Bigg)$$

$$\gamma_{12} = 2\cdot\varepsilon_{12}$$

GRADIENT MATRIX (3D)

$$\underset{6\times 3}{[R]} = \begin{bmatrix} \frac{\partial}{\partial x_1} & 0 & 0 \\ 0 & \frac{\partial}{\partial x_2} & 0 \\ 0 & 0 & \frac{\partial}{\partial x_3} \\ \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & 0 \\ 0 & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} & 0 & \frac{\partial}{\partial x_1} \end{bmatrix}$$

MATRIX OF SHAPE FUNCTIONS :

$$\underset{3\times ne}{[N]} = \begin{bmatrix} N_{11} & N_{12} & \cdots & N_{1\text{-}ne} \\ N_{21} & N_{22} & \cdots & N_{2\text{-}ne} \\ N_{31} & N_{32} & \cdots & N_{3\text{-}ne} \end{bmatrix}$$

STRAIN - DISPLACEMENT MATRIX OF A FINITE ELEMENT :

$$\underset{6\times ne}{[B_\partial]} = \underset{6\times 3}{[R]} \cdot \underset{3\times ne}{[N]}$$

STRAIN INCLUDING NONLINEAR PART:

$$\underset{6\times 1}{\{\varepsilon\}} = \underbrace{\underset{6\times ne}{[B_0]} \cdot \underset{ne\times 1}{\{q\}_e}}_{\text{linear part}} + \underbrace{\frac{1}{2}\underset{6\times ne}{[B_1(q)]} \cdot \underset{ne\times 1}{\{q\}_e}}_{\text{nonlinear part}}$$

(large deformations, stress stiffening)

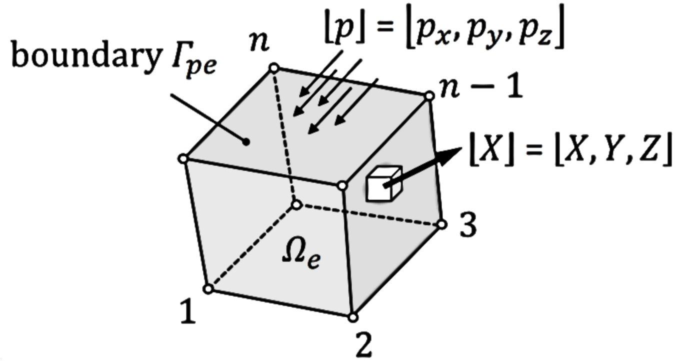
$$\underset{6\times ne}{[B_0]} = \begin{bmatrix} \dfrac{\partial \underset{1\times ne}{\lfloor N_{1j} \rfloor}}{\partial X_1} \\[2ex] \dfrac{\partial \underset{1\times ne}{\lfloor N_{2j} \rfloor}}{\partial X_2} \\[2ex] \dfrac{\partial \underset{1\times ne}{\lfloor N_{3j} \rfloor}}{\partial X_3} \\[2ex] \dfrac{\partial \underset{1\times ne}{\lfloor N_{1j} \rfloor}}{\partial X_2} + \dfrac{\partial \underset{1\times ne}{\lfloor N_{2j} \rfloor}}{\partial X_1} \\[2ex] \dfrac{\partial \underset{1\times ne}{\lfloor N_{2j} \rfloor}}{\partial X_3} + \dfrac{\partial \underset{1\times ne}{\lfloor N_{3j} \rfloor}}{\partial X_2} \\[2ex] \dfrac{\partial \underset{1\times ne}{\lfloor N_{3j} \rfloor}}{\partial X_1} + \dfrac{\partial \underset{1\times ne}{\lfloor N_{1j} \rfloor}}{\partial X_3} \end{bmatrix}$$

$$j = 1, \dots, ne$$

$$\frac{\partial \underset{1\times ne}{\lfloor N_{1j} \rfloor}}{\partial x_1} = \left\lfloor \frac{\partial N_{11}}{\partial X_1}, \frac{\partial N_{12}}{\partial X_1}, \dots, \frac{\partial N_{1\,ne}}{\partial x_1} \right\rfloor$$

$$
\underset{6\times ne}{\left[B_1(q)\right]} =
\begin{bmatrix}
\underset{1\times ne}{\lfloor q \rfloor_e} \cdot \dfrac{\partial \underset{ne\times 3}{[N]^T}}{\partial x_1} \cdot \dfrac{\partial \underset{3\times ne}{[N]}}{\partial x_1} \\[2ex]
\underset{1\times ne}{\lfloor q \rfloor_e} \cdot \dfrac{\partial \underset{ne\times 3}{[N]^T}}{\partial x_2} \cdot \dfrac{\partial \underset{3\times ne}{[N]}}{\partial x_2} \\[2ex]
\underset{1\times ne}{\lfloor q \rfloor_e} \cdot \dfrac{\partial \underset{ne\times 3}{[N]^T}}{\partial x_3} \cdot \dfrac{\partial \underset{3\times ne}{[N]}}{\partial x_3} \\[2ex]
\underset{1\times ne}{\lfloor q \rfloor_e} \cdot \left(\dfrac{\partial \underset{ne\times 3}{[N]^T}}{\partial x_1} \cdot \dfrac{\partial \underset{3\times ne}{[N]}}{\partial x_2} + \dfrac{\partial \underset{ne\times 3}{[N]^T}}{\partial x_2} \cdot \dfrac{\partial \underset{3\times ne}{[N]}}{\partial x_1} \right) \\[2ex]
\underset{1\times ne}{\lfloor q \rfloor_e} \cdot \left(\dfrac{\partial \underset{ne\times 3}{[N]^T}}{\partial x_2} \cdot \dfrac{\partial \underset{3\times ne}{[N]}}{\partial x_3} + \dfrac{\partial \underset{ne\times 3}{[N]^T}}{\partial x_3} \cdot \dfrac{\partial \underset{3\times ne}{[N]}}{\partial x_2} \right) \\[2ex]
\underset{1\times ne}{\lfloor q \rfloor_e} \cdot \left(\dfrac{\partial \underset{ne\times 3}{[N]^T}}{\partial x_1} \cdot \dfrac{\partial \underset{3\times ne}{[N]}}{\partial x_3} + \dfrac{\partial \underset{ne\times 3}{[N]^T}}{\partial x_3} \cdot \dfrac{\partial \underset{3\times ne}{[N]}}{\partial x_1} \right)
\end{bmatrix}
$$

EQUILIBRIUM EQUATION FOR A FINITE ELEMENT



ACCORDING TO THE PRINCIPLE OF VIRTUAL WORK

$$\int_{\Omega_e} \underset{1\times 6}{\lfloor \delta \varepsilon \rfloor} \cdot \underset{6\times 1}{\{\sigma\}} \, d\Omega_e - \int_{\Gamma_{pe}} \underset{1\times 3}{\lfloor \delta u \rfloor} \cdot \underbrace{\underset{3\times 1}{\{p\}}}_{\text{surface load}} \, d\Gamma_{pe} - \underset{1\times n_e}{\lfloor \delta q \rfloor_e} \cdot \underbrace{\underset{n_e\times 1}{\{F\}^n}}_{\text{nodal load}} +$$

$$- \int_{\Omega_e} \underset{1\times 3}{\lfloor \delta u \rfloor} \cdot \underbrace{\left(-\rho \underset{3\times n_e}{[N]} \cdot \underset{n_e\times 1}{\{\ddot{q}\}_e}\right)}_{\text{inertia load}} d\Omega_e = 0$$

VIRTUAL DISPLACEMENT:

$$\underset{1\times 3}{\lfloor \delta u \rfloor} = \underset{1\times ne}{\lfloor \delta q \rfloor_e} \cdot [N]^T$$

VIRTUAL STRAIN:

$$\underset{1\times 6}{\lfloor \delta \varepsilon \rfloor} = \underset{1\times ne}{\lfloor \delta q \rfloor_e} \cdot \underset{ne\times 6}{[B_0]^T} + \frac{1}{2}\underset{1\times ne}{\lfloor q \rfloor_e} \cdot [B_1(\delta q)]^T + \frac{1}{2}\underset{1\times ne}{\lfloor \delta q \rfloor_e} \cdot \underset{ne\times 6}{[B_1(q)]^T}$$

because of linearity of matrix $[B_1(q)]$ with respect to $\{q\}_e$:

$$\underset{1\times ne}{\lfloor q \rfloor_e} \cdot \underset{ne\times 6}{[B_1(\delta q)]^T} = \underset{1\times ne}{\lfloor \delta q \rfloor_e} \cdot \underset{ne\times 6}{[B_1(q)]^T}$$

thus:

$$\lfloor \delta \varepsilon \rfloor = \lfloor \delta q \rfloor_e \cdot \left([B_0]^T + [B_1(q)]^T \right)$$

$$\underbrace{\underset{1\times n_e}{\lfloor \delta q \rfloor_e}}_{\neq \lfloor 0 \rfloor} \cdot \Bigg(\int_{\Omega_e} \Big(\underset{n_e\times 6}{[B_0]^T} + \underset{n_e\times 6}{[B_1(q)]^T} \Big) \cdot \underset{6\times 1}{\{\sigma\}} \, d\Omega_e - \int_{\Gamma_{pe}} \underset{n_e\times 3}{[N]^T} \cdot \underset{3\times 1}{\{p\}} \, d\Gamma_{pe} +$$

$$- \underset{n_e\times 1}{\{F\}^n} + \int_{\Omega_e} \rho \cdot \underset{n_e\times 3}{[N]^T} \underset{3\times n_e}{[N]} \cdot \underset{n_e\times 1}{\{\ddot{q}\}_e} \, d\Omega_e \Bigg) = 0$$

finally (including damping): $[C]_e = \alpha [m]_e + \beta [K(q)]_e$

$$\underset{n_e\times n_e}{[m]_e} \cdot \underset{n_e\times 1}{\{\ddot{q}\}_e} + \underset{n_e\times n_e}{[C]_e} \cdot \underset{n_e\times 1}{\{\dot{q}\}_e} + \underset{n_e\times n_e}{[K(q)]_e} \cdot \underset{n_e\times 1}{\{q\}_e} = \underset{n_e\times 1}{\{F\}}$$

where: $$[M]_e = \int_{\Omega_e} \rho \, [N]^T [N] \cdot d\Omega_e$$

$$[k(q)]_e \cdot \{q\}_e = \int_{\Omega_e} \Big(\underset{n_e \times 6}{[B_0]^T} + \underset{n_e \times 6}{[B_1(q)]^T} \Big) \underset{6 \times 1}{\{\sigma\}} \, d\Omega_e$$

$$\underset{n_e \times 1}{\{F\}} = \int_{\Gamma_{pe}} \cdot \underset{n_e \times 3}{[N]^T} \cdot \underset{3 \times 1}{\{p\}} \, d\Gamma_{pe} + \underset{n_e \times 1}{\{F\}^n}$$

TANGENT MATRIX:

$$[K_T]_e = \frac{\partial \left([K(q)]_e \cdot \{q\}_e \right)}{\partial \{q\}_e} = \underbrace{[K_0]_e}_{\text{linear part}} + \underbrace{[K_\sigma]_e}_{\text{stress stiffening}} + \underbrace{[K_L]_e}_{\text{large deformation}}$$

$$\underset{n_e \times n_e}{[K_0]_e} = \int_{\Omega_e} \underset{n_e \times 6}{[B_0]^T} \cdot \underset{6 \times 6}{[D]} \cdot \underset{6 \times n_e}{[B_0]} \, d\Omega_e$$

$$\underset{n_e \times n_e}{[K_\sigma]_e} = \int_{\Omega_e} \left[\frac{\underset{n_e \times 3}{\partial [N]^T}}{\partial x_1}, \frac{\underset{n_e \times 3}{\partial [N]^T}}{\partial x_2}, \frac{\underset{n_e \times 3}{\partial [N]^T}}{\partial x_3} \right] \cdot \underset{9 \times 9}{[S]} \cdot \left\{ \begin{matrix} \dfrac{\underset{3 \times n_e}{\partial [N]}}{\partial x_1} \\ \dfrac{\underset{3 \times n_e}{\partial [N]}}{\partial x_2} \\ \dfrac{\partial [N]}{\partial x_3} \end{matrix} \right\} d\Omega_e$$

$$
\underset{9 \times 9}{[S]} = \begin{bmatrix}
\sigma_{11} & 0 & 0 & \sigma_{12} & 0 & 0 & \sigma_{13} & 0 & 0 \\
0 & \sigma_{11} & 0 & 0 & \sigma_{12} & 0 & 0 & \sigma_{13} & 0 \\
0 & 0 & \sigma_{11} & 0 & 0 & \sigma_{12} & 0 & 0 & \sigma_{13} \\
\sigma_{12} & 0 & 0 & \sigma_{22} & 0 & 0 & \sigma_{23} & 0 & 0 \\
0 & \sigma_{12} & 0 & 0 & \sigma_{22} & 0 & 0 & \sigma_{23} & 0 \\
0 & 0 & \sigma_{12} & 0 & 0 & \sigma_{22} & 0 & 0 & \sigma_{23} \\
\sigma_{13} & 0 & 0 & \sigma_{23} & 0 & 0 & \sigma_{33} & 0 & 0 \\
0 & \sigma_{13} & 0 & 0 & \sigma_{23} & 0 & 0 & \sigma_{33} & 0 \\
0 & 0 & \sigma_{13} & 0 & 0 & \sigma_{23} & 0 & 0 & \sigma_{33}
\end{bmatrix}
$$

large deformations :

$$
\underset{n_e \times n_e}{[K_L]_e} = \int_{\Omega_e} \Big(\underset{n_e \times 6}{[B_0]^T} \underset{6\times 6}{[D]} \underset{6 \times n_e}{[B_1(q)]} + \underset{n_e \times 6}{[B_1(q)]^T} \underset{6\times 6}{[D]} \cdot \underset{6 \times n_e}{[B_0]} +
$$

$$
+ \underset{n_e \times 6}{[B_1(q)]^T} \cdot \underset{6\times 6}{[D]} \cdot \underset{6 \times n_e}{[B_1(q)]} \Big) d\Omega_e
$$