



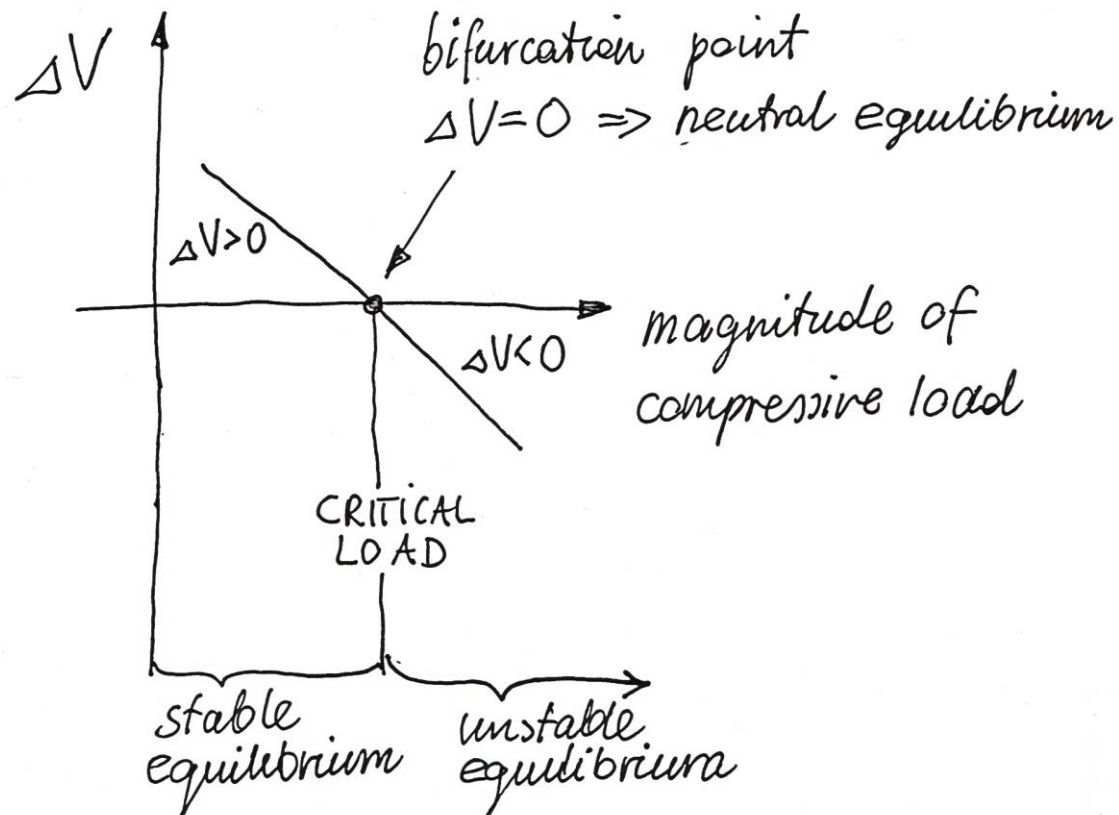
Institute of Aeronautics and Applied Mechanics

Finite element method 2 (FEM 2)

Buckling analysis

BUCKLING ANALYSIS

The stability of a linear structure depends on the change of total potential energy due to a very small disturbance from the equilibrium state.



$$\Delta V = \delta V + \frac{1}{2!} \delta^2 V + \frac{1}{3!} \delta^3 V \quad (\text{Taylor series})$$

1st variation

$\delta V = 0$ - necessary

Condition of equilibrium
 $V \rightarrow \min$

2nd variation
 included in the Lagrange-Dirichlet
 criterion

3rd variation
 (neglected)

For a very small disturbance of the displacement vector:

$$[dq]_{1 \times N} = [dq_1, dq_2, \dots, dq_N]$$

The change of V :

$$\Delta V = \underbrace{\sum_{i=1}^N \frac{\partial V}{\partial q_i} dq_i}_0 + \underbrace{\frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \frac{\partial^2 V}{\partial q_i \partial q_j} dq_i dq_j}_{\text{NONLINEAR PART}} =$$

$\nwarrow$ LINEAR PART

$$\begin{aligned}
 & \underbrace{([K] \{q\} = \{F\})}_{\substack{N \times N \quad N \times 1 \quad N \times 1}} \\
 &= 0 + \frac{1}{2} [dq]_{1 \times N} \underbrace{\left[\frac{\partial^2 V}{\partial q_i \partial q_j} \right]}_{N \times N} \cdot \underbrace{\{dq\}}_{N \times 1}
 \end{aligned}$$

$$\begin{bmatrix} \frac{\partial^2 V}{\partial q_i \partial q_j} \end{bmatrix}_{N \times N} = \begin{bmatrix} \frac{\partial^2 V}{\partial q_1^2} & \frac{\partial^2 V}{\partial q_1 \partial q_2} & \cdot & \cdot & \cdot & \frac{\partial^2 V}{\partial q_1 \partial q_N} \\ \frac{\partial^2 V}{\partial q_2 \partial q_1} & \frac{\partial^2 V}{\partial q_2^2} & & & & \frac{\partial^2 V}{\partial q_2 \partial q_N} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{\partial^2 V}{\partial q_N \partial q_1} & \frac{\partial^2 V}{\partial q_N \partial q_2} & \cdot & \cdot & \cdot & \frac{\partial^2 V}{\partial q_N \partial q_N} \end{bmatrix}$$

$\Delta V = 0$ for a neutral (or critical) equilibrium :

$$\left[\frac{\partial^2 V(\lambda_*)}{\partial q_i \partial q_j} \right]_{N \times N} \cdot \underbrace{\{dq\}}_{N \times 1} = \underbrace{\{0\}}_{N \times 1}$$

where :

λ_* - load multiplier , for which $\Delta V = 0$

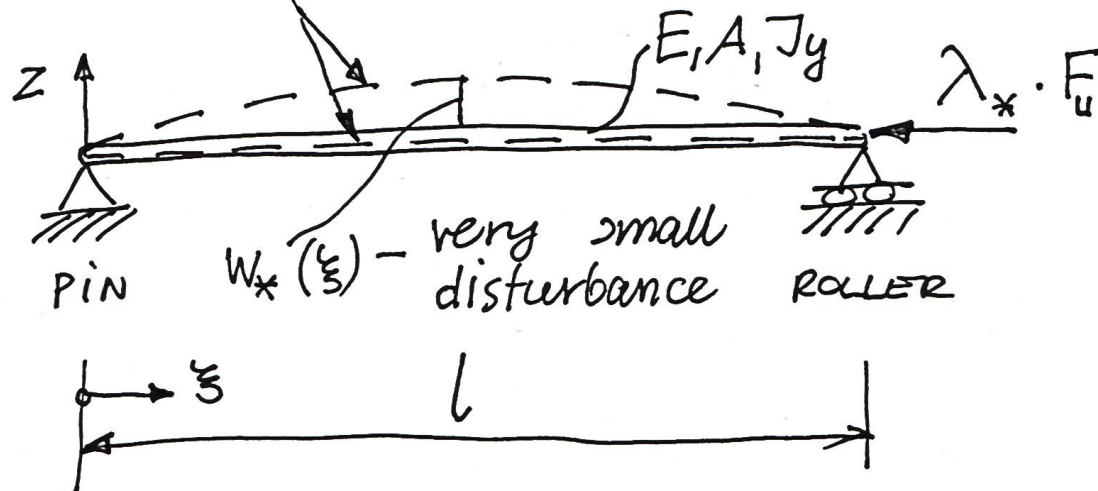
finally :

$$\det \left(\left[\frac{\partial^2 V(\lambda_*)}{\partial q_i \partial q_j} \right]_{N \times N} \right) = 0$$

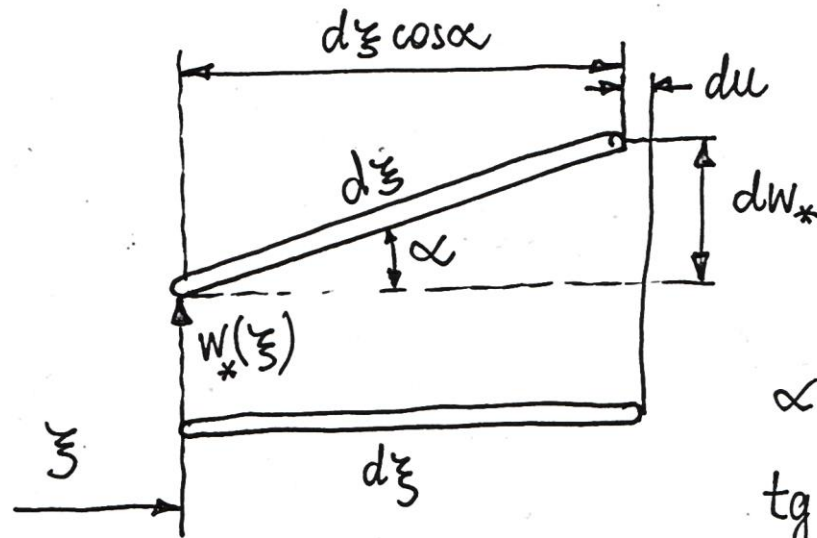
(eigen problem)

EXAMPLE . FIND THE CRITICAL LOAD AND BUCKLING MODES
FOR A COLUMN . USE ONE FINITE ELEMENT .

neutral equilibrium - two deformations are possible
for the critical load ($\Delta V = 0$)



$\bar{F}_u = 1N$ - unit load

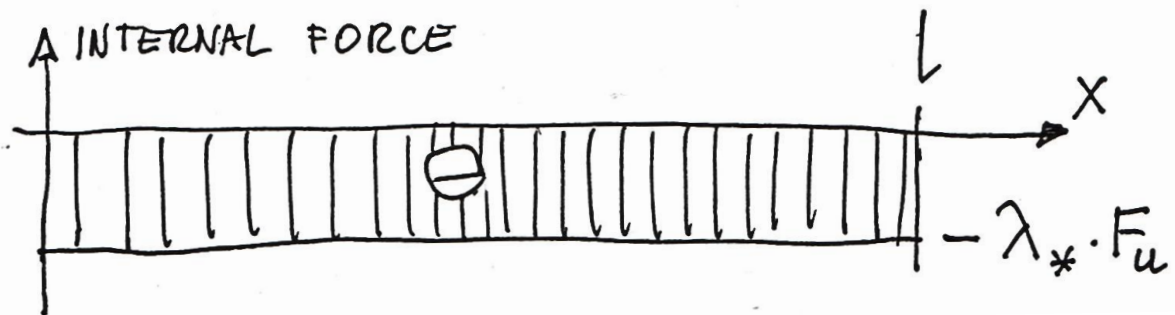
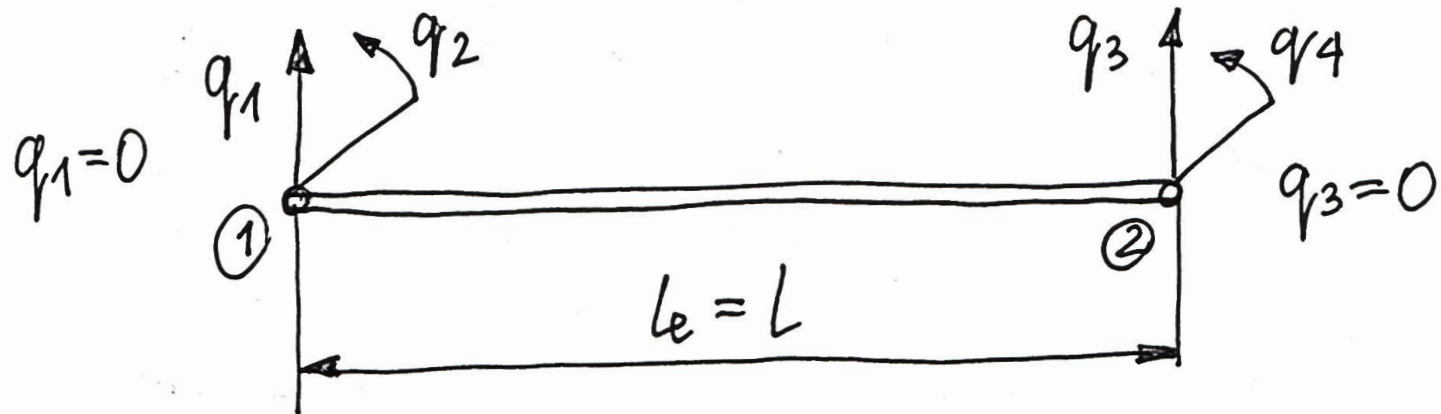


α - small angle
 $\operatorname{tg} \alpha \approx \alpha = \frac{dw_*}{d\xi}$

$$du = d\xi - d\xi \cdot \cos \alpha = d\xi (1 - \cos \alpha) =$$

$$= \left| \begin{array}{l} \cos 2\varphi = \cos^2 \varphi - \sin^2 \varphi \Rightarrow \cos 2\varphi = 1 - 2\sin^2 \varphi \\ \cos^2 \varphi = 1 - \sin^2 \varphi \qquad \qquad \qquad 1 - \cos 2\varphi = 2\sin^2 \varphi \\ 2\varphi = \alpha \Rightarrow \varphi = \frac{\alpha}{2} \end{array} \right| =$$

$$= d\xi \cdot 2\sin^2\left(\frac{\alpha}{2}\right) \approx d\xi \cdot 2 \cdot \frac{\alpha^2}{4} = \frac{1}{2} \left(\frac{dw_*}{d\xi}\right)^2 d\xi$$



Nonlinear part of strain energy :

$$\Delta U^{NL} = \frac{1}{2} \int_0^l EJ_y \left(\frac{d^2 W_*}{d\xi^2} \right)^2 d\xi$$

We consider only the nonlinear part of ΔV

so the strain energy of axial loading

on the axial displacement is neglected :

Nonlinear part of potential energy of loading :

$$\Delta W^{NL} = \int_0^L (\lambda_* \cdot F_u) du = \frac{1}{2} \int_0^L (\lambda_* \cdot F_u) \left(\frac{dw_*}{d\xi} \right)^2 d\xi$$

We consider only the nonlinear part of ΔV
so the potential energy of axial loading
on the axial displacement is neglected :

$$W_*(\xi) = [N_1(\xi), N_2(\xi), N_3(\xi), N_4(\xi)] \cdot \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}$$

$$\frac{dW_*}{d\xi} = \left[\frac{dN_1}{d\xi}, \frac{dN_2}{d\xi}, \frac{dN_3}{d\xi}, \frac{dN_4}{d\xi} \right] \cdot \underbrace{\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}_e}_{4 \times 1} = \underbrace{[N']}_{1 \times 4} \cdot \underbrace{\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}_e}_{4 \times 1}$$

$$\frac{d^2W_*}{d\xi^2} = \left[\frac{d^2N_1}{d\xi^2}, \frac{d^2N_2}{d\xi^2}, \frac{d^2N_3}{d\xi^2}, \frac{d^2N_4}{d\xi^2} \right] \cdot \underbrace{\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}_e}_{4 \times 1} = \underbrace{[N'']}_{1 \times 4} \cdot \underbrace{\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}_e}_{4 \times 1}$$

$$\Delta V^{NL} = \Delta U^{NL} - \Delta W^{NL} =$$

$$= \frac{1}{2} \int_0^L EJ_y \cdot \frac{d^2 W_*}{d\xi^2} \cdot \frac{d^2 W_*}{d\xi^2} d\xi - \frac{1}{2} \int_0^L (\lambda_* \cdot F_u) \cdot \frac{dW_*}{d\xi} \cdot \frac{dW_*}{d\xi} d\xi =$$

$\begin{matrix} \uparrow\uparrow \\ LqJ_e \cdot \{N''\} \\ 1 \times 4 \quad 4 \times 1 \end{matrix}$
 $\begin{matrix} \nwarrow\swarrow \\ [N''] \cdot \{q\}_e \\ 1 \times 4 \quad 4 \times 1 \end{matrix}$

$\begin{matrix} \uparrow\uparrow \\ LqJ_e \cdot \{N'\} \\ 1 \times 4 \quad 4 \times 1 \end{matrix}$
 $\begin{matrix} \nwarrow\swarrow \\ [N'] \cdot \{q\}_e \\ 1 \times 4 \quad 4 \times 1 \end{matrix}$

$$= \frac{1}{2} LqJ_e \int_0^L EJ_y \{N''\} [N''] d\xi \cdot \{q\}_e - \frac{1}{2} LqJ_e \cdot \lambda_* \cdot \int_0^L \{N'\} \cdot F_u \cdot [N'] d\xi \cdot \{q\}_e =$$

$\underbrace{\hspace{15em}}$
 stiffness matrix $[k]_e$
 4×4

$\underbrace{\hspace{15em}}$
 prestress matrix $[k_\sigma]_e$
 4×4

$$= \frac{1}{2} LqJ_e ([k]_e - \lambda_* [k_\sigma]_e) \cdot \{q\}_e$$

stiffness matrix

$$[k]_e = \frac{2EJ_y}{l^3} \begin{bmatrix} 6 & 3l & -6 & 3l \\ 3l & 2l^2 & -3l & l^2 \\ -6 & -3l & 6 & -3l \\ 3l & l^2 & -3l & 2l^2 \end{bmatrix}$$

prestress matrix

$$[k_0]_e = \frac{F_u}{30l} \cdot \begin{bmatrix} 36 & 3l & -36 & 3l \\ 3l & 4l^2 & -3l & -l^2 \\ -36 & -3l & 36 & -3l \\ 3l & -l^2 & -3l & 4l^2 \end{bmatrix}$$

FOR THE CRITICAL LOAD: $\Delta V = 0$

$$q_1 = q_3 = 0 \Rightarrow \left(\underset{2 \times 2}{[K]_e} - \lambda_* \underset{2 \times 2}{[K_\sigma]_e} \right) \cdot \underset{2 \times 1}{\{q\}_e} = \underset{2 \times 1}{\{0\}}$$

$$\left(\frac{2EJ_y}{l^3} \begin{bmatrix} 2l^2 & l^2 \\ l^2 & 2l^2 \end{bmatrix} - \lambda_* \cdot \frac{F_u}{30l} \cdot \begin{bmatrix} 4l^2 & -l^2 \\ -l^2 & 4l^2 \end{bmatrix} \right) \cdot \underset{2 \times 1}{\{q\}_e} = \underset{4 \times 1}{\{0\}}$$

$$\det \left(\underset{2 \times 2}{[K]_e} - \lambda_* \underset{2 \times 2}{[K_\sigma]_e} \right) = 0$$

$$\det \left(\frac{2EJ_y}{l^3} \begin{bmatrix} 2l^2 & l^2 \\ l^2 & 2l^2 \end{bmatrix} - \lambda_* \cdot \frac{F_u}{30l} \cdot \begin{bmatrix} 4l^2 & -l^2 \\ -l^2 & 4l^2 \end{bmatrix} \right) = 0$$

after introducing a new constant : $\lambda = \lambda_* \cdot \frac{F_u \cdot L^2}{60 E J_y}$

$$\det \left(\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix} \right) = 0$$

$$\det \left(\begin{bmatrix} 2-4\lambda & 1+\lambda \\ 1+\lambda & 2-4\lambda \end{bmatrix} \right) = 0$$

$$(2-4\lambda)^2 - (1+\lambda)^2 = 0$$

$$4 - 16\lambda + 16\lambda^2 - 1 - 2\lambda - \lambda^2 = 0$$

$$15\lambda^2 - 18\lambda + 3 = 0$$

$$\Delta = (-18)^2 - 4 \cdot 15 \cdot 3 = 324 - 180 = 144$$

$$\sqrt{\Delta} = 12$$

$$\lambda_1 = \frac{18-12}{30} = \frac{1}{5} \quad \lambda_2 = \frac{18+12}{30} = 1$$

$$\lambda_{*1} = \lambda_1 \cdot \frac{60 E J_y}{F_u \cdot l^2} = \frac{12 E J_y}{F_u \cdot l^2}$$

$$\lambda_{*2} = \lambda_2 \cdot \frac{60 E J_y}{F_u \cdot l^2} = \frac{60 E J_y}{F_u \cdot l^2}$$

FIRST CRITICAL LOAD

$$F_{\text{CRIT}_1} = \lambda_{*1} \cdot F_u = \frac{12 E J_y}{l^2}$$

SECOND CRITICAL LOAD

$$F_{\text{CRIT}_2} = \lambda_{*2} \cdot F_u = \frac{60 E J_y}{l^2}$$

ANALYTICAL SOLUTION - EULER'S CRITICAL FORCE : $\overline{F}_{crit i} = i^2 \frac{\pi^2 EJ_y}{L^2}$

$$\overline{F}_{crit 1} = \frac{\pi^2 EJ_y}{L^2} = 9.87 \frac{EJ_y}{L^2}$$

$$\overline{F}_{crit 2} = \frac{4\pi^2 EJ_y}{L^2} = 39.48 \frac{EJ_y}{L^2}$$

$$\Delta F_{crit 1} = \frac{F_{crit 1} - \overline{F}_{crit 1}}{\overline{F}_{crit 1}} = 22\%$$

$$\Delta F_{crit 2} = \frac{F_{crit 2} - \overline{F}_{crit 2}}{\overline{F}_{crit 2}} = 52\%$$

BUCKLING MODES :

$$\begin{bmatrix} 2-4\lambda_i & 1+\lambda_i \\ 1+\lambda_i & 2-4\lambda_i \end{bmatrix} \begin{Bmatrix} q_2(\lambda_i) \\ q_4(\lambda_i) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

two linearly dependent equations

1st : $(2-4\lambda_i) \cdot q_2(\lambda_i) + (1+\lambda_i) \cdot q_4(\lambda_i) = 0$

$$q_4(\lambda_i) = -\frac{2-4\lambda_i}{1+\lambda_i} \cdot q_2(\lambda_i)$$

LET'S ASSUME : $q_2(\lambda_1) = q_2(\lambda_2) = \theta$
(any angle)

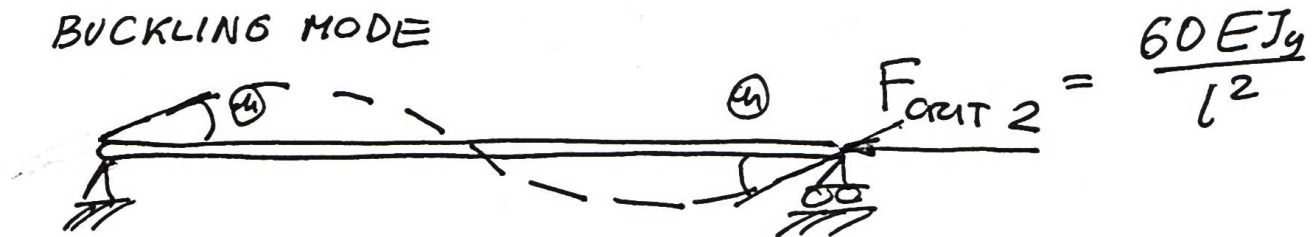
$$q_4(\lambda_1) = - \frac{2 - 4 \cdot \frac{4}{5}}{1 + \frac{4}{5}} \cdot \textcircled{u} = - \textcircled{u}$$

$$q_4(\lambda_2) = - \frac{2 - 4 \cdot 1}{1 + 1} \cdot \textcircled{u} = \textcircled{u}$$

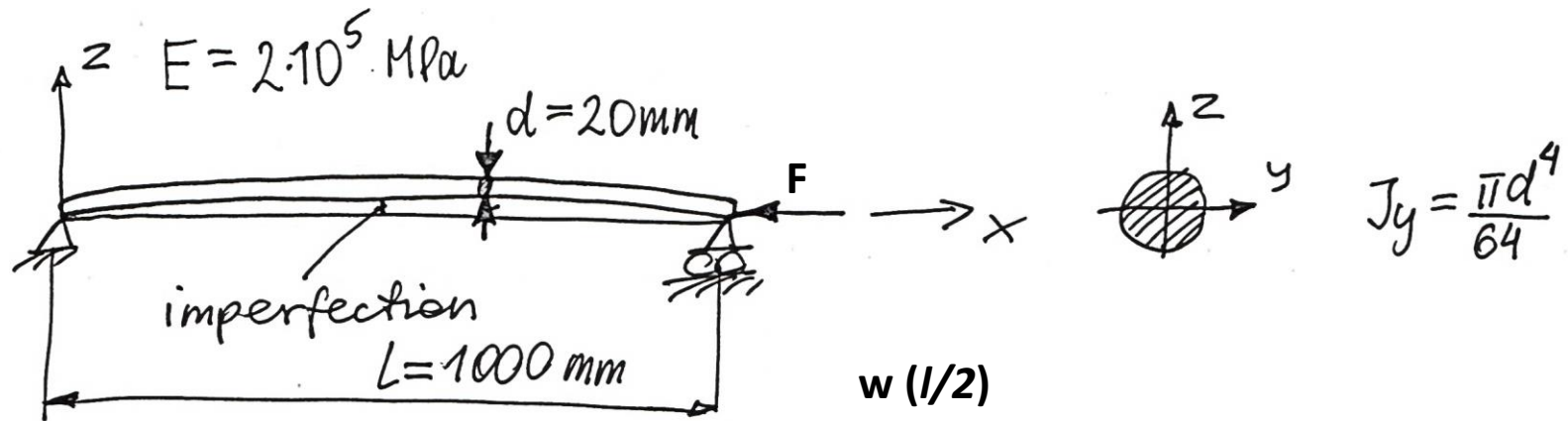
1ST BUCKLING MODE



2ND BUCKLING MODE

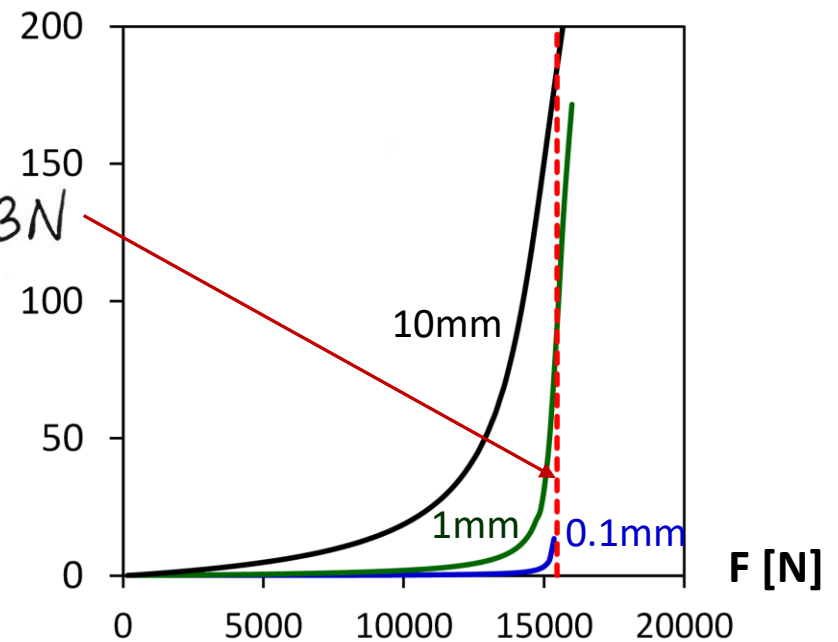


NONLINEAR SOLUTION

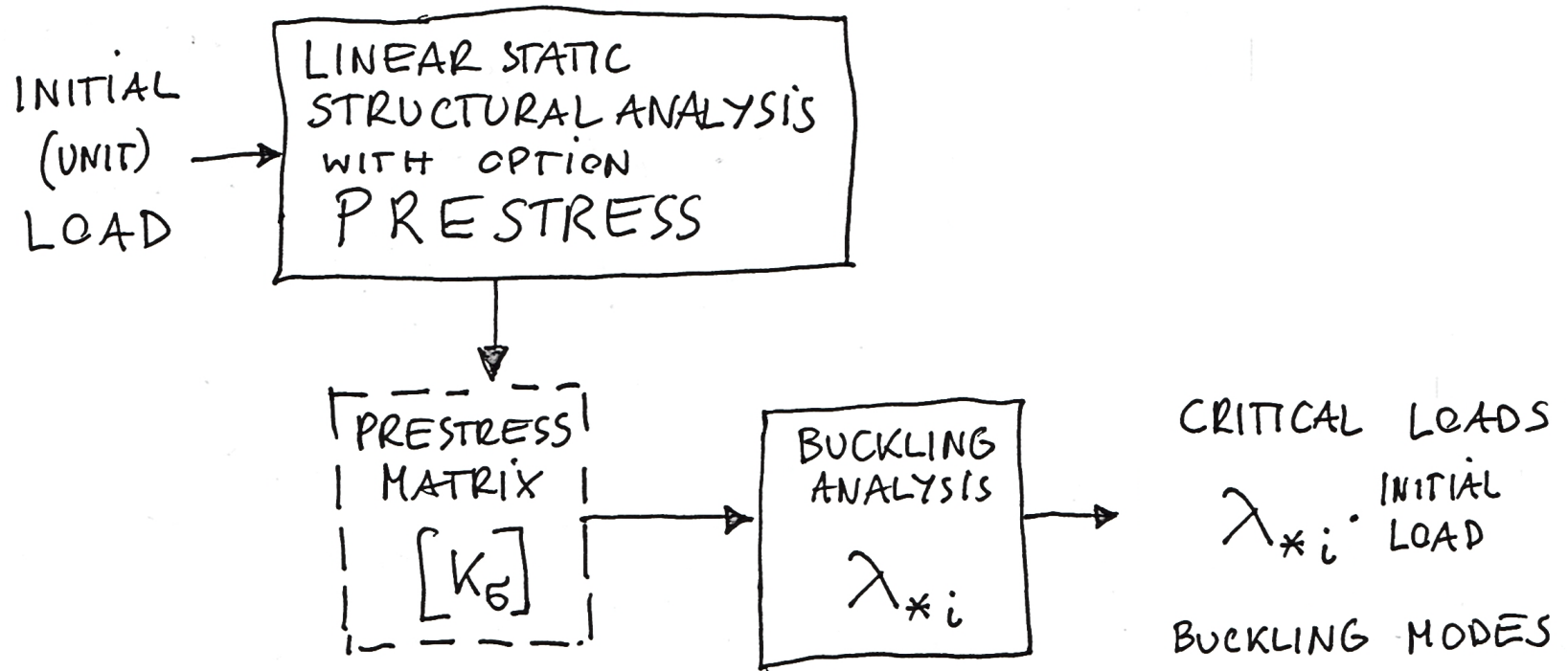


$$\bar{F}_{\text{crit}1} = \frac{\pi^2 E J_y}{L^2} = \frac{\pi^3 E d^4}{64 L^2} = 15503 \text{ N}$$

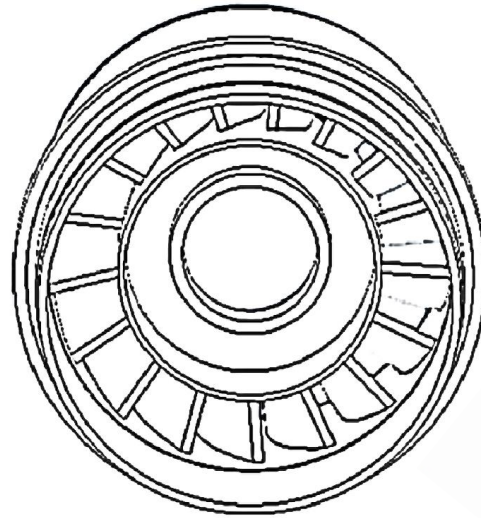
$w(l/2)$



BUCKLING ANALYSIS



EXAMPLE . BUCKLING OF THE NOZZLE GUIDE VANE
DUE TO TEMPERATURE INCREMENT



$$\Delta T_{\text{CRIT}} = ?$$

surface load
 $p = \text{const}$

$$\Delta T_{\text{CRIT}} = \lambda_{*1} \cdot \underbrace{\Delta T_u}_{1^\circ\text{C}}$$

↑
FIRST CRITICAL LOAD

$$p_{\text{CRIT}} = \lambda_{*1} \cdot p \quad \text{but} \quad p = \text{const} \Rightarrow \lambda_{*1} = 1$$

ITERATIVE SOLUTION :

STEP	INITIAL LOAD	PRESTRESS MATRIX	BUCKLING ANALYSIS
1	$\Delta T_u^1 = 1^\circ\text{C}, p$	$[K_5]_1$	λ_{*1}^1
2	$\Delta T_u^2 = \lambda_{*1}^1 \cdot \Delta T_u^1, p$	$[K_5]_2$	λ_{*1}^2
.	.	.	.
.	.	.	.
.	.	.	.
j	$\Delta T_u^j = \lambda_{*1}^{j-1} \cdot \Delta T_u^{j-1}, p$	$[K_5]_j$	λ_{*1}^j
.	.	.	.
.	.	.	.
LAST	$\Delta T_u^{\text{LAST}} = \lambda_{*1}^{\text{LAST}-1} \cdot \Delta T_u^{\text{LAST}-1}, p$	.	$\lambda_{*1}^{\text{LAST}} \cong 1.0$