Warsaw University of Technology

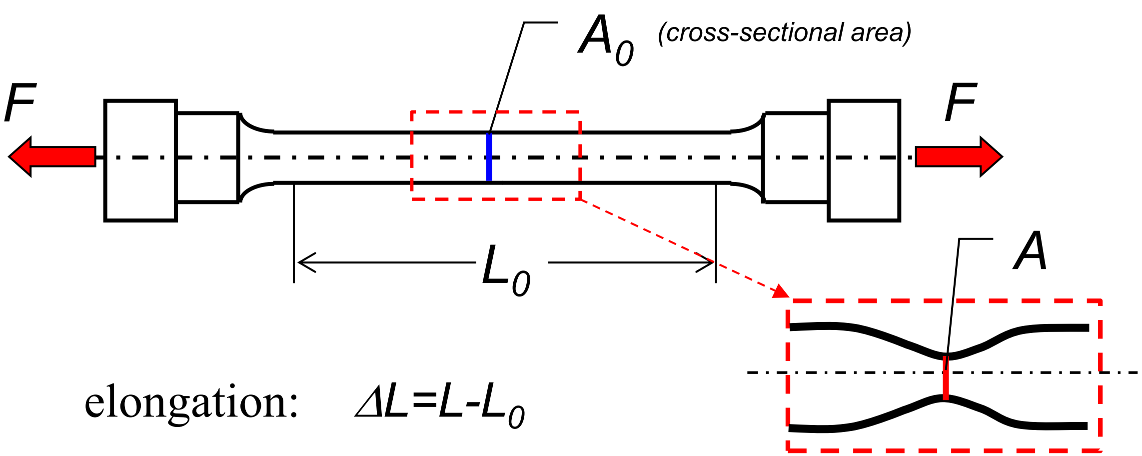
Faculty of Power and Aeronautical Engineering
WARSAW UNIVERSITY OF TECHNOLOGY

Institute of Aeronautics and Applied Mechanics

Finite element method 2 (FEM 2)

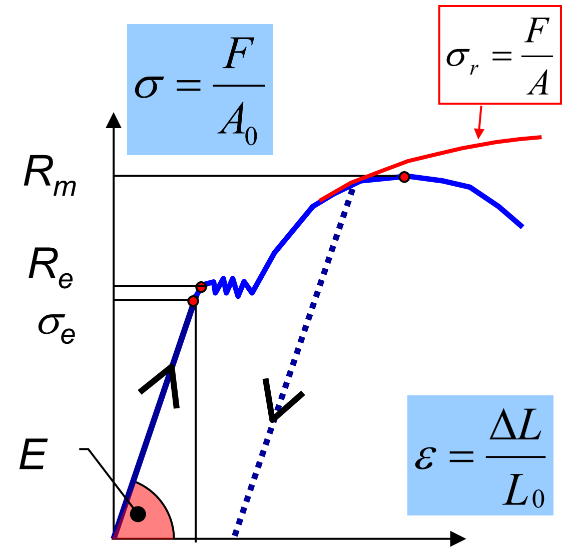
Material nonlinearities

Uniaxial tensile test



elongation: $\Delta L = L - L_0$

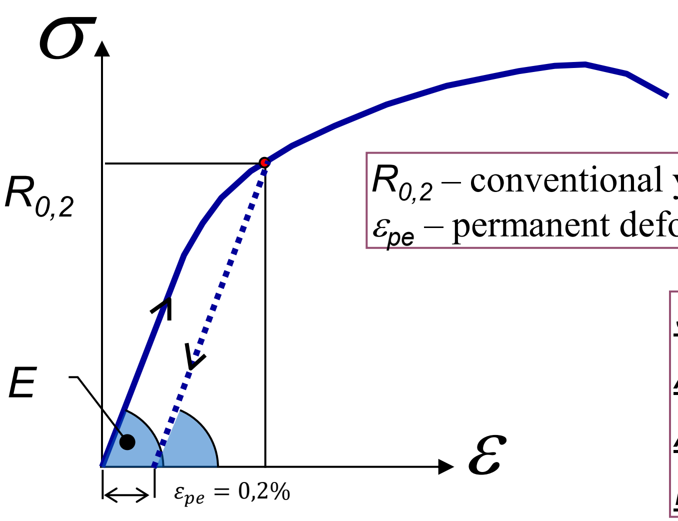
Material with a clear yield point:



E – Young's modulus
σ_e – elastic limit
R_e – yield stress
R_m – ultimate tensile stress

σ_r – true stress
A – area of a neck

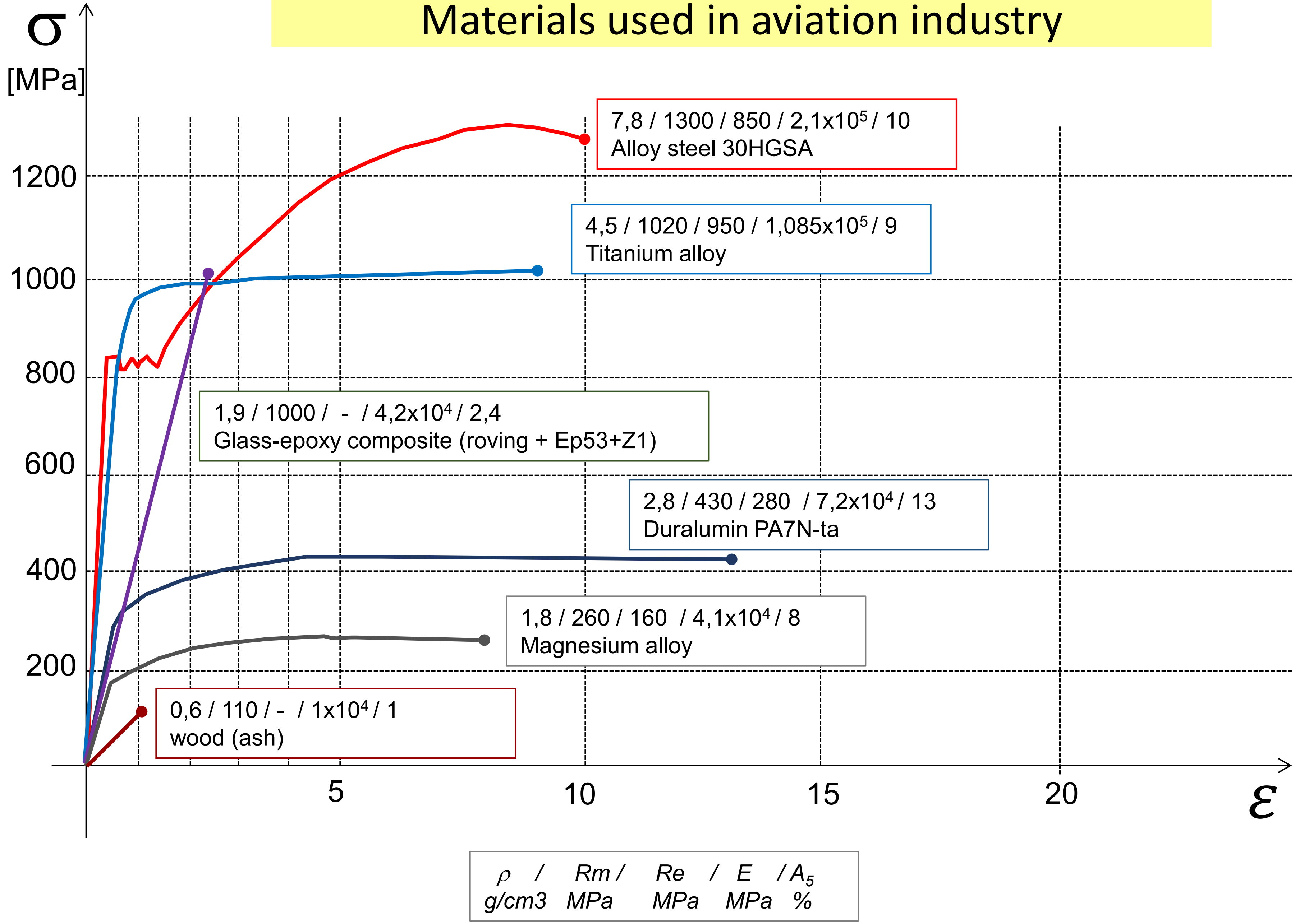
Material without a clear yield point:



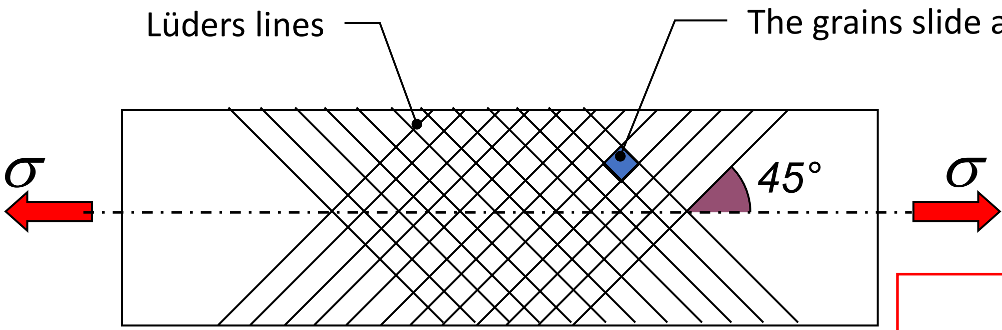
$R_{0,2}$ – conventional yield strength
ε_{pe} – permanent deformation

Steel: ST3S: R_e= 200 MPa R_m= 450 MPa
Alloy steel: (Cr-Ni-Mo): R_e= 870 MPa R_m= 1020 MPa
Aluminium: $R_{0,2}$ = 120 MPa R_m= 140 MPa
Duralumin (PA9): $R_{0,2}$ = 490 R_m= 570 MPa

Materials used in aviation industry

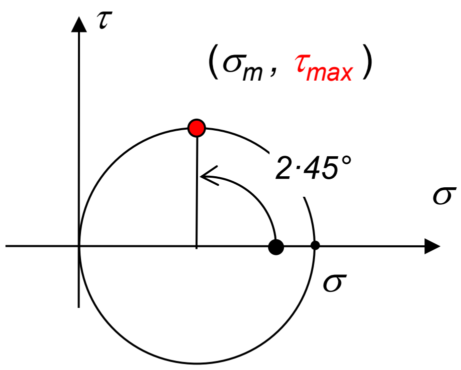


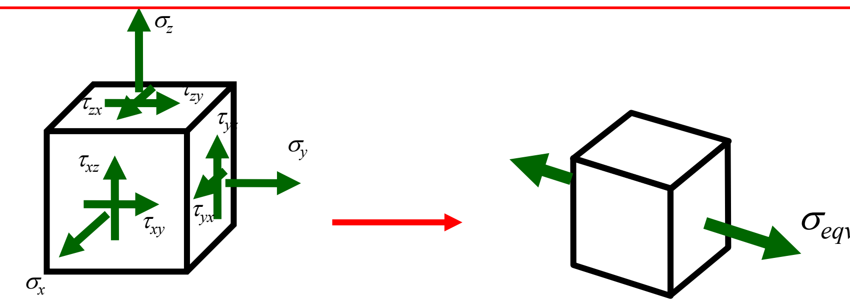
Equivalent stress hypothesis by Tresca (INTENSITY)



All plastic deformation is caused by slippage

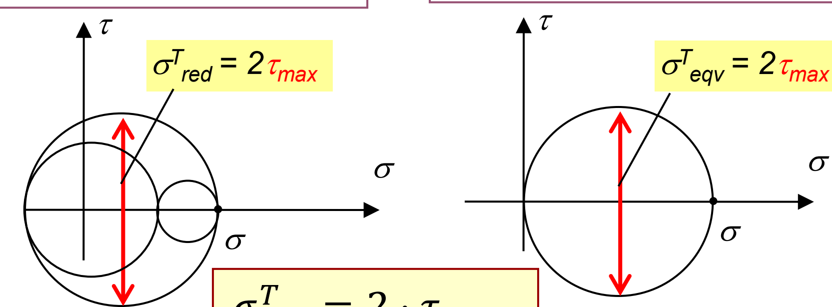
A sample made of mild steel with a clear yield point





3-D stress state

Equivalent uniaxial stress state



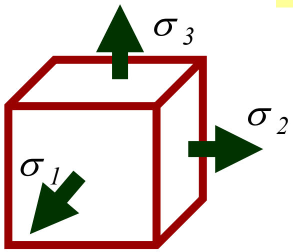
$$\sigma^{T}_{eqv} = 2 \cdot \tau_{\max}$$

Equivalent stress characterize a substitute uniaxial state of tension, which in terms of safety corresponds to the 3-D state.

Hypothesis of *Tresca*:

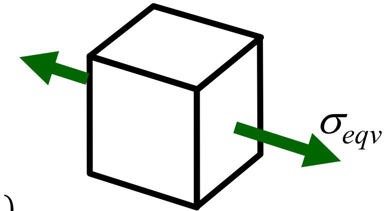
The τ_{max} value is a measure of the material effort for a given stress state indicating the first permanent deformations.

Equivalent stress hypothesis by Huber-Mises-Hencke (SEQV)



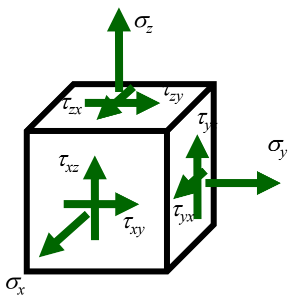
Energy density for 3-D stress state

$$U' = \frac{1+\nu}{3E} \cdot \left\{ \frac{1}{2} [(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2] \right\}$$



Energy density for 1-D tension

$$U' = \frac{1+\nu}{3E} \cdot \left\{ \frac{1}{2} \left[\left(\sigma_{eqv}\right)^2 + \left(\sigma_{eqv}\right)^2 \right] \right\} = \frac{1+\nu}{3E} \cdot \sigma_{eqv}^2$$

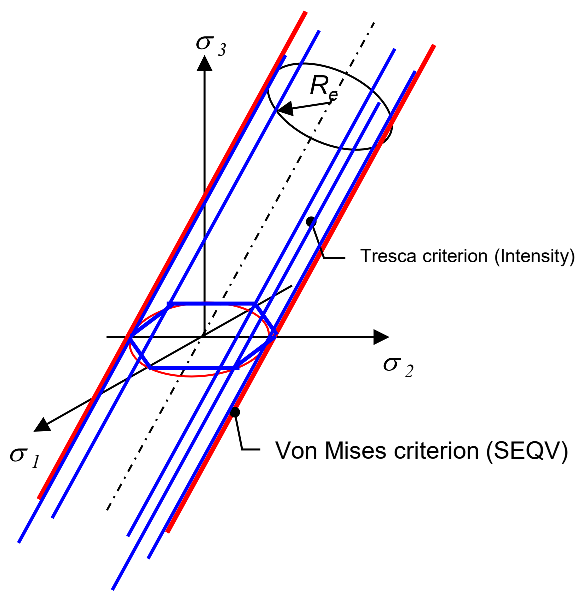


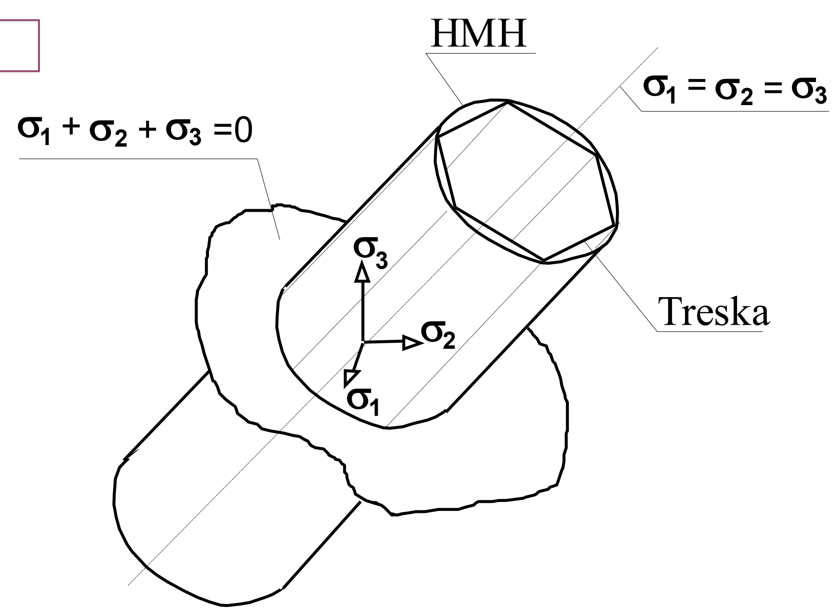
Plastic defomation occurs when the shear strain energy density (per unit volume) equals or exceeds the energy density at which the same material plasticizes in a uniaxial tensile test.

$$\sigma_{eqv}^{HMH} = \sqrt{\frac{1}{2}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]}$$

$$\sigma_{eqv}^{HMH} = \sqrt{\frac{1}{2}\left[(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2\right] + 3\left(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2\right)}$$

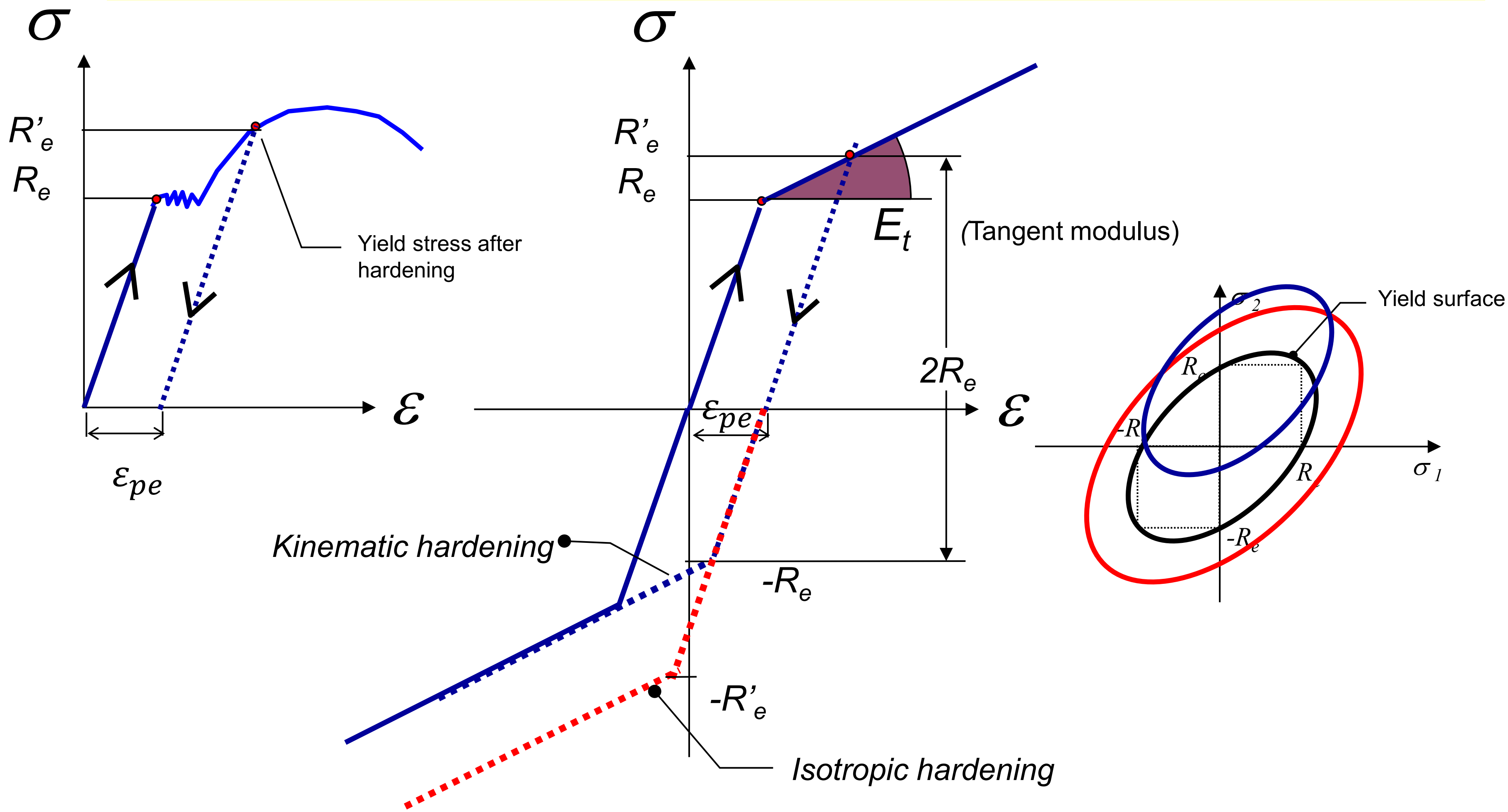
Plastic deformation starts when $\sigma_{eqv} = R_e$

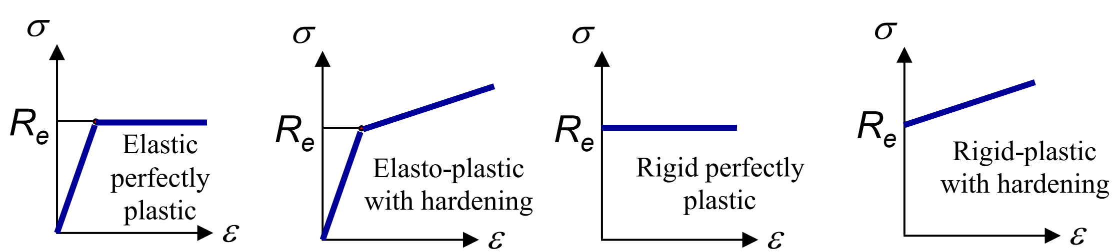




Yield surface in a 3-D space of principal stresses

Hardening models





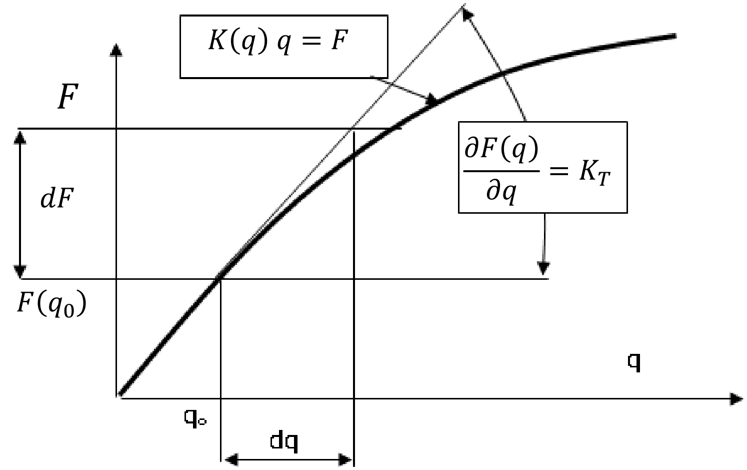
Elasto-plastic deformation as a source of nonlinearity

Nonlinear set of FEM equation:

$$\boxed{K\,(q)\;q = F}$$

the matrix coefficients depend on the deformation caused by development of the plastic zone in the structure

For a single $\boldsymbol{q}$ parameter



The tangent stiffness matrix: $$\boldsymbol{K}_T = \frac{\partial \boldsymbol{F}(\boldsymbol{q})}{\partial \boldsymbol{q}} = \int_V \boldsymbol{B}_0^T\,\boldsymbol{D}^*\boldsymbol{B}_0 dV$$

$\boldsymbol{B^o}$ - strain-displacement matrix $(\boldsymbol{\varepsilon} = \boldsymbol{B}^0\boldsymbol{q})$

Constitutive matrix including plastic strain: $\boldsymbol{D}^* = \dfrac{\partial \boldsymbol{\sigma}}{\partial \boldsymbol{\varepsilon}}$

$$\boldsymbol{D}^* = \boldsymbol{D} - \boldsymbol{D}\frac{\partial f}{\partial \boldsymbol{\sigma}}\frac{\partial f}{\partial \boldsymbol{\sigma}^T}\boldsymbol{D}\left(E_t + \frac{\partial f}{\partial \boldsymbol{\sigma}^T}\boldsymbol{D}\frac{\partial f}{\partial \boldsymbol{\sigma}}\right)^{-1}$$

$\mathbf{D}$ – constitutive matrix for the elastic state

E_t – tangent modulus

The yield function can be expressed as:

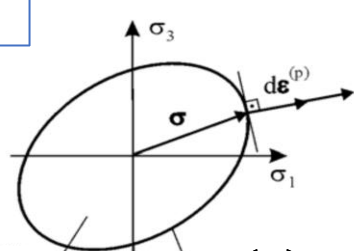
$$f = \sigma_{eqv} - R_e(\kappa) = 0$$

Equivalent stress. Yield stress

The flow rule: $$\boldsymbol{d\varepsilon^p} = \boldsymbol{d\lambda}\frac{\partial f}{\partial \boldsymbol{\sigma}}$$

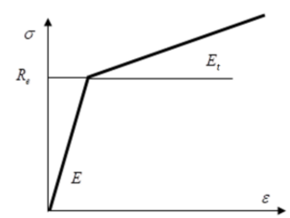
Plastic strain increment

Plastic multiplier



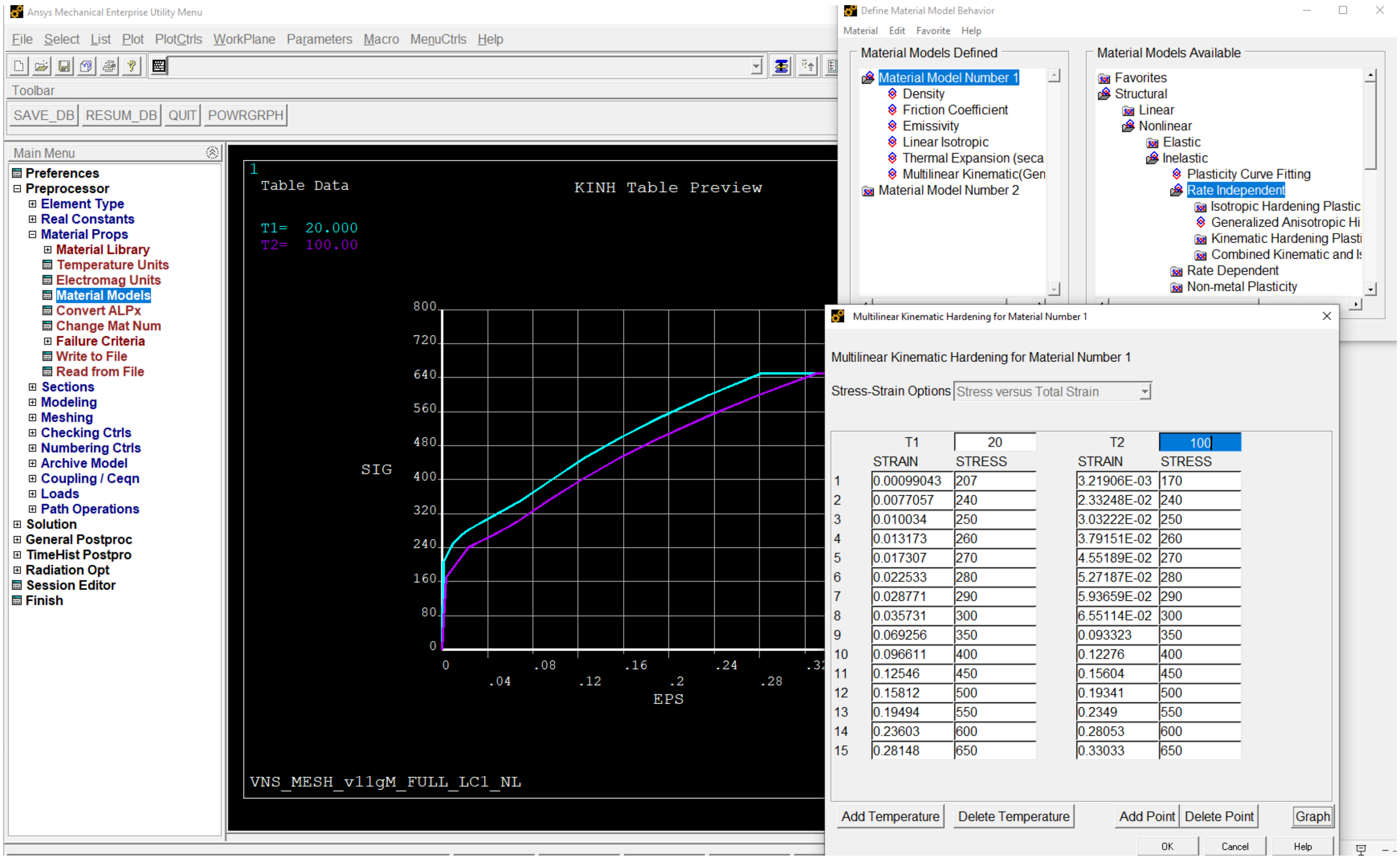
$\dfrac{\partial f}{\partial \boldsymbol{\sigma}}$

$\boldsymbol{f(\sigma) < 0}$ $\boldsymbol{f(\sigma) = 0}$

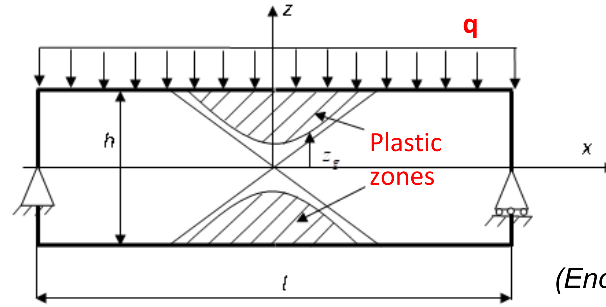


A nonlinear system of FEM equations is usually solved using iterative techniques (Newton-Raphson method).

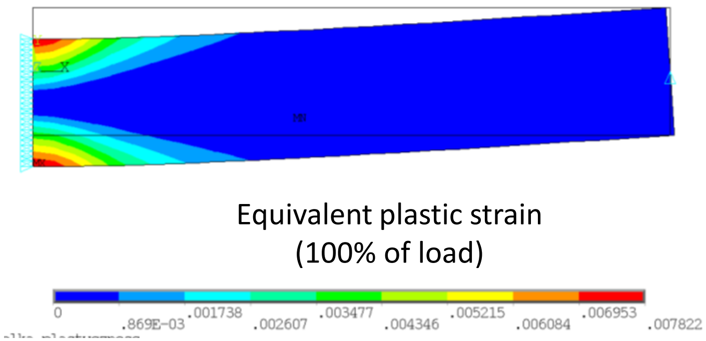
Example of elasto-plastic properties in ANSYS (Multilinear Characteristics with Kinematic Hardening)



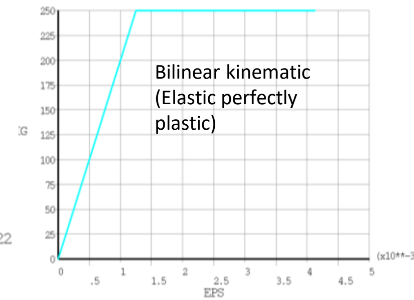
Example 1. Bending of a beam loaded by traction.
(half model)

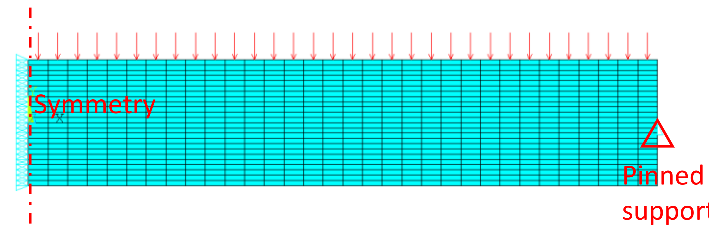


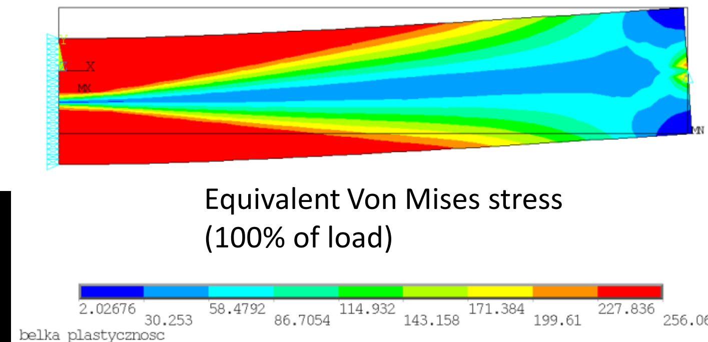
Solid182
(Enchanced strain)



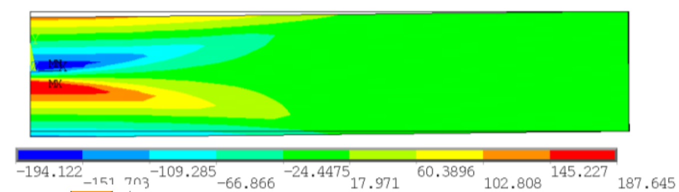
Equivalent plastic strain
(100% of load)

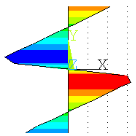




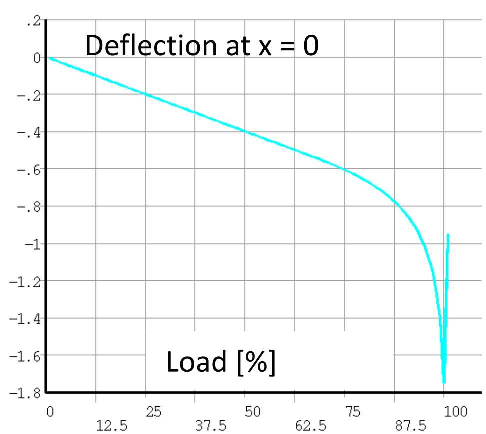


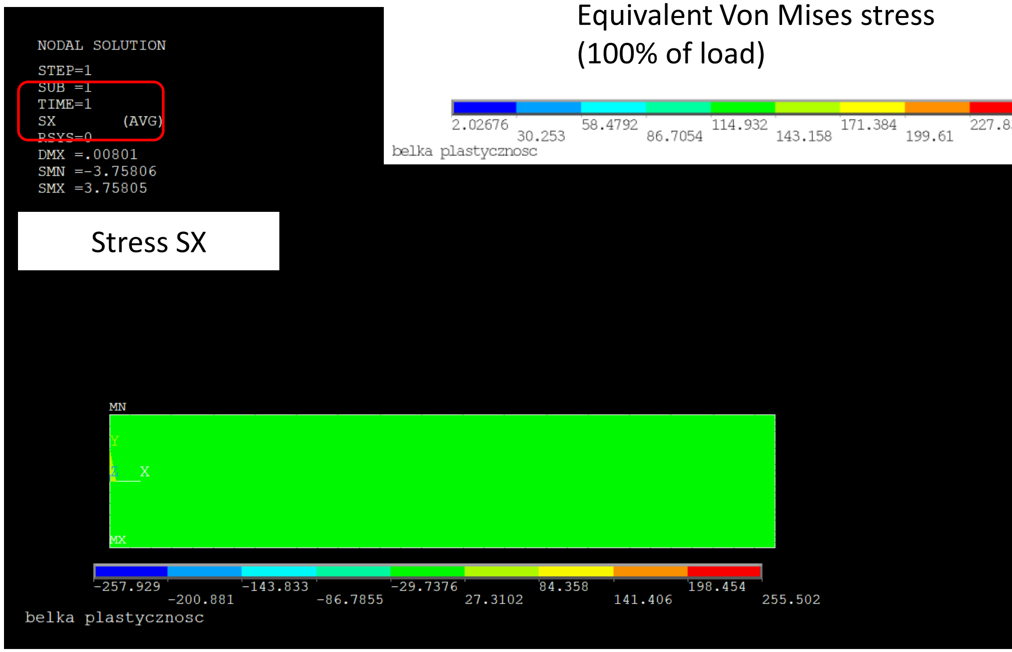
Equivalent Von Mises stress
(100% of load)





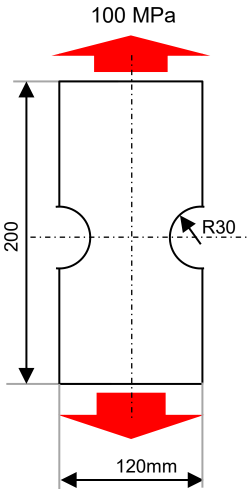
Residual stress after unloading





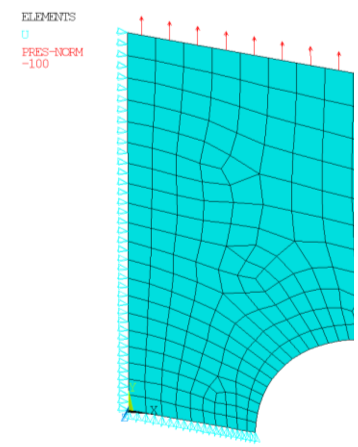
Example 2a. 2-D structures with a notch *(linear elastic model)*

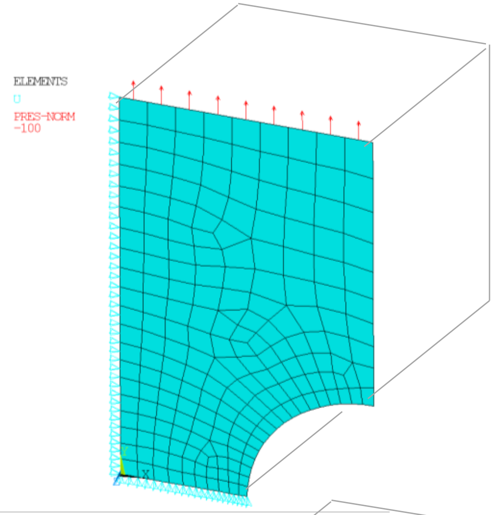
Plane stress | Plane strain | Axis-symmetry

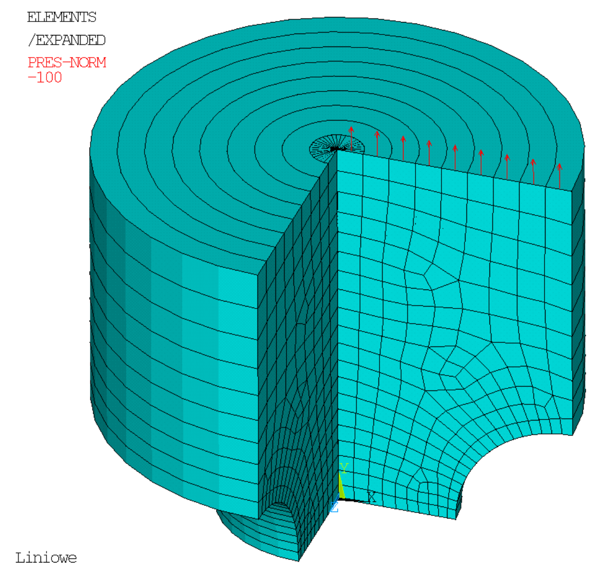


E=7*10⁴ MPa

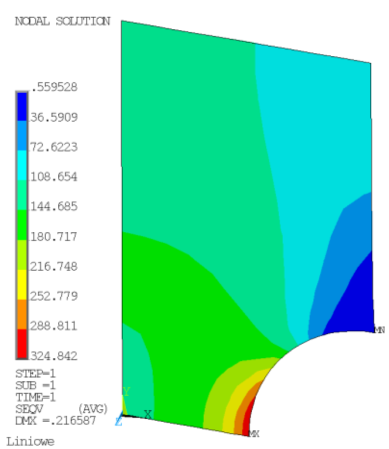
ν = 0.32

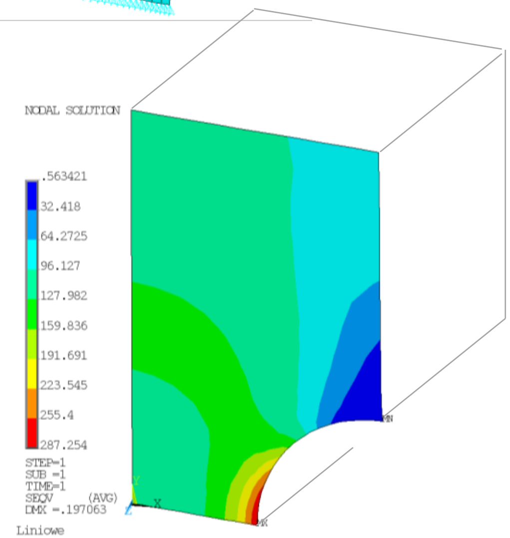


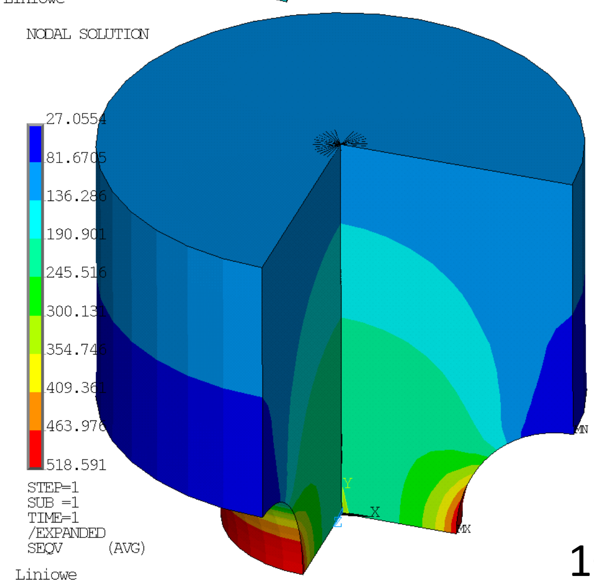




Solid183



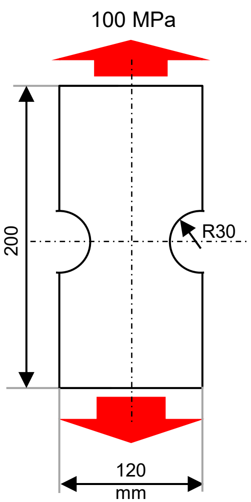


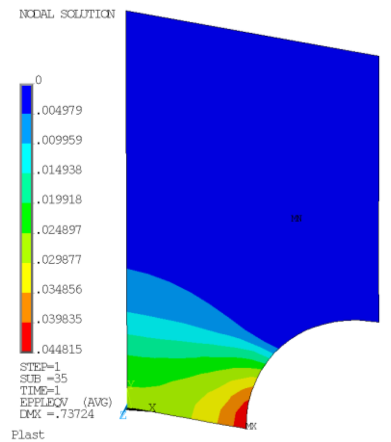


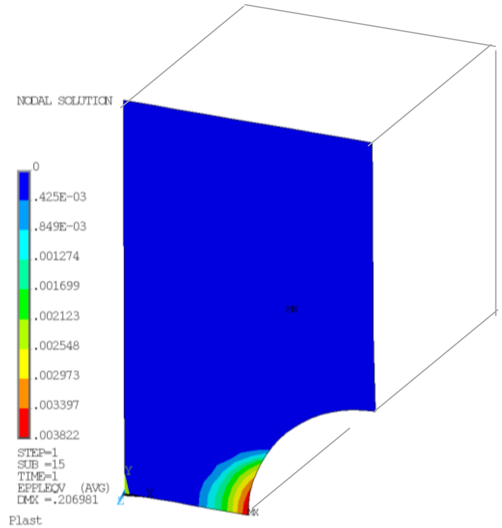
Equivalent Von Mises stress

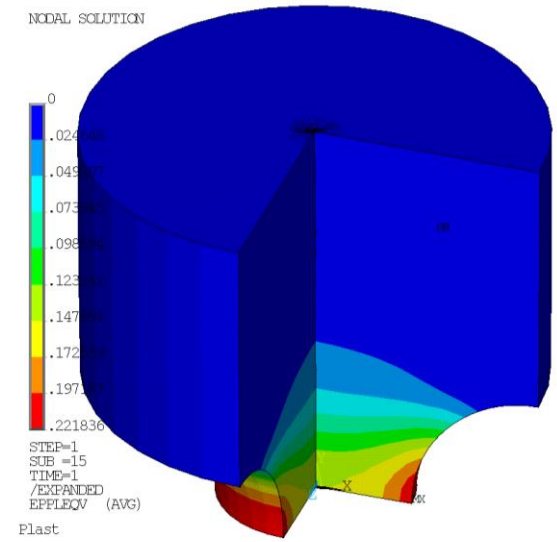
Example 2b. 2-D structures with a notch *(elasto-plastic model with hardening)*

Plane stress	Plane strain	Axis-symmetry









Solid183

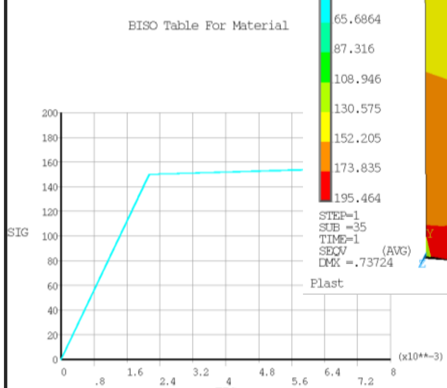
Equivalent Von Mises strain

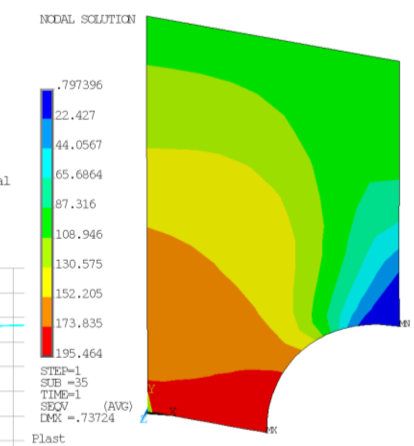
E=7·10⁴ MPa

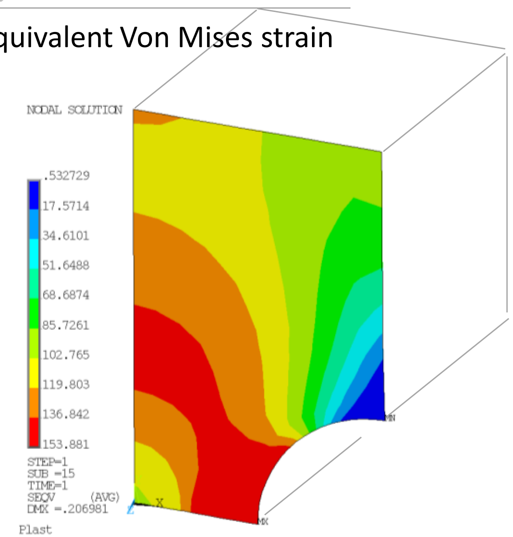
ν = 0.32

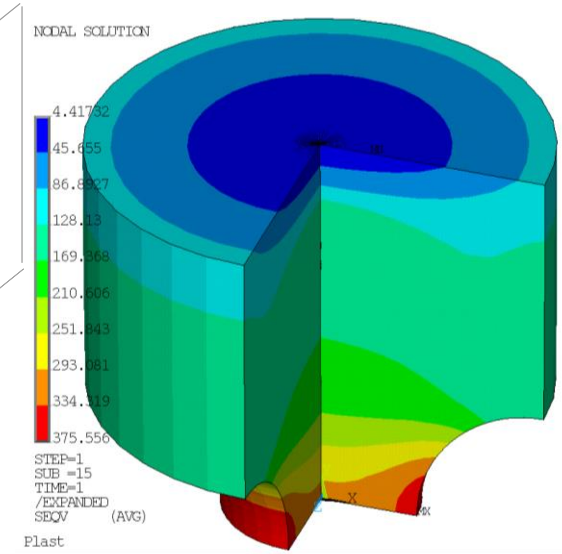
$R_{0,2}$ = 150 MPa

Eu=1000 MPa









Equivalent Von Mises stress