Maximum Rate of Growth of Enstrophy in the Navier-Stokes System on 2D Bounded Domains

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Given $\nu > 0$, $d = 3$, prove (or disprove) the existence and smoothness of the solution of

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} + \nabla p - \nu \Delta \mathbf{v} = 0$$

in $\Omega \subset \mathbb{R}^d$, 

$$\nabla \cdot \mathbf{v} = 0$$

in $\Omega \subset \mathbb{R}^d$, 

$$\mathbf{v} \cdot \mathbf{n} = \mathbf{v} \cdot \mathbf{s} = 0$$

on $\partial \Omega$, 

$$\mathbf{v}(\mathbf{x}, t = 0) = \mathbf{v}_0(\mathbf{x})$$

in $\Omega \subset \mathbb{R}^d$, 

for all $t > 0$.

Award: $1M$.

Clay Mathematics Institute:
http://www.claymath.org/sites/default/files/navierstokes.pdf
Solutions of the 2D Periodic NSE are analytic in time, but in the 3D periodic case this is true only for a very small interval of time (Foias and Temam, 1989),

If the amplitude of $v_0$ is sufficiently small, then unique and smooth solutions are proven to exist for all time (Foias and Temam, 1989),

The first-ever estimate showing how rapidly the enstrophy can grow in a 3D periodic setting (Lu and Doering, 2008),

There exist a couple of similar results involving both the Periodic Burgers Equation (Ayala and Protas, 2011) and the 2D Periodic NSE (Ayala and Protas, 2014),
Bounded domains may lead to a finite-time blow-up in the case of the 3D Euler Equation (Hou and Luo, 2014),
Lack of analogous results of the 3D Navier-Stokes,
Lack of relevant estimates/computational results for the 2D/3D Navier-Stokes.
The two-dimensional vorticity transport equation with no-slip boundary conditions:

\[
\frac{\partial \omega}{\partial t} + (\mathbf{v} \cdot \nabla) \omega = \nu \Delta \omega \quad \text{in } \Omega \subset \mathbb{R}^2, \\
\mathbf{v} \cdot \mathbf{n} = \mathbf{v} \cdot \mathbf{\tau} = 0 \quad \text{on } \partial \Omega,
\]

where

\[
\mathbf{v}(x, y, t) = [u(x, y, t), v(x, y, t)], \\
\omega(x, y, t) = \nabla^\perp \mathbf{v} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}.
\]
Streamfunction

- Velocity vs. streamfunction
  \[ \mathbf{v} = \nabla \times \psi \mathbf{k}, \]

- Streamfunction vs. vorticity
  \[ \Delta \psi = -\omega, \]

- Boundary conditions for the streamfunction
  \[ \psi = \frac{\partial \psi}{\partial n} = 0, \]

- Zero mean property
  \[ \int_{\Omega} \omega d\Omega = -\int_{\Omega} \Delta \psi d\Omega = \int_{\partial \Omega} \frac{\partial \psi}{\partial n} d\sigma = 0. \]
Enstrophy and its Growth Rate

Enstrophy as an $L_2$-norm of the vorticity:

\[ E(\omega) = \frac{1}{2} \int_{\Omega} \omega^2 d\Omega. \]

Instantaneous rate of growth of enstrophy

\[ \frac{dE}{dt} = \frac{1}{2} \frac{d}{dt} \int_{\Omega} \omega^2 d\Omega = \int_{\Omega} \omega \frac{\partial \omega}{\partial t} d\Omega = -\int_{\Omega} \omega (v \cdot \nabla) \omega d\Omega \]

\[ + \nu \int_{\Omega} \omega \nabla \omega d\Omega \]
Since we impose no-slip boundary conditions on the velocity field,

\[ \int_{\Omega} \omega (\mathbf{v} \cdot \nabla) \omega d\Omega = 0. \]

Therefore,

\[ \frac{d\mathcal{E}}{dt} = \nu \int_{\Omega} \omega \Delta \omega d\Omega. \]
Given the initial value of the enstrophy, $E_0$, we want to maximize

$$\mathcal{J}(\omega) = \nu \int_{\Omega} \omega \Delta \omega \, d\Omega,$$

subject to

$$\frac{1}{2} \int_{\Omega} \omega^2 \, d\Omega = E_0,$$
$$\Delta \psi = -\omega \quad \text{in } \Omega,$$
$$\psi = \frac{\partial \psi}{\partial n} = 0 \quad \text{on } \partial \Omega.$$
Gradient-Based Method

Goal: To find the vorticity field $\tilde{\omega}$ that maximizes $\mathcal{J}$,
Solution: Steepest-Ascent Method,

$$\omega^{(n+1)} = \omega^{(n)} + \tau_n \nabla^H \mathcal{J}(\omega^{(n)}),$$
$$\omega^{(1)} = \omega_0,$$

where

$$\tau_n = \arg\max_{\tau > 0} P \left( \omega^{(n)} + \tau \nabla^H \mathcal{J}(\omega^{(n)}) \right).$$
Riesz Representation Theorem:

\[ \mathcal{J}'(\omega, \omega') = \left\langle \nabla^{H^1} \mathcal{J}(\omega), \omega' \right\rangle_{H^1} \]

Expand the inner product:

\[ \left\langle \nabla^{H^1} \mathcal{J}(\omega), \omega' \right\rangle_{H^1} = \int_{\Omega} \left[ (\text{Id} - \Delta) \nabla^{H^1} \mathcal{J}(\omega) \right] \omega' d\Omega + \int_{\partial \Omega} \frac{\partial}{\partial n} \left[ \nabla^{H^1} \mathcal{J}(\omega) \right] \omega' d\sigma. \]
Computing the Gradient

Use the definition of $\mathcal{J}$ and perturb it,

$$\mathcal{J}'(\omega, \omega') = \int_{\Omega} 2\nu \Delta \omega \omega' d\Omega - \nu \oint_{\partial \Omega} \omega' \frac{\partial \omega}{\partial n} - \omega \frac{\partial \omega'}{\partial n} d\sigma$$

$$\int_{\Omega} 2\nu \Delta \omega \omega' d\Omega - \nu \oint_{\partial \Omega} \omega' \frac{\partial \omega}{\partial n} d\sigma + \oint_{\partial \Omega} \frac{\partial \omega}{\partial s} p' d\sigma.$$  

Compare two integrals.
Computing the Gradient

Remedy:

- Use the Poisson Pressure Equation (PPE),
- Define $f(\omega)$, st.

\[
\Delta f = 0 \quad \text{in } \Omega,
\]
\[
\frac{\partial f}{\partial n} = \frac{\partial \omega}{\partial s} \quad \text{on } \partial \Omega,
\]

- Define $k$, s.t.

\[
\Delta k = \frac{\partial^2}{\partial x \partial y} (fs_{11}) + \left( \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial x^2} \right) (fs_{12}) \quad \text{in } \Omega,
\]
\[
k = \text{arbitrary} \quad \text{on } \partial \Omega.
\]
Computing the Gradient

Derive the final form of $J'$,

$$J' (ω, ω') = ν \int_Ω [2Δω + k + fω] ω' dΩ - ν \int_{∂Ω} \left[ \frac{∂f}{∂s} + \frac{∂ω}{∂n} \right] ω' dσ,$$

and, by comparison,

$$(\text{Id} - Δ) \nabla^H J (w) = ν (2Δω + k + fω) \text{ in } Ω,$$

$$\frac{∂}{∂n} \nabla^H J (w) = -ν \left( \frac{∂f}{∂s} + \frac{∂ω}{∂n} \right) \text{ on } ∂Ω.$$
Euler-Lagrange Equations

Augment the cost functional,

\[ J_A(\omega) = J(\omega) + \lambda \left( \frac{1}{2} \int_\Omega \omega^2 d\Omega - E_0 \right) + \int_\Omega \varphi(\Delta \psi + \omega) d\Omega, \]

perturb it, and set all terms proportional to \( \omega' \) to 0...
\begin{align*}
2\nu \Delta \omega + \lambda \omega + \varphi + \omega & \left[ \int_{\partial \Omega} \frac{\partial \omega}{\partial s} G(x, x') ds \right] \\
+ \int_{\partial \Omega} \frac{\partial \omega}{\partial s} \int_{\Omega} G(x, x'') \left[ 4 \frac{\partial u}{\partial x} \left( D_1 \cdot K(x'', x') + D_1 \cdot \nabla L(x'', x') \right) \right. \\
+ \left. \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \left( D_2 \cdot K(x'', x') + D_2 \cdot \nabla L(x'', x') \right) \right] d\sigma d\Omega = 0 \Omega,
\end{align*}

\begin{align*}
\Delta \psi &= -\omega \quad \Omega, \\
\Delta \varphi &= 0 \quad \Omega, \\
\psi &= \frac{\partial \psi}{\partial n} = 0 \partial \Omega, \\
\frac{\partial \omega}{\partial n} &= \frac{\partial}{\partial s} M^* \frac{\partial \omega}{\partial s} \partial \Omega, \\
\frac{1}{2} \int_{\Omega} \omega^2 d\Omega &= \mathcal{E}_0 \quad \text{(initial enstrophy constraint)}. \end{align*}
Chebyshev Collocation Method

- Chebyshev approximation,
  \[ u(x_i) = u_N(x_i) = \sum_{k=0}^{N} \hat{u}_k T_k(x_i), \quad i = 0, \ldots, N, \]

- Gauss-Lobatto grid,
  \[ x_i = \cos \left( \frac{\pi i}{k} \right), \quad i = 0, \ldots, k, \]

- Chebyshev polynomial,
  \[ T_k(x) = \cos(k \cos^{-1} x), \quad k = 0, 1, 2, \ldots \]
Differentiation

- Two methods of differentiation: in Chebyshev space and in real space,

- The latter prevents us from aliasing errors and Chebyshev transforms/cosine FFT (cost $N \log N$),

- Differentiation matrix is full and poorly-conditioned.

- Differentiation in real space in flexible,

  \[ u' = D_N u, \]
Gauss-Lobatto/Gauss-Lobatto-Fourier Grid
Spectral Accuracy

Error decreases as $O(c^N)$, $0 < c < 1$. 
\( \kappa(\epsilon) = \epsilon^{-1} \left( \mathcal{J}(\omega + \epsilon\omega') - \mathcal{J}(\omega) \right) \langle \nabla^{H^1} \mathcal{J}(\omega), \omega' \rangle_{H^1} \).
Initial Vorticity Field

How to find the initial vorticity field $\omega_0$?

Solve

$$-\Delta \omega - \varphi = \lambda \omega,$$
$$-\Delta \psi - \omega = 0,$$
$$-\Delta \varphi = 0,$$
$$-\Delta p^* = 0,$$

with the following boundary conditions:

$$\psi = \frac{\partial \psi}{\partial n} = 0,$$
$$\frac{\partial \omega}{\partial n} = -\frac{\partial p^*}{\partial s},$$
$$\frac{\partial \omega}{\partial s} = \frac{\partial p^*}{\partial n}.$$

The above eigenvalue problem is derived from the original Euler-Lagrange system.
Sparsity diagrams

Generalized Eigenvalue Problem,

$$Ay = \lambda By.$$
Initial Vorticity Field - Square Domain

- $\lambda_1 = 16.96$
- $\lambda_2 = 27.95$
- $\lambda_3 = 27.95$
- $\lambda_4 = 39.48$
Initial Vorticity Field - Circular Domain

\[ \lambda_1 = 14.68 \]
\[ \lambda_2 = 29.61 \]
\[ \lambda_3 = 29.61 \]
\[ \lambda_4 = 46.36 \]
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Motivation
Preliminaries
Maximum Enstrophy Growth as an Optimization Method
Numerical Methods and Results

Chebyshev Collocation Method
\(\kappa\)-test
dE/dt

max \(\text{dE/dt}\)

max \(\text{dE/dt}\) vs. \(E_0\), N=48, Square

1st branch
2nd branch
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**Motivation**

**Preliminaries**

**Maximum Enstrophy Growth as an Optimization Method**

**Chebyshev Collocation Method**

**κ-test**

**dE/dt**

\[ \max \frac{dE}{dt} \]

\[ \text{max } \frac{dE}{dt} \text{ vs. } E_0, N=48, \text{ Square, Logarithmic} \]

1st branch
2nd branch

\[ \sim E_0^2 \]

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Extreme States of the Vorticity

\begin{align*}
    E_0 &= 0.001 \\
    E_0 &= 0.01
\end{align*}
Extreme States of the Vorticity

$E_0 = 0.1$

$E_0 = 1$
Thank you!