LECTURE 14

NORMAL SHOCK WAVE
Gas dynamics admits existence of strongly discontinuous flow. The normal shock wave is the simplest example of such flow.

\[ \partial \Omega = S_0 \cup S_1 \cup S_2 \]

Conservation laws for the NSW

1. **Mass**

\[ \int_{\partial \Omega} \rho u_n \, ds = 0 \quad \Rightarrow \quad \rho_1 u_1 = \rho_2 u_2 \]

1D case

2. **Linear momentum**

\[ \int_{\partial \Omega} (\rho u_n v + p n) \, ds = 0 \quad \Rightarrow \quad \rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2 \]

1D case

3. **Energy**

\[ \frac{\kappa p_1}{(\kappa - 1) \rho_1} + \frac{1}{2} u_1^2 = \frac{\kappa p_2}{(\kappa - 1) \rho_2} + \frac{1}{2} u_2^2 \]
Divide (2) by (1) …

\[ u_1 + \frac{p_1}{\rho_1 u_1} = u_2 + \frac{p_2}{\rho_2 u_2} \Rightarrow u_1 - u_2 = \frac{p_2}{\rho_2 u_2} - \frac{p_1}{\rho_1 u_1} \]  

(4)

New form of (3) is

\[ (u_1 - u_2)(u_1 + u_2) = \frac{2\kappa}{\kappa - 1} \left( \frac{p_2}{\rho_2} - \frac{p_1}{\rho_1} \right) \]  

(5)

Using (4) the Eq. (5) can be rewritten as

\[ \left( \frac{p_2}{\rho_2 u_2} - \frac{p_1}{\rho_1 u_1} \right)(u_1 + u_2) = \frac{2\kappa}{\kappa - 1} \left( \frac{p_2}{\rho_2} - \frac{p_1}{\rho_1} \right) \]  

(6)

The LHS of (6) can be transformed using the mass conservation equation (1) …

\[ \text{LHS}_{(6)} = \frac{p_2}{\rho_2 u_2} y_1 + \frac{p_2}{\rho_2 y_2} - \frac{p_1}{\rho_1 y_1} y_1 - \frac{p_1}{\rho_1 u_1} y_2 = \frac{p_2}{\rho_1} + \frac{p_2}{\rho_2} - \frac{p_1}{\rho_1} - \frac{p_1}{\rho_2} \]

\[ = \rho_1 y_1 \]

\[ = \rho_1 y_1 \]

\[ = \rho_2 y_2 \]
Thus, we get from (6) 
\[
\frac{p_2}{\rho_1} + \frac{p_2}{\rho_2} - \frac{p_1}{\rho_1} - \frac{p_1}{\rho_2} = \frac{2\kappa}{\kappa - 1}\left(\frac{p_2}{\rho_2} - \frac{p_1}{\rho_1}\right) \tag{7}
\]

We multiply (7) by \(\rho_1/p_1\) and get the formula which involves only the ratios \(\rho_1/\rho_2\) and \(p_2/p_1\) .... 

\[
\begin{align*}
\frac{p_2}{p_1} &= \frac{\kappa+1}{\kappa-1} \frac{\rho_1}{\rho_2} - 1 = \frac{\kappa+1}{\kappa-1} \frac{\rho_2}{\rho_1} - 1 \\
&= \frac{\kappa+1}{\kappa-1} \frac{\rho_2}{\rho_1} - 1
\end{align*}
\tag{8}
\]

We have obtained the formula describing the thermodynamic process which affects the gas passing through the NSW. Note that it is different that the isentropic process! We call the above formula the Hugoniot Adiabat (HA).

Let us analyze some properties of the HA. Note that for \(\frac{\rho_2}{\rho_1} = \frac{\kappa+1}{\kappa-1} > 1\) we have the vertical asymptote. If \(\kappa = 1.4\) then corresponding ratio is \(\frac{\rho_2}{\rho_1} = 6\).

Thus, at the shock wave the density of gas always increases but never more than \(\frac{\kappa+1}{\kappa-1}\) times.
For brevity we introduce $y = p_2/p_1$, $x = \rho_2/\rho_1$ and $\alpha = \frac{\kappa+1}{\kappa-1}$. The formula (8) can be now written as

$$y(x) = \frac{\alpha x - 1}{\alpha - x}$$

Then, we have

$$y'(x) = \frac{(\alpha + l)(\alpha - 1)}{(\alpha - x)^2} \Rightarrow y'(I) = \frac{\alpha + l}{\alpha - 1} = \kappa$$

Note that in case of the isentropic process, we have $y(x) = x^\kappa$, so $y'(x) = \kappa x^{\kappa-1}$ and $y'(I) = \kappa$. Moreover

$$y''_{Hugoniot}(x) = \frac{2(\alpha + l)(\alpha - 1)}{(\alpha - x)^3} \Rightarrow y''_{Hugoniot}(I) = \kappa(\kappa - 1)$$

$$y''_{isentropic}(x) = \kappa(\kappa - 1) x^{\kappa-2} \Rightarrow y''_{isentropic}(I) = \kappa(\kappa - 1)$$

We see that the isentropic and normal shock wave adiabats fit very well to each other in the vicinity $x = I$ (they have the same values of $y(I)$, $y'(I)$ and $y''(I)$). We say that these lines are strictly tangent at $x = I$. 
Physically it means that weak shock waves are nearly isentropic and

\[
\frac{p_2}{p_1} \bigg|_{\text{Hugoniot}} - \frac{p_2}{p_1} \bigg|_{\text{isentropic}} = C \left( \frac{\rho_2}{\rho_1} - 1 \right)^3 + \text{h.o.t}
\]
From the 1st Principle of Thermodynamics we have
\[ ds = \frac{dq}{T} = \frac{1}{T} (di - \frac{dp}{\rho}) = c_p \frac{dT}{T} - \frac{dp}{T \rho} = c_p \frac{dT}{T} - R \frac{dp}{p} \]

From the Clapeyron equation . . . .
\[ p = \rho RT \Rightarrow \ln p = \ln \rho + \ln T + \text{const} \]

Thus
\[ \frac{dp}{p} = \frac{d\rho}{\rho} + \frac{dT}{T} \Rightarrow \frac{dT}{T} = \frac{dp}{p} - \frac{d\rho}{\rho} \]

and the differential of (mass specific) entropy can be written as
\[ ds = c_p \left( \frac{dp}{p} - \frac{d\rho}{\rho} \right) - R \frac{dp}{p} = c_p \frac{dp}{p} - c_v \frac{d\rho}{\rho} \]

After integration we get
\[ s = c_v \ln p - c_p \ln \rho + \text{const} = c_v \ln(p / \rho^\kappa) + \text{const} = \kappa c_v \]

Thus, the change of entropy between two thermodynamic states can be expressed as follows

\[
\frac{\Delta s}{c_v} \equiv \frac{s_2 - s_1}{c_v} = \ln\left(\frac{p_2}{\rho_2^\kappa}\right) - \ln\left(\frac{p_1}{\rho_1^\kappa}\right) = \ln\left(\frac{p_2}{p_1}\right) - \ln\left(\frac{\rho_2^\kappa}{\rho_1^\kappa}\right)
\]

Note that for the Hugoniot adiabat we have

\[
\frac{\rho_2}{\rho_1} < 1 \implies \left.\frac{p_2}{p_1}\right|_{\text{Hugoniot}} < \left(\frac{\rho_2}{\rho_1}\right)^\kappa \implies \Delta s < 0
\]

i.e., the shock wave cannot expand the gas (it would lead to entropy decrease which contradicts the 2\textsuperscript{nd} Principle of Thermodynamics). Thus the (nontrivial) shock wave must be a compression wave! Indeed, in such case

\[
\frac{p_2}{p_1} > \left(\frac{\rho_2}{\rho_1}\right)^\kappa \implies \Delta s > 0
\]
We further observe that \[
\frac{\rho_2}{\rho_1} > 1 \text{ and } \rho_1 u_1 = \rho_2 u_2 \implies u_2 = \left(\frac{\rho_2}{\rho_1}\right)^{-1} u_1 < u_1
\]

Using the energy equation (integral) we also conclude that
\[
T_2 - T_1 = (u_1^2 - u_2^2) / 2c_p > 0 \implies T_2 > T_1
\]
i.e., after crossing the shock wave the gas warms up.

We would like to know what kind of flow exists at different sides of the NSW.

Writing the energy equation in the following form
\[
\frac{1}{2} u^2 + \frac{\kappa p}{(\kappa - 1)\rho} = \left(\frac{1}{2} u^2 + \frac{\kappa p}{(\kappa - 1)\rho}\right)_{M=1} \equiv \frac{\kappa + 1}{2(\kappa - 1)} a_*^2
\]
we obtain the following
\[
\frac{1}{2} u_1 + \frac{\kappa p_1}{(\kappa - 1)u_1 \rho_1} = \frac{\kappa + 1}{2(\kappa - 1)} a_*^2
\]
\[
\frac{1}{2} u_2 + \frac{\kappa p_2}{(\kappa - 1)u_2 \rho_2} = \frac{\kappa + 1}{2(\kappa - 1)} a_*^2
\]
Subtracting the above equations we get

\[ u_1 - u_2 + \frac{2\kappa}{\kappa - 1} \left( \frac{p_1}{\rho_1 u_1} - \frac{p_2}{\rho_2 u_2} \right) = \frac{\kappa + 1}{\kappa - 1} \left( \frac{a_2^2}{u_1} - \frac{a_2^2}{u_2} \right) \]

\[ = u_2 - u_1 \text{(Eq.4)} \]

After some algebra we derive the Prandtl’s Relation

\[ u_2 - u_1 = \frac{u_2 - u_1}{u_1 u_2} a_2^2 \quad \text{and} \quad u_2 \neq u_1 \quad \Rightarrow \quad u_1 u_2 = a_2^2 \]

The immediate conclusion from PR is that \( u_1 > a_* \quad \text{and} \quad u_2 < a_* \)

What about the Mach number?

The energy equation

\[ \frac{1}{2} u^2 + \frac{a^2}{\kappa - 1} = \frac{\kappa + 1}{2(\kappa - 1)} a_*^2 \]

can be divided by the square of velocity to obtain

\[ 1 + \frac{2}{\kappa - 1} \frac{1}{M^2} = \frac{\kappa + 1}{\kappa - 1} \frac{a_*^2}{u^2} \]
After simple manipulations we have
\[
\left( \frac{u}{a_*} \right)^2 = \frac{\kappa + 1}{\kappa - 1 + \frac{2}{M^2}}
\]

From the above the following we infer
\[
u_1 > a_* \Rightarrow M_1 > 1 \quad \text{and} \quad u_2 < a_* \Rightarrow M_2 < 1
\]

Thus, the flow in front of the NSW is always supersonic, while the flow behind it – always subsonic.

The quantity \( \lambda = \frac{u}{a_*} \) is called the velocity coefficient. In contrast to the Mach number, the velocity coefficient assumes values in the bounded interval, namely
\[
\lim_{M_1 \to \infty} \lambda_1 = \sqrt{\frac{\kappa + 1}{\kappa - 1}}, \quad \lim_{M_1 \to \infty} \lambda_2 = \sqrt{\frac{\kappa - 1}{\kappa + 1}}
\]

The Mach number of the flow behind the wave \( M_2 \) can be expressed as the function of the Mach number in front of the wave \( M_1 \). To obtain this formula we use the Prandtl’s Relation …
\[
\frac{u_1}{a_*} \frac{u_2}{a_*} = 1 \quad \Rightarrow \quad \sqrt{\frac{\kappa+1}{\kappa-1 + \frac{2}{M_1^2}}} \sqrt{\frac{\kappa+1}{\kappa-1 + \frac{2}{M_2^2}}} = 1
\]

After some algebra we get

\[
M_2 = \frac{\sqrt{2 + (\kappa-1)M_1^2}}{\sqrt{2\kappa M_1^2 - \kappa + 1}} < 1
\]

Similarly, we can expressed the ratios of density, pressure and temperature values. The density ratio can be evaluated as follows

\[
\frac{\rho_2}{\rho_1}(M_1) = \frac{u_1}{u_2}(M_1) = \frac{M_1}{M_2(M_1)} \frac{a_1}{a_2}(M_1) = \frac{M_1}{M_2(M_1)} \left( \frac{a}{a_0} \right)_{is} (M_1) \left( \frac{a}{a_0} \right)_{is}^{-1} [M_2(M_1)] > 1
\]
To evaluate the pressure ratio (as the function of $M_1$) we rewrite the momentum equation in the following way

$$
p + \rho u^2 = p \left(1 + \frac{u^2}{p/\rho}\right) = p \left(1 + \frac{\kappa u^2}{a^2}\right) = p(1 + \kappa M^2) = \text{const}
$$

Since the above expression has the same value at both sides of the shock wave, we get

$$\frac{p_2(M_1)}{p_1} = \frac{1 + \kappa M_1^2}{1 + \kappa M_2^2(M_1)} > 1$$

The flow through the shock is adiabatic (total energy is conserved) and the total (or stagnation) temperature $T_0$ remains the same.

Thus, we can write

$$\frac{T_2(M_1)}{T_1} = \frac{(T/T_0)(M_2)}{(T/T_0)[M_2(M_1)]} > 1$$

where the ratio $T/T_0$ can be calculated from the formula derived in the Lecture 13.

$$\frac{T}{T_0}(M) = \left(1 + \frac{\kappa - 1}{2} M^2\right)^{-1}$$
In the end, we will explore what happens to the stagnation pressure. Conceptually, we consider the process described as follows:

\[ p_{01} \rightarrow p_1 \rightarrow p_2 \rightarrow p_{02} < p_{01} \]

- acceleration (isentropic)
- shock wave
- deceleration (isentropic)
We claim that the total (stagnation) pressure diminishes on the shock wave.

The justification of this fact goes as follows.

We know that the entropy of the gas increases on the shock wave. The formula derived earlier can be written for stagnation parameters, namely

\[ 0 < \frac{s_2 - s_1}{c_v} = \ln \left( \frac{p_{02}}{p_{01}} \right) - \kappa \ln \left( \frac{\rho_{02}}{\rho_{01}} \right) \]

Since the total temperature at both sides is the same then

\[ T_{01} = T_{02} \equiv T_0 \quad \Rightarrow \quad \frac{\rho_{02}}{\rho_{01}} = \frac{p_{02}}{p_{01}} \]

Thus

\[ 0 < \frac{\Delta s}{c_v} = (1 - \kappa) \ln \left( \frac{p_{02}}{p_{01}} \right) \quad \Rightarrow \quad \frac{p_{02}}{p_{01}} < 1. \]