Maximum Rate of Growth of Enstrophy in the Navier-Stokes System on 2D Bounded Domains

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PW MEIL ZA Seminar, April 28, 2017

Millennium Problem

Given $\nu > 0$, $d = 3$, prove (or disprove) the existence and smoothness of the solution of

$$\begin{aligned} &\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} + \nabla p - \nu \Delta \mathbf{v} = 0 && \text{in } \Omega \subset \mathbb{R}^d, \\ &\nabla \cdot \mathbf{v} = 0 && \text{in } \Omega \subset \mathbb{R}^d, \\ &\mathbf{v} \cdot \mathbf{n} = \mathbf{v} \cdot \mathbf{s} = 0 && \text{on } \partial\Omega, \\ &\mathbf{v}(\mathbf{x}, t = 0) = \mathbf{v}_0(\mathbf{x}) && \text{in } \Omega \subset \mathbb{R}^d, \end{aligned}$$

for all $t > 0$.
Award: \$1M.
Clay Mathematics Institute:
http://www.claymath.org/sites/default/files/navierstokes.pdf

Literature Review - Periodic Domains

- Solutions of the 2D Periodic NSE are analytic in time, but in the 3D periodic case this is true only for a very small interval of time (Foias and Temam, 1989),
- If the amplitude of $\mathbf{v}_0$ is sufficiently small, then unique and smooth solutions are proven to exist for all time (Foias and Temam, 1989),
- The first-ever estimate showing how rapidly the enstrophy can grow in a 3D periodic setting (Lu and Doering, 2008),
- There exist a couple of similar results involving both the Periodic Burgers Equation (Ayala and Protas, 2011) and the 2D Periodic NSE (Ayala and Protas, 2014),

Literature Review - Bounded Domains

- Bounded domains may lead to a finite-time blow-up in the case of the 3D Euler Equation (Hou and Luo, 2014),
- Lack of analogous results of the 3D Navier-Stokes,
- Lack of relevant estimates/computational results for the 2D/3D Navier-Stokes.

The Vorticity Transport Equation

The two-dimensional vorticity transport equation with no-slip boundary conditions:

$$\begin{aligned} &\frac{\partial \omega}{\partial t} + (\mathbf{v} \cdot \nabla)\omega = \nu \Delta \omega && \text{in } \Omega \subset \mathbb{R}^2, \\ &\mathbf{v} \cdot \mathbf{n} = \mathbf{v} \cdot \boldsymbol{\tau} = 0 && \text{on } \partial\Omega, \end{aligned}$$

where

$$\begin{aligned} \mathbf{v}(x, y, t) &= [u(x, y, t), v(x, y, t)], \\ \omega(x, y, t) &= \nabla^{\perp}\mathbf{v} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}. \end{aligned}$$

Streamfunction

- Velocity vs. streamfunction

$$\mathbf{v} = \nabla \times \psi \mathbf{k},$$

- Streamfunction vs. vorticity

$$\Delta \psi = -\omega,$$

- Boundary conditions for the streamfunction

$$\psi = \frac{\partial \psi}{\partial n} = 0,$$

- Zero mean property

$$\int_\Omega \omega \mathrm{d}\Omega = -\int_\Omega \Delta \psi \mathrm{d}\Omega = \oint_{\partial\Omega} \frac{\partial \psi}{\partial n} \mathrm{d}\sigma = 0.$$

Enstrophy and its Growth Rate

Enstrophy as an L_2-norm of the vorticity:

$$\mathcal{E}(\omega) = \frac{1}{2}\int_\Omega \omega^2 \mathrm{d}\Omega.$$

Instantaneous rate of growth of enstrophy

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \omega^2 \mathrm{d}\Omega = \int_\Omega \omega \frac{\partial \omega}{\partial t}\mathrm{d}\Omega = -\int_\Omega \omega(\mathbf{v}\cdot\nabla)\omega \mathrm{d}\Omega$$

$$+\nu\int_\Omega \omega \nabla \omega \mathrm{d}\Omega$$

Enstrophy and its Growth Rate

Since we impose no-slip boundary conditions on the velocity field,

$$\int_\Omega \omega(\mathbf{v} \cdot \nabla)\omega \mathrm{d}\Omega = 0.$$

Therefore,

$$\boxed{\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} = \nu \int_\Omega \omega \Delta \omega \mathrm{d}\Omega}.$$

Optimization Problem

Given the initial value of the enstrophy, $\mathcal{E}_0$, we want to maximize

$$\mathcal{J}(\omega) = \nu \int_\Omega \omega \Delta \omega \mathrm{d}\Omega,$$

subject to

$$\begin{aligned} &\tfrac{1}{2}\int_\Omega \omega^2 \mathrm{d}\Omega = \mathcal{E}_0, && \\ &\Delta\psi = -\omega && \text{in } \Omega, \\ &\psi = \frac{\partial \psi}{\partial n} = 0 && \text{on } \partial\Omega. \end{aligned}$$

Gradient-Based Method

Goal: To find the vorticity field $\tilde{\omega}$ that maximizes $\mathcal{J}$,
Solution: Steepest-Ascent Method,

$$\omega^{(n+1)} = \omega^{(n)} + \tau_n \nabla^{H^1} \mathcal{J}(\omega^{(n)}),$$
$$\omega^{(1)} = \omega_0,$$

where

$$\tau_n = \operatorname{argmax}_{\tau>0} \ \mathrm{P}\left(\omega^{(n)} + \tau \nabla^{H^1} \mathcal{J}(\omega^{(n)})\right).$$

Computing the Gradient

Riesz Representation Theorem:

$$\mathcal{J}'(\omega, \omega') = \left\langle \nabla^{H^1} \mathcal{J}(\omega), \omega' \right\rangle_{H^1}$$

Expand the inner product:

$$\left\langle \nabla^{H^1} \mathcal{J}(\omega), \omega' \right\rangle_{H^1} = \int_\Omega \left[(\mathrm{Id} - \Delta) \nabla^{H^1} \mathcal{J}(\omega) \right] \omega' \mathrm{d}\Omega + \oint_{\partial\Omega} \frac{\partial}{\partial n} \left[\nabla^{H^1} \mathcal{J}(\omega) \right] \omega' \mathrm{d}\sigma.$$

Computing the Gradient

Use the definition of $\mathcal{J}$ and perturb it,

$$\mathcal{J}'(\omega,\omega') = \int_\Omega 2\nu\Delta\omega\omega' \mathrm{d}\Omega - \nu \oint_{\partial\Omega} \omega' \frac{\partial \omega}{\partial n} - \omega \frac{\partial \omega'}{\partial n} \mathrm{d}\sigma$$

$$\int_\Omega 2\nu\Delta\omega\omega' \mathrm{d}\Omega - \nu \oint_{\partial\Omega} \omega' \frac{\partial \omega}{\partial n} \mathrm{d}\sigma + \oint_{\partial\Omega} \frac{\partial \omega}{\partial s} p' \mathrm{d}\sigma.$$

Compare two integrals.

Computing the Gradient

Remedy:

- Use the Poisson Pressure Equation (PPE),
- Define $f(\omega)$, st.

$$\begin{aligned} \Delta f &= 0 && \text{in } \Omega, \\ \frac{\partial f}{\partial n} &= \frac{\partial \omega}{\partial s} && \text{on } \partial\Omega, \end{aligned}$$

- Define k, s.t.

$$\begin{aligned} \Delta k &= \frac{\partial^2}{\partial x \partial y}(f s_{11}) + \left(\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial x^2}\right)(f s_{12}) && \text{in } \Omega, \\ k &= \text{arbitrary} && \text{on } \partial\Omega. \end{aligned}$$

Computing the Gradient

Derive the final form of $\mathcal{J}'$,

$$\mathcal{J}'(\omega,\omega') = \nu \int_\Omega \left[2\Delta\omega + k + f\omega\right]\omega' \mathrm{d}\Omega - \nu \oint_{\partial\Omega} \left[\frac{\partial f}{\partial s} + \frac{\partial \omega}{\partial n}\right]\omega' \mathrm{d}\sigma,$$

and, by comparison,

$$\begin{aligned} (\mathrm{Id} - \Delta)\,\nabla^{H^1}\mathcal{J}(w) &= \nu\,(2\Delta\omega + k + f\omega) && \text{in } \Omega, \\ \frac{\partial}{\partial n}\nabla^{H^1}\mathcal{J}(w) &= -\nu\left(\frac{\partial f}{\partial s} + \frac{\partial \omega}{\partial n}\right) && \text{on } \partial\Omega. \end{aligned}$$

Euler-Lagrange Equations

Augment the cost functional,

$$\mathcal{J}_A(\omega) = \mathcal{J}(\omega) + \lambda \left(\frac{1}{2} \int_\Omega \omega^2 \mathrm{d}\Omega - \mathcal{E}_0 \right) + \int_\Omega \varphi \left(\Delta\psi + \omega \right) \mathrm{d}\Omega,$$

perturb it, and set all terms proportional to ω' to 0...

$$
\begin{aligned}
&2\nu\Delta\omega + \lambda\omega + \varphi + \omega\left[\oint_{\partial\Omega}\frac{\partial\omega}{\partial s}\mathbf{G}(\mathbf{x},\mathbf{x}')\mathrm{d}s\right] \\
&+ \oint_{\partial\Omega}\frac{\partial\omega}{\partial s}\int_{\Omega}\mathbf{G}(\mathbf{x},\mathbf{x}'')\left[4\frac{\partial u}{\partial x}\left(\mathbf{D}_1\cdot\mathbf{K}(\mathbf{x}'',\mathbf{x}') + \mathbf{D}_1\cdot\nabla\mathcal{L}(\mathbf{x}'',\mathbf{x}')\right)\right. && \Omega, \\
&\left.+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\left(\mathbf{D}_2\cdot\mathbf{K}(\mathbf{x}'',\mathbf{x}') + \mathbf{D}_2\cdot\nabla\mathcal{L}(\mathbf{x}'',\mathbf{x}')\right)\right]\mathrm{d}\sigma\mathrm{d}\Omega = 0 \\
&\Delta\psi = -\omega && \Omega, \\
&\Delta\varphi = 0 && \Omega, \\
&\psi = \frac{\partial\psi}{\partial n} = 0 && \partial\Omega, \\
&\frac{\partial\omega}{\partial n} = \frac{\partial}{\partial s}M^{*}\frac{\partial\omega}{\partial s} && \partial\Omega, \\
&\frac{1}{2}\int_{\Omega}\omega^2\mathrm{d}\Omega = \mathcal{E}_0 \quad \text{(initial enstrophy constaint).}
\end{aligned}
$$

Chebyshev Collocation Method

- Chebyshev approximation,

$$u(x_i) = u_N(x_i) = \sum_{k=0}^{N} \hat{u}_k T_k(x_i), \qquad i = 0, \ldots, N,$$

- Gauss-Lobatto grid,

$$x_i = \cos\left(\frac{\pi i}{k}\right), \qquad i = 0, \ldots, k,$$

- Chebyshev polynomial,

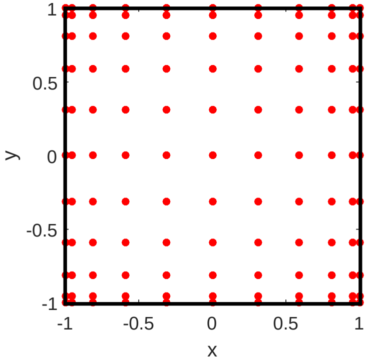
$$T_k(x) = \cos(k \cos^{-1} x), \qquad k = 0, 1, 2, \ldots$$

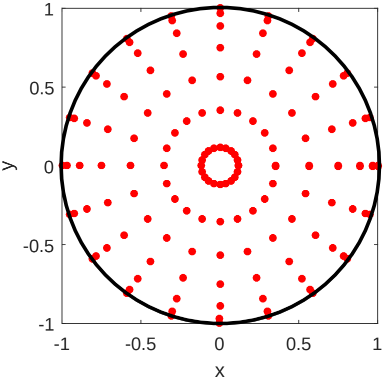
Differentiation

- Two methods of differentiation: in Chebyshev space and in real space,
- The latter prevents us from aliasing errors and Chebyshev transforms/cosine FFT (cost $N \log N$),
- Differentiation matrix is full and poorly-conditioned.
- Differentiation in real space in flexible,

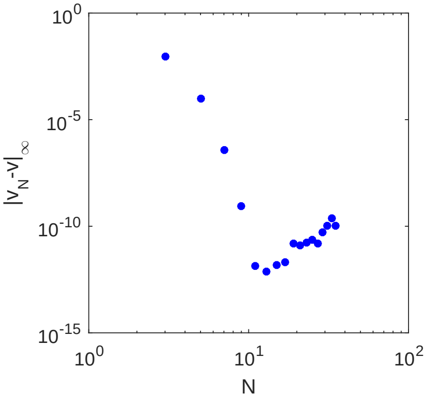
$$\mathbf{u}' = \mathbf{D}_N \mathbf{u},$$

Gauss-Lobatto/Gauss-Lobatto-Fourier Grid





Spectral Accuracy

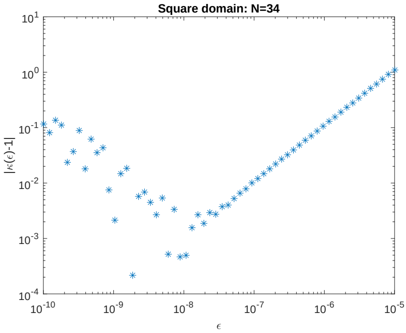


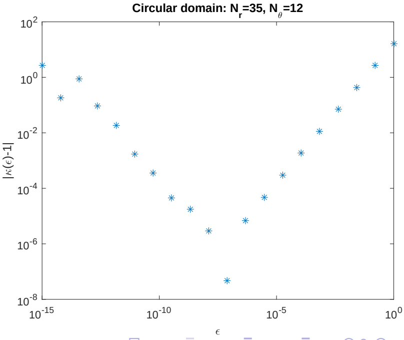
Error decreases as $\mathcal{O}(c^N)$, $0 < c < 1$.

κ-test

Check the correctness of the H^1 gradient,

$$\kappa(\epsilon) = \frac{\epsilon^{-1}\left(\mathcal{J}(\omega + \epsilon\omega') - \mathcal{J}(\omega)\right)}{\left\langle \nabla^{H^1}\mathcal{J}(\omega), \omega' \right\rangle_{H^1}}.$$





Initial Vorticity Field

How to find the initial vorticity field ω_0?
Solve

$$
\begin{aligned}
-\Delta\omega - \varphi &= \lambda\omega, \\
-\Delta\psi - \omega &= 0, \\
-\Delta\varphi &= 0, \\
-\Delta p^* &= 0,
\end{aligned}
$$

with the following boundary conditions:

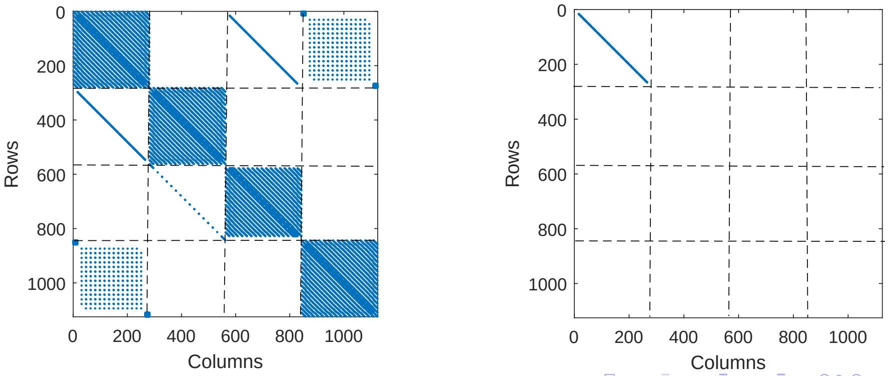
$$
\begin{aligned}
\psi = \frac{\partial\psi}{\partial n} &= 0, \\
\frac{\partial\omega}{\partial n} &= -\frac{\partial p^*}{\partial s}, \\
\frac{\partial\omega}{\partial s} &= \frac{\partial p^*}{\partial n}.
\end{aligned}
$$

The above eigenvalue problem is derived from the original Euler-Lagrange system.

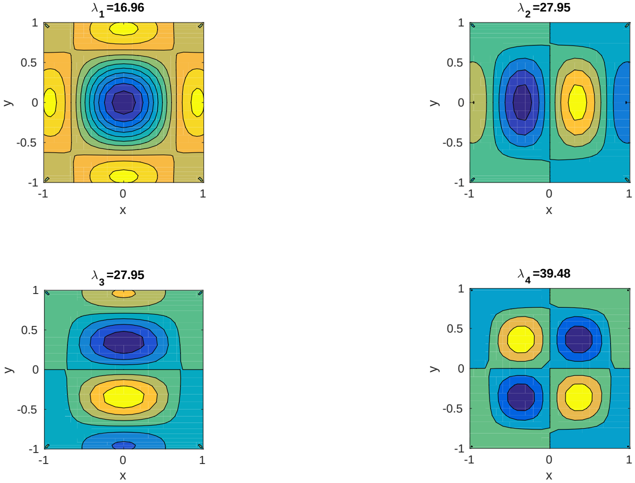
Sparsity diagrams

Generalized Eigenvalue Problem,

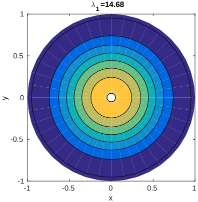
$$\mathbf{Ay} = \lambda \mathbf{By}.$$

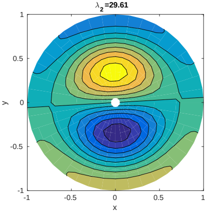


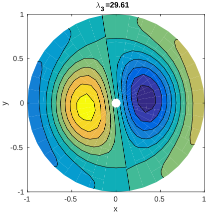
Initial Vorticity Field - Square Domain

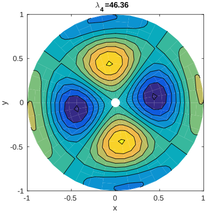


Initial Vorticity Field - Circular Domain

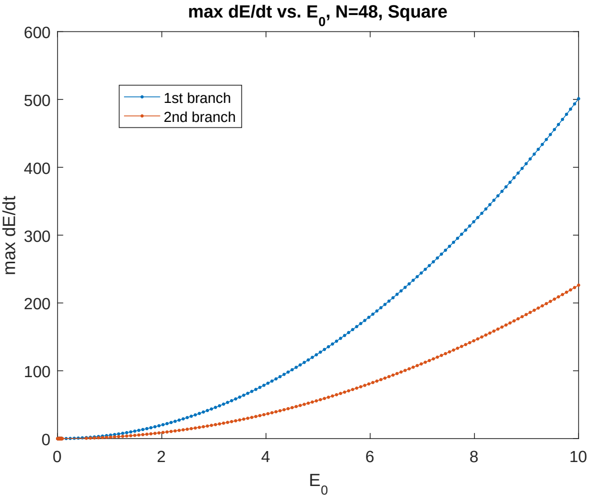




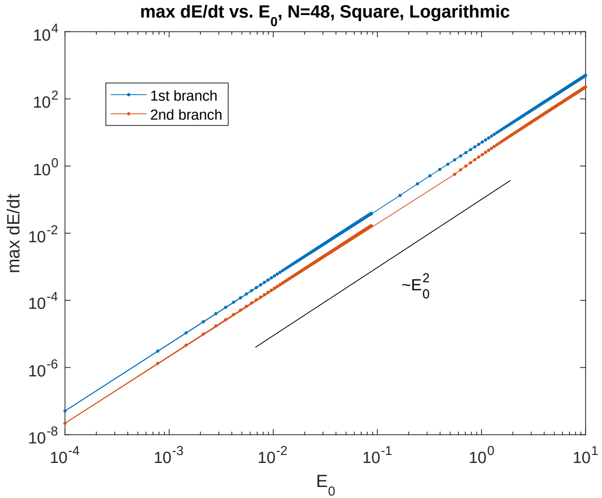




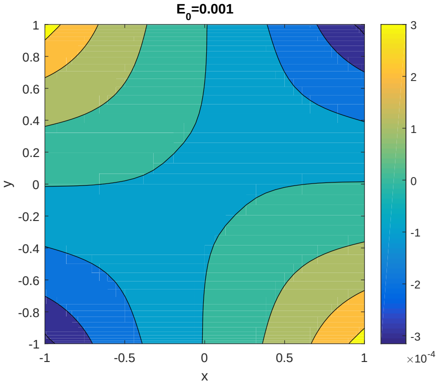
max dE/dt


max dE/dt vs. E0, N=48, Square
max dE/dt
E0
1st branch
2nd branch
0
100
200
300
400
500
600
0
2
4
6
8
10

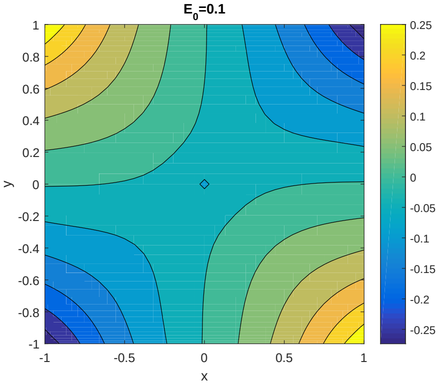
max dE/dt


max dE/dt vs. E0, N=48, Square, Logarithmic
1st branch
2nd branch
~E0^2
max dE/dt
E0
10^4
10^2
10^0
10^-2
10^-4
10^-6
10^-8
10^-4
10^-3
10^-2
10^-1
10^0
10^1

Extreme States of the Vorticity



Extreme States of the Vorticity



Thank you!