Aerodynamics I

Elements of gasdynamics

źródło:wikipedia.org

Elements of gasdynamics

Ideal gas

A collection of particles interacting with each other only by elastic collisions. Size of the particle is negligibly small in comparison with the mean free path.

Equation of state (Clapeyron):

$$pV = RT \tag{1.1}$$

variables of state:

p – pressure, T – temperature, V – volume

In gasdynamics we use specific volume (for unit of mass):

$$V = \frac{1}{\rho}$$

Equation of state (1.1) can be written:

$$\frac{p}{\rho} = RT \tag{1.2}$$

Perfect gas

Let us introduce two functions of state – internal energy e and enthalpy h:

$$h = e + \frac{p}{\rho} \tag{1.3}$$

For perfect gas, internal energy and enthalpy are linear functions of temperature:

$$e = e(T) = c_v\,T \quad \rightarrow \quad c_v = const \tag{1.4}$$

$$h = h(T) = c_p\,T \quad \rightarrow \quad c_p = const \tag{1.5}$$

Using (1.2):

$$\frac{p}{\rho} = R\,T = (c_p - c_v)\,T \quad \rightarrow \quad R = c_p - c_v \tag{1.6}$$

Let us define specific heat ratio $k = c_p/c_v$, then:

$$c_p = \frac{k\,R}{k-1} \qquad c_v = \frac{R}{k-1} \qquad h = \frac{k}{k-1}\frac{p}{\rho} \qquad e = \frac{1}{k-1}\frac{p}{\rho} \tag{1.7}$$

First law of thermodynamics

The change of the internal energy of a closed system is equal to the sum of the heat added to the system and work done by the system.

$$de = \delta q + \delta w \tag{1.8}$$

Types of thermodynamical processes:

- **adiabatic process** – no heat transfer between a system and its surroundings
- **reversible process** – it is possible after "reverted" process to recover initial state of gas. no dissipation (e.g., viscosity, conductivity)
- **isentropic process** – adiabatic and reversible. Entropy is preserved (see next slide)

For reversible process (no friction forces):

$$\delta w = -p\,dV \quad \rightarrow \quad de = \delta q - p\,dV \tag{1.9}$$

Second law of thermodynamics

First law of thermodynamics allows to determine a change of internal energy but not a direction of the process. E.g., transferring heat from A to B if $T_A < T_B$ does not violate the 1 L.T.

Let us introduce new function of state – entropy s:

$$ds = \frac{\delta q_{rev}}{T} \tag{1.10}$$

δq_{rev} – infinitesimal heat added to a system in reversible process.

For any process following holds:

$$ds = \frac{\delta q}{T} + ds_{irrev} \quad \rightarrow \quad \boxed{ds \geqslant \frac{\delta q}{T}} \tag{1.11}$$

in adiabatic process , where $\delta q = 0$:

$$\boxed{ds \geqslant 0} \tag{1.12}$$

(1.11) and (1.12) defines second law of thermodynamics:

Entropy of an isolated system must grow or stay preserved.

Second law of thermodynamics

In case of reversible process, using 1 L.T. (1.9) and (1.10):

$$T\,ds = de + p\,dV \tag{1.13}$$

using definition of enthalpy (1.3) and (1.13) :

$$dh = de + p\,dV + V\,dp \tag{1.14}$$

$$T\,ds = dh - V\,dp \tag{1.15}$$

Using equation of state (1.1) and equations (1.4), (1.5) :

$$ds = c_v \frac{dT}{T} + \frac{p\,dV}{T} = c_v \frac{dT}{T} + R\,\frac{dV}{V} \tag{1.16}$$

$$ds = c_p \frac{dT}{T} - \frac{V\,dp}{T} = c_p \frac{dT}{T} - R\,\frac{dp}{p} \tag{1.17}$$

Change of entropy for perfect gas in a process $1 \rightarrow 2$ can be defined as:

$$s_2 - s_1 = \int_{T_1}^{T_2} c_v \frac{dT}{T} + \int_{V_1}^{V_2} R\,\frac{dV}{V} = c_v \ln\frac{T_2}{T_1} + R\,\ln\frac{V_2}{V_1} \tag{1.18}$$

$$s_2 - s_1 = \int_{T_1}^{T_2} c_p \frac{dT}{T} - \int_{p_1}^{p_2} R\,\frac{dp}{p} = c_p \ln\frac{T_2}{T_1} - R\,\ln\frac{p_2}{p_1} \tag{1.19}$$

Isentropic process

Equation (1.19) can be transformed:

$$s_2 - s_1 = \frac{k\ R}{k-1}\ \ln\frac{p_2\ \rho_1}{p_1\ \rho_2} - R\ \ln\frac{p_2}{p_1} = \frac{R}{k-1}\ \left[k\ \ln\frac{p_2\ \rho_1}{p_1\ \rho_2} - (k-1)\ \ln\frac{p_2}{p_1}\right]$$

$$s_2 - s_1 = \frac{R}{k-1}\ \ln\left[\frac{p_2}{p_1}\left(\frac{\rho_1}{\rho_2}\right)^k\right] \qquad (1.20)$$

Relationships for change of temperature as a function of change of pressure or density can be obtained from (1.19) or (1.18). Eventually, for isentropic process ($s_2 - s_1 = 0$):

$$\frac{p_1}{p_2} = \left(\frac{\rho_1}{\rho_2}\right)^k = \left(\frac{T_1}{T_2}\right)^{\frac{k}{k-1}} \qquad (1.21)$$

Equations of motion for compressible inviscid fluid

For compressible fluid $\rho \neq const$!

Integral equations of motion for a control volume Ω bounded by boundary Γ (without body forces e.g., gravity):

Continuity equation

$$\frac{\partial}{\partial t}\int_\Omega \rho \, d\Omega + \oint_\Gamma \rho \, \mathbf{v}\cdot\mathbf{n} \, d\Gamma = 0 \qquad (1.22)$$

Momentum equation

$$\frac{\partial}{\partial t}\int_\Omega \rho \, \mathbf{v} \, d\Omega + \oint_\Gamma \rho \, \mathbf{v}\,(\mathbf{v}\cdot\mathbf{n}) \, d\Gamma = -\oint_\Gamma p \, \mathbf{n} \, d\Gamma \qquad (1.23)$$

Energy equation total energy $E = e + \frac{v^2}{2}$

$$\frac{\partial}{\partial t}\int_\Omega \rho \, E \, d\Omega + \oint_\Gamma \rho \, E \, \mathbf{v}\cdot\mathbf{n} \, d\Gamma = \underbrace{-\oint_\Gamma p \, \mathbf{v}\cdot\mathbf{n} \, d\Gamma}_{\text{work}} + \underbrace{\int_\Omega \rho \, \dot{q} \, d\Omega}_{\text{heat}} \qquad (1.24)$$

$\mathbf{v}$ – velocity
$\dot{q}$ – heat sources (usually we will assume $\dot{q} = 0$)

Equations of motion for compressible inviscid fluid

Differential equations in conservative form:

Continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \qquad (1.25)$$

Momentum equation

$$\frac{\partial (\rho \mathbf{v})}{\partial t} + \nabla \cdot (\mathbf{v} \otimes \rho \mathbf{v}) + \nabla p = 0 \qquad (1.26)$$

Energy equation

$$\frac{\partial (\rho E)}{\partial t} + \nabla \cdot (\mathbf{v}(\rho E + p)) = \rho\, \dot{q} \qquad (1.27)$$

Compressible vs. incompressible fluid in steady flow

Continuity equation

$$\nabla \cdot \mathbf{v} = 0 \quad \leftrightarrow \quad \nabla \cdot (\rho \mathbf{v}) = \rho \nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla \rho = 0 \tag{1.28}$$

Momentum equation

$$\nabla \cdot (\mathbf{v} \otimes \mathbf{v}) + \frac{1}{\rho} \nabla p = 0 \quad \leftrightarrow \quad \nabla \cdot (\mathbf{v} \otimes \rho \mathbf{v}) + \nabla p = 0 \tag{1.29}$$

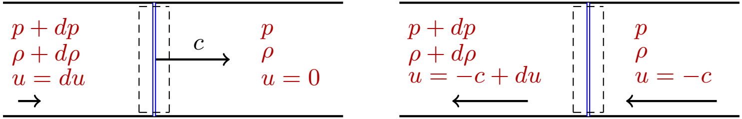
Energy equation

$$\text{none} \quad \leftrightarrow \quad \nabla \cdot (\rho E \, \mathbf{v}) + \nabla p = 0 \tag{1.30}$$

In case of incompressible flow, the Bernoulli (derived by integrating the momentum equation) equation can be treated as substitute of energy equation. It describes the change of the work of pressure forces into kinetic energy:

$$\frac{v^2}{2} + \frac{p}{\rho} = const \tag{1.31}$$

Speed of sound



external coordinate system *coordinate system moving with the wave*

Continuity equation:

$$(\rho + d\rho)(c - du)A = \rho\, c\, A \quad \rightarrow \quad du = \frac{c\, d\rho}{\rho + d\rho} \tag{1.32}$$

Momentum equation:

$$\dot{m}(c - du) - (p + dp)A = \dot{m}\, c - p\, A \quad \rightarrow \quad \rho\, c\, du = dp \tag{1.33}$$

$$c^2 \frac{\rho\, \delta\rho}{\rho + d\rho} = dp \quad \rightarrow \quad c^2 = \frac{dp}{d\rho}\left(1 - \frac{d\rho}{\rho}\right) \tag{1.34}$$

for small disturbances $d\rho \ll \rho$:

$$c^2 = \frac{dp}{d\rho} \quad \xrightarrow{\text{isentropic}} \quad c^2 = k\,\frac{p}{\rho} = k\, R\, T \tag{1.35}$$

Mach number

Mach number is a non-dimensional quantity used for related to the compressible effects in fluid dynamics.

$$M = \frac{u}{c} \tag{1.36}$$

Example

Standard conditions are: $p = 1000\ hPa$, $T = 25^{\circ}C = 298.15^{\circ}K$
for air (two-atom molecules) $k = 1.4$, $R = 287.05\ \frac{J}{kg\ K}$
density $\rho = \frac{p}{R\,T} = 1.168\ kg/m^3$
speed of sound $c = \sqrt{k\,T} = 292.547\ m/s$

for motion with velocity $u = 100\ km/h = 27.778\ m/s$ Mach number is $M = 0.095$

Classification of regimes related to Mach number

incompressible	$M < 0.3$	flow can be treated as fully incompressible (density change $< \sim 5\%$)
subsonic	$M < 0.7$	flow is fully subsonic
transonic	$0.7 < M < 1.2$	occur significant compressible effects; large parts of the flow are sub/supersonic
supersonic	$1.2 < M < 5$	flow is almost completely supersonic
hypersonic	$5 < M$	interaction between shock-waves and boundary layer; high temperatures; dissociation and ionization of the gas

Integral of energy equation

Using continuity eq. (1.25) the energy eq. (1.27) can be transformed ($\dot{q} = 0$):

$$\rho \frac{dE}{dt} + \nabla \cdot \mathbf{v}\, p = 0 \tag{1.37}$$

Using:

$$\rho \frac{d}{dt}\left(\frac{p}{\rho}\right) = \frac{dp}{dt} - \frac{p}{\rho}\frac{d\rho}{dt} = \frac{dp}{dt} + p\,\nabla \cdot \mathbf{v} = \frac{\partial p}{\partial t} + \nabla \cdot \mathbf{v}\, p \tag{1.38}$$

and (1.37):

$$\rho \frac{d}{dt}\left(e + \frac{u^2}{2} + \frac{p}{\rho}\right) = \frac{\partial p}{\partial t} \tag{1.39}$$

For steady problems we can obtain a relation along a streamline:

$$e + \frac{u^2}{2} + \frac{p}{\rho} = h + \frac{u^2}{2} = const \tag{1.40}$$

Integral of energy equation

Let us introduce parameters at the stagnation point ($u = 0$): p_0, ρ_0, i T_0. Additionally, we can define a critical state for which $u = c$ which means that $M = 1$. Parameters of the gas are denoted by $*$: p_*, ρ_* i T_*.

Equation (1.40) can be written in following form:

$$h + \frac{u^2}{2} = c_p T + \frac{u^2}{2} = \frac{c^2}{k-1} + \frac{u^2}{2} = const \tag{1.41}$$

The equation at the stagnation point ($u_0 = 0$):

$$\frac{c^2}{k-1} + \frac{u^2}{2} = \frac{c_0^2}{k-1} = const \tag{1.42}$$

for critical state ($u_* = c_*$):

$$\frac{c^2}{k-1} + \frac{u^2}{2} = \frac{c_*^2}{k-1} + \frac{c_*^2}{2} = \frac{k+1}{2(k-1)} c_*^2 = const \tag{1.43}$$

From (1.42) and (1.43) it is obvious that for a given streamline for perfect gas in adiabatic process c_0 i c_* are constant.

Relations for isentropic process

The change of the parameters of state for isentropic process (from point 0 to 1) can be described using (1.40):

$$h_0 = h_1 + \frac{u_1^2}{2} \quad \rightarrow \quad \frac{k}{k-1}\frac{p_0}{\rho_0} = \frac{k}{k-1}\frac{p_1}{\rho_1} + \frac{u_1^2}{2} \tag{1.44}$$

Using isentropic relation (1.21) eq. above can be transformed to:

$$\frac{p_0}{p_1} = \left[1 + \frac{k-1}{2}M_1^2\right]^{\frac{k}{k-1}} \tag{1.45}$$

In similar way the relations for density and temperature can be obtained:

$$\frac{\rho_0}{\rho_1} = \left[1 + \frac{k-1}{2}M_1^2\right]^{\frac{1}{k-1}} \tag{1.46}$$

$$\frac{T_0}{T_1} = 1 + \frac{k-1}{2}M_1^2 \tag{1.47}$$

Relations for isentropic process

If we define the Mach number related to the speed of sound at critical point $M_* = u/c_*$ then equation (1.43) can be transformed by dividing both sides by u^2:

$$\frac{c^2}{u^2}\frac{1}{k-1} + \frac{1}{2} = \frac{k+1}{2(k-1)}\frac{c_*^2}{u^2} \quad \rightarrow \quad \frac{1}{M^2}\frac{1}{k-1} + \frac{1}{2} = \frac{k+1}{2(k-1)}\frac{1}{M_*^2}$$

Then following relation can be obtained:

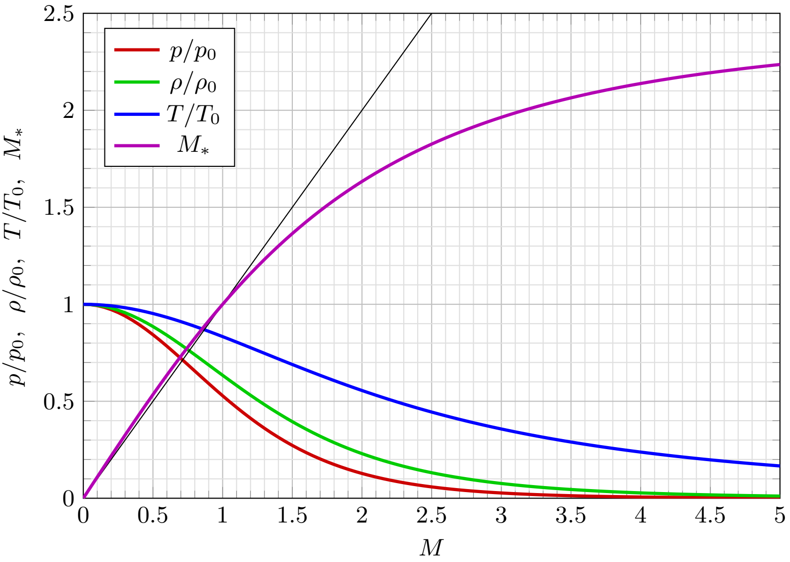
$$M_*^2 = \frac{(k+1)\,M^2}{2 + (k-1)\,M^2} \qquad (1.48)$$

which means:

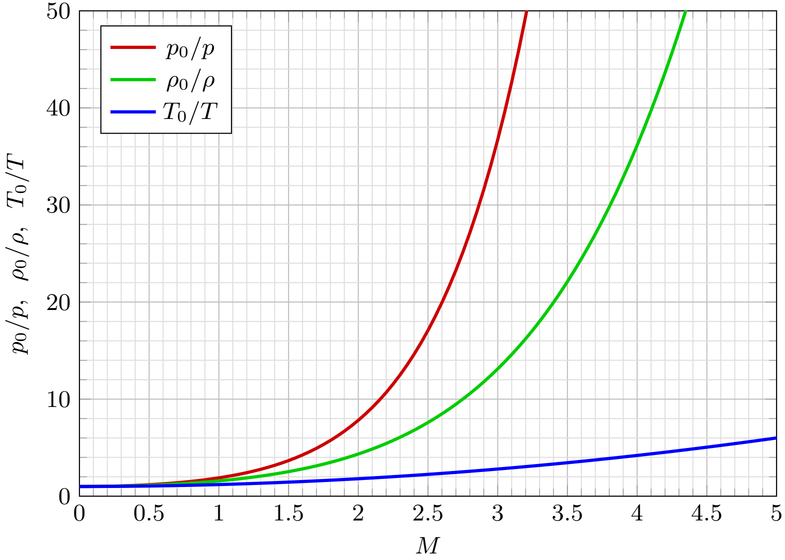
$$M < 1 \quad \Rightarrow \quad M_* < 1$$
$$M = 1 \quad \Rightarrow \quad M_* = 1$$
$$M > 1 \quad \Rightarrow \quad M_* > 1$$

$$M \rightarrow \infty \quad \Rightarrow \quad M_* = \sqrt{\frac{k+1}{k-1}} = \sqrt{6} \qquad (1.49)$$

Relations for isentropic process



Relations for isentropic process



Normal shock wave

Normal shock wave

external coordinate system *coordinate system related to the shock wave*

continuity equation:

$$\rho_1\, u_1 = \rho_2\, u_2 \tag{2.1}$$

momentum equation:

$$\rho_1\, u_1^2 + p_1 = \rho_2\, u_2^2 + p_2 \tag{2.2}$$

energy equation:

$$e_1 + \frac{p_1}{\rho_1} + \frac{u_1^2}{2} = e_2 + \frac{p_2}{\rho_2} + \frac{u_2^2}{2} \quad \rightarrow \quad h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2} \tag{2.3}$$

equation of state:

$$\frac{p_1}{\rho_1\, T_1} = \frac{p_2}{\rho_2\, T_2} \tag{2.4}$$

Normal shock wave

Equation of energy (2.3) can be written as:

$$c_p\, T_1 + \frac{u_1^2}{2} = c_p\, T_2 + \frac{u_2^2}{2} \quad \rightarrow \quad \frac{c_1^2}{k-1} + \frac{u_1^2}{2} = \frac{c_2^2}{k-1} + \frac{u_2^2}{2}$$

$$\rightarrow \quad \frac{c_1^2}{k-1} + \frac{u_1^2}{2} = \frac{c_2^2}{k-1} + \frac{u_2^2}{2} = \frac{k+1}{2(k-1)} c_*^2 \tag{2.5}$$

Dividing both sides of the momentum eq. (2.2) by continuity eq. (2.1):

$$\frac{p_1}{\rho_1\, u_1} + u_1 = \frac{p_2}{\rho_2\, u_2} + u_2 \quad \rightarrow \quad u_2 - u_1 = \frac{c_1^2}{k\, u_1} - \frac{c_2^2}{k\, u_2} \tag{2.6}$$

Obtaining c_1^2 and c_2^2 from (2.5) and substituting into (2.6):

$$\frac{k+1}{2\,k}\, \frac{c_*^2}{u_1\, u_2} + \frac{k-1}{2\,k} = 1 \quad \rightarrow \quad \boxed{c_*^2 = u_1\, u_2} \tag{2.7}$$

Normal shock wave

Equation (2.7) can be transformed:

$$\frac{u_1\ u_2}{c_*^2} = M_{*1}\ M_{*2} = 1 \quad \rightarrow \quad M_{*2} = \frac{1}{M_{*1}} \tag{2.8}$$

Using energy eq. (2.3):

$$\frac{c^2}{k-1} + \frac{u^2}{2} = \frac{k+1}{2(k-1)} c_*^2 \quad \rightarrow \quad \frac{1}{k-1}\frac{c^2}{u^2} + \frac{1}{2} = \frac{k+1}{2(k-1)}\frac{c_*^2}{u^2} \quad \rightarrow$$

$$\frac{1}{k-1}\frac{1}{M^2} + \frac{1}{2} = \frac{k+1}{2(k-1)}\frac{1}{M_*^2} \quad \rightarrow \quad M_*^2 = \frac{(k+1)\ M^2}{2+(k-1)\ M^2} \tag{2.9}$$

Using relation (2.8) and (2.9):

$$M_2^2 = \frac{2+(k-1)\ M_1^2}{2\ k\ M_1^2 - (k-1)} \tag{2.10}$$

Transforming continuity eq. (2.1) and using relation for M_*:

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{u_1^2}{u_2\ u_1} = \frac{u_1^2}{c_*^2} = M_{*1}^2 \quad \rightarrow \quad \frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{(k+1)\ M_1^2}{2+(k-1)\ M_1^2} \tag{2.11}$$

Normal shock wave

Using momentum eq. (2.2) and continuity eq. (2.1) :

$$p_2 - p_1 = \rho_2\, u_2^2 - \rho_1\, u_1^2 = \rho_1\, u_1 (u_2 - u_1)$$

$$\frac{p_2 - p_1}{p_1} = \frac{k\; u_1^2}{c_1^2} \left(1 - \frac{u_2}{u_1}\right) = k\; M_1^2 \left(1 - \frac{u_2}{u_1}\right) \qquad (2.12)$$

Using (2.11) and applying simplification:

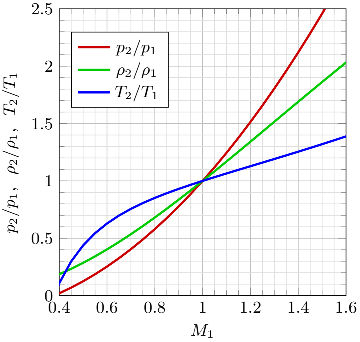
$$\frac{p_2}{p_1} = 1 + \frac{2\; k}{k+1}\; (M_1^2 - 1) \qquad (2.13)$$

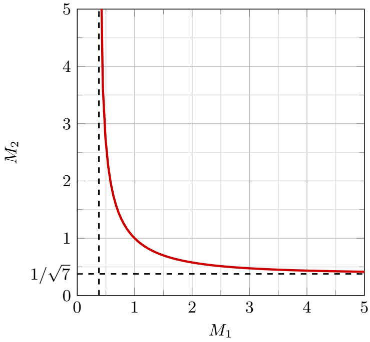
From eq. of state (2.4) and already known relations (2.11) and (2.13):

$$\frac{T_2}{T_1} = \frac{c_2^2}{c_1^2} = \left[1 + \frac{2\; k}{k+1}\; (M^2 - 1)\right] \frac{2 + (k-1)\; M_1^2}{(k+1)\; M_1^2} \qquad (2.14)$$

Normal shock wave

The relations (2.10), (2.11), (2.13) and (2.14) can describe compression shock wave ($M_1 > 1$) and also expansion shock wave ($M_1 < 1$).





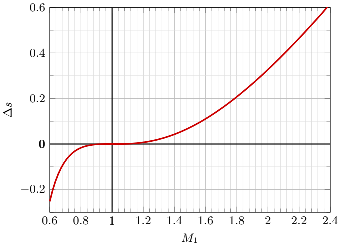
Normal shock wave

Are both types of shock waves physical ?

Let us find the change of entropy using eq. (1.20), (2.13) and (2.11):

$$\Delta s = s_2 - s_1$$
$$= \frac{R}{k-1} \ \ln \left[\left[1 + \frac{2\,k}{k+1}\,(M_1^2 - 1) \right] \left[\frac{(k+1)\,M_1^2}{2 + (k-1)\,M_1^2} \right]^{-k} \right] \quad (2.15)$$

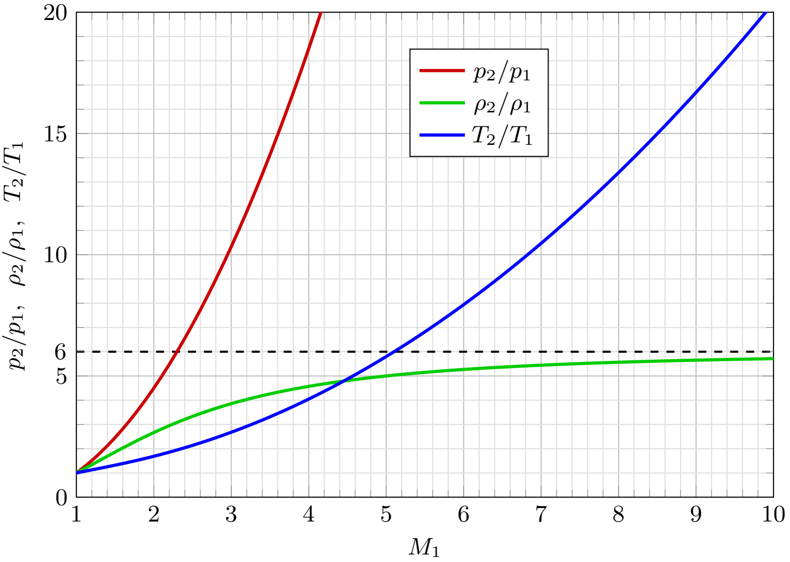
The entropy chane is a function of the Mach number M_1:



If $M_1 < 1$ (expansion shock wave) then $\Delta s < 0$. According to 2 L.T. it is unphysical.

In physical flows only compression shock waves can exist

Normal shock wave

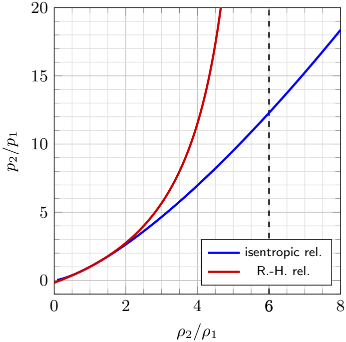


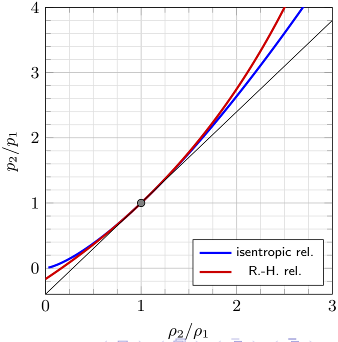
Normal shock wave as adiabatic process

The relations for normal shock wave can expressed (similarly to the isentropic relation) as $\frac{p_2}{p_1} = f\left(\frac{\rho_2}{\rho_1}\right)$:

$$\frac{p_2}{p_1} = -\frac{(k+1)\frac{\rho_2}{\rho_1} - (k-1)}{(k-1)\frac{\rho_2}{\rho_1} - (k+1)} \qquad (2.16)$$

This relation defines adiabatic process related to the shock wave and is called Rankine-Hugoniot relation.





Normal shock wave - total pressure loss

Energy equation (2.3) for parameters at stagnation point:

$$c_p\ T_{01} = c_p\ T_{02} \quad \rightarrow \quad T_{01} = T_{02} \tag{2.17}$$

Eq. (1.19) at stagnation point:

$$s_2 - s_1 = c_p\ \ln\frac{T_{02}}{T_{01}} - R\ \ln\frac{p_{02}}{p_{01}} \quad \rightarrow \quad s_2 - s_1 = R\ \ln\frac{p_{01}}{p_{02}} \tag{2.18}$$

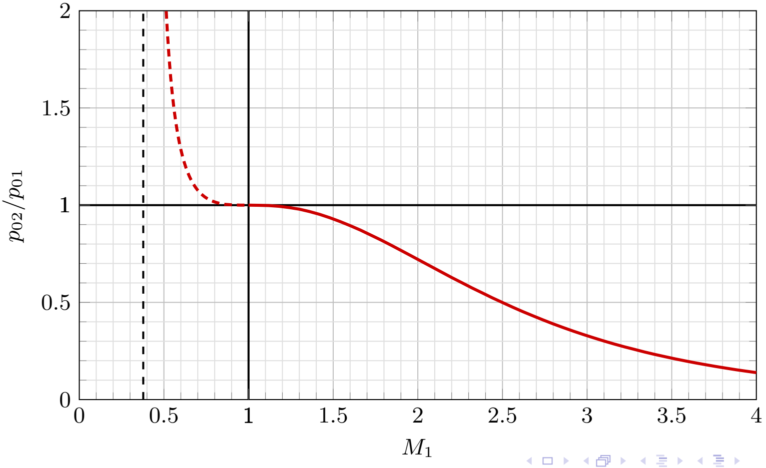
Change of total pressure can be obtained using (2.15):

$$R\ \ln\frac{p_{02}}{p_{01}} = -\frac{R}{k-1}\ \ln\left[\left[1+\frac{2\ k}{k+1}\ (M_1^2-1)\right]\left[\frac{(k+1)\ M_1^2}{2+(k-1)\ M_1^2}\right]^{-k}\right]$$

$$\frac{p_{02}}{p_{01}} = \left[1+\frac{2\ k}{k+1}\ (M_1^2-1)\right]^{-\frac{1}{k-1}}\left[\frac{(k+1)\ M_1^2}{2+(k-1)\ M_1^2}\right]^{\frac{k}{k-1}} \tag{2.19}$$

Normal shock wave

As can be found from (2.18) the change of entropy is directly related to the change of total pressure. The change of total pressure can be presented on a plot using (2.19):



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