PROGRAM ROZWOJOWY
POLITECHNIKI WARSZAWSKIEJ

FLUID MECHANICS 3 - LECTURE 8

BOUNDARY LAYER – PART 2

KAPITAŁ LUDZKI
NARODOWA STRATEGIA SPÓJNOŚCI

UNIA EUROPEJSKA
EUROPEJSKI
FUNDUSZ SPOŁECZNY

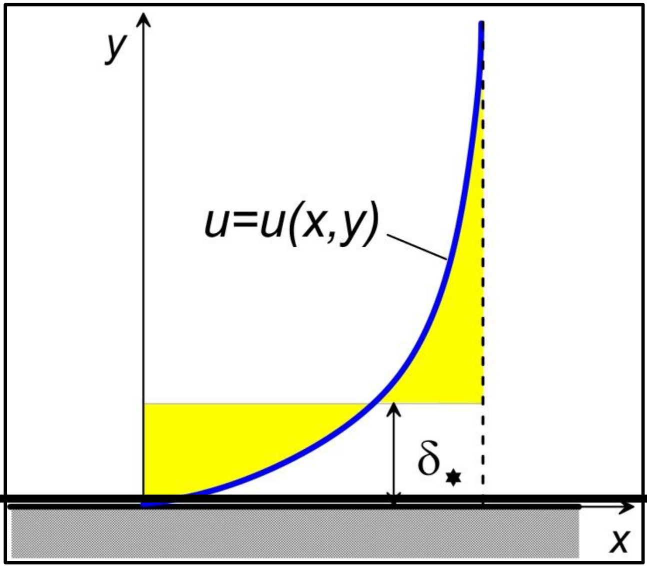
Integral approach to the BL

In the first part of the lecture, we derived the Prandtl Equation and discussed existence and some properties of its the self-similar solutions. Here, we consider an alternative approach based on the usage of integral quantities.

One of the main problems is to develop “objective” measures of the BL thickness. Probably the most basic definition is 99% BL thickness, denoted as δ_{99}. By definition, it is such distance from the wall that

$$u(x, y=\delta_{99}) := 0.99 \cdot U_0(x)$$

The other (more physically relevant) concept is the **displacement thickness**. In we consider the BL velocity profile at a given section (x is fixed), we can define the deficit in the volumetric flux inside the BL (as compared to ideal uniform velocity profile $u \equiv U_0$) by the following integral



$$\Delta Q(x) = \int_0^{\infty} [U_0(x) - u(x,y)]dy$$

Then, the displacement thickness is defined by the formula

$$\delta_*(x) := \frac{\Delta Q(x)}{U_0(x)} = \int_0^\infty \left[1 - \frac{u(x,y)}{U_0(x)}\right] dy$$

Hence, the quantity $\delta_*(x)$ tells us how big vertical displacement of the wall would cause equivalent flus deficit in an ideal fluid flow.

Analogous analysis can be applied to the flux of the linear momentum. If one considers the layer of fluid of a thickness δ, then the deficit of the linear momentum due to non-uniformity of the velocity profile seems to be equal

$$\Delta M_{x,1} = \rho \int_0^\delta (U_0^2 - u^2) dy$$

However, the actual deficit is smaller, because certain amount of fluid leaves the layer through the upper edge while moving with the velocity close to its external value U_0 . The amount of linear momentum carries by this fluid is equal

$$\Delta M_{x,2} \approx \rho \Delta Q U_0 = \rho \int_0^\delta (U_0^2 - U_0 u) dy$$

Hence, the actual local deficit of the x-component of the linear momentum in the boundary layer is equal

$$\Delta M_x = \Delta M_{x,1} - \Delta M_{x,2} = \rho \int_0^\delta (U_0^2 - u^2 - U_0^2 + U_0 u) dy = \rho \int_0^\delta u(U_0 - u) dy$$

The momentum thickness of the BL is then defined as follows

$$\delta_{**} = \lim_{\delta \to \infty} \frac{\Delta M_x}{\rho U_0^2} = \int_0^\infty \frac{u(x,y)}{U_0(x)} \left[1 - \frac{u(x,y)}{U_0(x)} \right] dy$$

The ratio between displacement and momentum thicknesses is called the shape factor. Obviously, the shape factor is always larger than unity (why?)

$$H := \frac{\delta_*}{\delta_{**}} > 1$$

Example:

Assume that the local velocity profile (x – fixed) is well approximated by the exponential law

$$u(y) = U_0(1 - e^{-\alpha y})$$

Then

$$\delta_* = \lim_{\delta\to\infty} \int_0^{\delta} (1 - 1 + e^{-\alpha y})dy = -\tfrac{1}{\alpha} \lim_{\delta\to\infty} [e^{-\alpha\delta} - e^0] = \tfrac{1}{\alpha}$$

$$\delta_{**} = \lim_{\delta\to\infty} \int_0^{\delta} (1 - e^{-\alpha y})e^{-\alpha y} dy = \lim_{\delta\to\infty} \int_0^{\delta} e^{-\alpha y} dy - \lim_{\delta\to\infty} \int_0^{\delta} e^{-2\alpha y} dy = \tfrac{1}{\alpha} - \tfrac{1}{2\alpha} = \tfrac{1}{2\alpha}$$

Hence, the shape factor is $H = \dfrac{\delta_*}{\delta_{**}} = 2$

Exercise: do the same for $u(y) = U_0[1 - (1 + y)^{-\alpha}]$, $\alpha > 1$

Von Karman Equation

Consider again the Prandtl Equation

$$u\,\partial_x u + \upsilon\,\partial_y u = U_0(x)U_0'(x) + \tfrac{\mu}{\rho}\partial_{yy}u$$

Let us integrate this equation with respect to the spatial coordinate y in the interval $[0,\delta]$, where $y=0$ corresponds to the wall and $y=\delta$ is sufficiently far away from the wall. The result is

$$\underbrace{\int_0^{\delta}\left(u\,\partial_x u + \upsilon\,\partial_y u - U_0 U_0'\right)dy}_{L_\delta} = \underbrace{\tfrac{\mu}{\rho}\int_0^{\delta}\partial_{yy}u\,dy}_{R_\delta}$$

Let us focus on the right-hand side of the above integral equality. We can write

$$R_\delta = \tfrac{\mu}{\rho}\int_0^{\delta}\partial_{yy}u\,dy = \tfrac{1}{\rho}\left(\mu\,\partial_y u\Big|_{y=\delta} - \mu\,\partial_y u\Big|_{y=0}\right)$$

Taking the limit $\delta \to \infty$, we obtain

$$R = \lim_{\delta \to \infty} R_\delta = -\tfrac{1}{\rho}\mu \partial_y u\Big|_{wall} = -\tfrac{1}{\rho}\tau_w$$

In the above, the symbol τ_w denotes the tangent stress at the wall.

Next, using the continuity equation we can express the vertical velocity component as follows

$$\upsilon\Big|_{y=0} = 0 \;\;\Rightarrow\;\; \upsilon = \int_0^y \partial_y \upsilon dy = -\int_0^y \partial_x u\, dy$$

Hence

$$\int_0^\delta \upsilon \partial_y u\, dy = \int_0^\delta (-\int_0^y \partial_x u\, dy')\partial_y u\, dy = (-\int_0^y \partial_x u\, dy')\, u\Big|_0^\delta + \int_0^\delta u \partial_x u\, dy =$$

$$= -u(x,\delta)\int_0^\delta \partial_x u\, dy + \int_0^\delta u \partial_x u\, dy = -\int_0^\delta [u(x,\delta)\partial_x u - u(x,y)\partial_x u]\, dy$$

Inserting obtained expression to the left-hand side of the equation and integrating by parts, we get

$$L_\delta = \int_0^\delta (u\partial_x u + \upsilon\partial_x u - U_0 U_0')dy =$$

$$= \int_0^\delta (u\partial_x u - U_0 U_0')dy - \int_0^\delta [u(\delta)\partial_x u - u\partial_x u]dy =$$

$$= \int_0^\delta [u\partial_x u - U_0 U_0' - u(\delta)\partial_x u + u\partial_x u]dy =$$

$$= \int_0^\delta [u\partial_x u - U_0 U_0' - u(\delta)\partial_x u + u\partial_x u + uU_0' - uU_0']dy =$$

$$= -\int_0^\delta \{u\tfrac{\partial}{\partial x}(U_0 - u) + [u(\delta) - u]\tfrac{\partial}{\partial x}u\}dy - \int_0^\delta (U_0 - u)U_0' dy$$

Since $\lim_{\delta\to\infty} u(\delta) = U_0$, one gets

$$L = \lim_{\delta \to \infty} L_\delta = -\int_0^\infty \{u \tfrac{\partial}{\partial x}(U_0 - u) + [U_0 - u] \tfrac{\partial}{\partial x} u\} dy - \int_0^\infty (U_0 - u) U_0' dy =$$

$$= -\tfrac{d}{dx} \int_0^\infty u(U_0 - u) dy - U_0' \int_0^\infty (U_0 - u) dy =$$

$$= -\tfrac{d}{dx} \left[U_0^2 \int_0^\infty \frac{u}{U_0}(1 - \frac{u}{U_0}) dy \right] - U_0' U_0 \int_0^\infty (1 - \frac{u}{U_0}) dy$$

Finally, we have obtained the **von Karman Equation**

$$\tfrac{d}{dx}[U_0^2(x)\delta_{**}(x)] + U_0(x)U_0'(x)\delta_*(x) = \tfrac{1}{\rho}\tau_W$$

This equation can be used for approximate solution of the BL flows. It can be also used as a tool to estimate the local value of the tangent stress without actually measuring friction force he wall gradient of the velocity (direct and reliable measurement of this quantity is very difficult).

The procedure is particularly straightforward when the pressure gradient is zero. Then from the von Karman Equation follows that

$$\tau_W = \rho \tfrac{d}{dx}[U_0^2 \delta_{**}(x)] = \rho U_0^2(x) \tfrac{d}{dx} \delta_{**}(x)$$

The total friction force developed along the wall segment $[x_1, x_2]$ is obtained by means of integration

$$F_\tau = \int_{x_1}^{x_2} \tau_W dx = \rho U_0^2 \int_{x_1}^{x_2} \tfrac{d}{dx} \delta_{**} dx = \rho U_0^2 [\delta_{**}(x_2) - \delta_{**}(x_1)] \equiv \rho U_0^2 \Delta \delta_{**}$$

Let $\rho = 1\, kg/m^3$, $U_0 = 100\, m/s$, $\Delta\delta_{**} = 1\, mm$ we obtain $F_\tau = 10\, N/m$ (force per 1 meter of span).

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