PROGRAM ROZWOJOWY
POLITECHNIKI WARSZAWSKIEJ

LECTURE 10

DYNAMICAL SIMILTUDE AND DIMENSIONAL ANALYSIS

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DIMENSIONLESS FORM OF THE NAVIER-STOKES EQUATION

The Navier-Stokes equation for an incompressible flow can be written in the following form

$$\frac{\partial \boldsymbol{v}}{\partial t}+(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}=-\frac{1}{\rho}\nabla p+\nu\Delta\boldsymbol{v}+\boldsymbol{f}$$

In order to make comparisons between various flow it is necessary to introduce the **dimensionless form** of the Navier-Stokes Equation. To this end we choose reference or **scaling quantities** for:

- time $t=T\,\tilde{t}$,
- linear dimensions $x_j=L\,\tilde{x}_j$,
- velocity $\boldsymbol{v}=V\,\tilde{\boldsymbol{v}}$,
- pressure $p=P\,\tilde{p}$,
- volume force $\boldsymbol{f}=F\,\tilde{\boldsymbol{f}}$.

In the above, all symbols with wave are dimensionless quantities.

Consequently, we have also **dimensionless differential operators**

$$\frac{\partial}{\partial t}=\frac{d\tilde{t}}{dt}\frac{\partial}{\partial \tilde{t}}=\frac{1}{T}\frac{\partial}{\partial \tilde{t}} \quad , \qquad \frac{\partial}{\partial x_j}=\frac{d\tilde{x}_j}{dx_j}\frac{\partial}{\partial \tilde{x}_j}=\frac{1}{L}\frac{\partial}{\partial \tilde{x}_j}.$$

The Navier-Stokes equations can be now written as

$$\frac{V}{T}\frac{\partial \tilde{\boldsymbol{v}}}{\partial \tilde{t}}+\frac{V^2}{L}(\tilde{\boldsymbol{v}}\cdot\tilde{\nabla})\tilde{\boldsymbol{v}}=-\frac{P}{\rho L}\tilde{\nabla}\tilde{p}+\frac{\nu V}{L^2}\tilde{\Delta}\tilde{\boldsymbol{v}}+F\,\tilde{\boldsymbol{f}},$$

or, after multiplication by L/V^2

$$\frac{L}{VT}\frac{\partial \tilde{\boldsymbol{v}}}{\partial \tilde{t}}+(\tilde{\boldsymbol{v}}\cdot\tilde{\nabla})\tilde{\boldsymbol{v}}=-\frac{P}{\rho V^2}\tilde{\nabla}\tilde{p}+\frac{\nu}{VL}\tilde{\Delta}\tilde{\boldsymbol{v}}+\frac{FL}{V^2}\tilde{\boldsymbol{f}}.$$

In the above equation, four nondimensional combinations of the scaling quantities have appeared.

We can define the following **similarity numbers**

- Strouhal number $St = \frac{V\,T}{L}$,
- Euler number $Eu = \frac{\rho V^2}{P}$,
- Reynolds number $Re = \frac{V\,L}{\nu}$,
- Froude number $Fr = \frac{V^2}{F\,L}$.

Finally, the Navier-Stokes equation can be written in dimensionless form as follows

$$\frac{1}{St}\frac{\partial \tilde{\boldsymbol{v}}}{\partial \tilde{t}} + (\tilde{\boldsymbol{v}} \cdot \tilde{\nabla})\tilde{\boldsymbol{v}} = -\frac{1}{Eu}\tilde{\nabla}\tilde{p} + \frac{1}{Re}\tilde{\Delta}\tilde{\boldsymbol{v}} + \frac{1}{Fr}\tilde{\boldsymbol{f}}$$

Note that the coefficient at dimensionless convective acceleration is equal 1. The remaining terms in this equations are multiplied by reciprocals of the similitude numbers. We can say that **each similitude number is a measure of significance of a corresponding term in comparison with the convective acceleration term**.

More physically: **the similitude numbers tell us how important (or large) are effects related to flow unsteadiness, pressure forces, viscous forces and gravity forces in comparison to the inertial forces (related to the convective part of the fluid acceleration)**.

For instance, the **Reynolds number** tells us how important are **viscous effects** – apparently they become negligible if the Reynolds number is very large. Similarly, the effect of the volume forces becomes not much important when the Froude number is large, and so on. **However, this general interpretation is correct providing the time and space scaling quantities are relevant for the physical effect of interest**!

Let's consider a typical example:

In a viscous flow past a wing of an airplane the **boundary layer** exists in the close vicinity of the wing's surface. Typically, the length scale in define as the wing's chord and the reference velocity is the velocity of the free stream far from the airplane. Since the kinematic viscosity of air is of the order 10^{-5} (m^2/s) the Reynolds number is typically very large, say, of the order of millions. It might suggest that the viscous term can be neglected and such conclusion is essentially correct for the **flow outside the boundary layer**. For that reason, the large scale flow around the airplane can be adequately model by the Euler equations. However, **inside the boundary layer viscous effects are never negligible**!.

The misinterpretations of the large Reynolds number in such case is the result of choice of an irrelevant length scale: what really matters for the boundary later flow is not a wing's chord but rather the boundary layer's thickness which is smaller by several orders of magnitude than the wing's chord!.

Conditions for dynamic similitude of flows:

We say that two flows are dynamically similar if:

- **they are geometrically similar, e.g. the shapes (but usually not dimensions) of the flow domains are the same,**

- **all similitude numbers computed on the basis of the corresponding scaling quantities are the same for both flows. It means that the dimensionless governing equations in both cases are identical,**

- **the dimensionless form of the initial and boundary conditions are identical.**

The above conditions are very important in **Experimental Fluid Mechanics**. where investigations are usually carried out with the use of re-scaled models of a real technical object (a model of an aircraft, a model of a car and so on). On the other hand, **these conditions are very rigorous and it is usually very difficult (or impossible) to meet all of them simultaneously.**

Consider the investigation of the sailing boat model made in the scale 1:16, which is carried out in the towing basin. The aim of the investigation is to assess a hydrodynamic drag of the boat.

Clearly, the geometry of the model and the real flows are not strictly similar – anyway, the real object is not be design to sail inside a 16-times magnified copy of the towing basin! In fact, the experimentalists assume (reasonably) that the measurements they conduct will give relevant results because the boat's model of the boat is relatively small (or, equivalently, the basin is sufficiently large) and all kinds of "side walls" and "bottom" effects are negligible.

Secondly, it is nearly impossible to keep both Reynolds and Froude numbers the same as in a real flow – it is actually a kind of **self-contradictory demand**. Indeed, to keep the same value of the Froude number, the model should move **4 times slower** than the real object, while keeping the same Reynolds number would require to move the model **16 times faster**!

The latter statement assumes that the fluid inside the experimental basin is the same water as in real conditions. It should be noted that - at least in principle – we could play with fluid viscosity (but water is already the rather low-viscosity fluid!) or with the gravity (towing basin inside the dropping elevator?)

What is actually done is the splitting of the experiment into parts devoted to a different regime of the yacht’s motion: for small velocity the frictions drag dominates and the similitude with respect to viscous effect is crucial, while for the fast motion the drag is mostly due to gravitational effects (surface waves generated by a moving boat) and then the Froude number should be kept the same (or close).

APPLICATION OF DIMENSIONAL ANALYSIS FOR PREDICTION OF MATHEMATICAL FORM OF PHYSICAL LAWS

A majority of physical quantities are dimensional ones, i.e., they are expressed in certain **physical units**. In Mechanics, there are **3 basic physical units**:

- the mass unit $[M]$ (in the SI system of units $[M] = kg$)
- the time unit $[T]$ (in the SI system of units $[T] = s$)
- the length unit $[L]$ (in the SI system of units $[L] = m$)

A unit of any other mechanical quantity is the **product of different powers of basic units**. For instance:

- the unit of acceleration is $[A] = [L] \cdot [T]^{-2} = m \cdot s^{-2}$
- the unit of energy is $[E] = [M][L]^2[T]^{-2} = kg \cdot m^2 \cdot s^{-2} = J$
- the unit of pressure is $[P] = [M][L]^{-1}[T]^{-2} = N / m^2 = Pa$

Imagine that certain mechanical problem involves the following set of physical quantities $\{q_1, q_2, .., q_n\}$. Let us assume that there exist at most $\boldsymbol{r}$ dimensionally independent quantities in this set, i.e., such that the unit of any of them can be expressed by the units of remaining ones (the example will follow shortly). Clearly, in mechanics, $\boldsymbol{r \leq 3}$.

Next, let us assume that the physical law related to this problem can be written in the implicit functional form as follows:

$$f(q_1, q_2, ..., q_n) = 0$$

The famous **Buckingham's Pi Theorem** (1914) states that this law can be equivalently written in the dimensionless form

$$\Phi(\Pi_1, \Pi_2, ..., \Pi_{n-r}) = 0$$

where ***n-r*** numbers $\Pi_1, \Pi_2, ..., \Pi_{n-r}$ are dimensionless combinations of the quantities $q_1, q_2, .., q_n$.

Consider two practical examples.

<u>Example 1</u>: Stationary flow of a viscous liquid in the straight pipe

We want to predict the form of the formula expressing the pressure drop Δp in the viscous liquid flow (density ρ and viscosity μ) along the pipe segment of the length l and diameter d. Let the average flow velocity is w. Additionally assume that the pipe wall is rough (not smooth) and the characteristic high of the surface roughness is s.

We look for the functional formula of the form

$$f(\Delta p, w, \rho, \mu, l, d, s) = 0$$

According to **Pi Theorem** we should be able to write this law in the dimensionless

$$\Phi(\Pi_1, \Pi_2, \Pi_3, \Pi_4) = 0$$

Indeed, the maximal number of dimensionally independent quantities is 3 – it is enough to take the velocity ***w***, the density $\boldsymbol{\rho}$ and the diameter ***d*** (explain!).

Now, four nondimensional parameters $\Pi_k,\ k = 1,..,4$ will be constructed as follows.

We seek for the first parameter is the form of $\Pi_1 = w^{\alpha}\rho^{\beta}d^{\gamma}\Delta p$. The appropriate calculations are:

$$[\Pi_1] = (\tfrac{m}{s})^{\alpha}(\tfrac{kg}{m^3})^{\beta}(m)^{\gamma}\,\frac{kg}{ms^2} = kg^{\beta+1}m^{\alpha-3\beta+\gamma-1}s^{-\alpha-2} \equiv kg^0 m^0 s^0$$

$$\beta + 1 = 0\ ,\ \alpha - 3\beta + \gamma - 1 = 0\ ,\ -\alpha - 2 = 0$$

$$\Downarrow$$

$$\alpha = -2\ ,\ \beta = -1\ ,\ \gamma = 0$$

Hence we have obtained the first dimensionless number $\Pi_1 = \frac{\Delta p}{\rho w^2}$

Similarly, the second parameter is sought in the form $\Pi_2 = w^\alpha \rho^\beta d^\gamma l$. Then

$$[\Pi_2] = (\tfrac{m}{s})^\alpha (\tfrac{kg}{m^3})^\beta (m)^\gamma m = kg^\beta m^{\alpha+\gamma+1} s^{-\alpha} \equiv kg^0 m^0 s^0$$

$$\beta = 0 \;\;,\;\; \alpha + \gamma + 1 = 0 \;\;,\;\; -\alpha = 0$$

$$\Downarrow$$

$$\alpha = 0 \;\;,\;\; \beta = 0 \;\;,\;\; \gamma = -1$$

Hence, the second dimensionless number is $\Pi_2 = \frac{l}{d}$. Analogous arguments lead also to the third number, which is $\Pi_3 = \frac{s}{d}$

It is left for the Reader to show that the last number is defined as $\Pi_4 = \frac{\mu}{w\rho d} = \frac{\nu}{wd}$ and this is simply the inverse of the **Reynolds number**.

We conclude that the functional relation between the parameters of our problem can be written in the form of

$$\Phi\left(\frac{\Delta p}{\rho w^2}, \frac{l}{d}, \frac{s}{d}, \frac{\nu}{wd}\right) = 0$$

or (replacing inverse of *Re* by *Re* itself)

$$\frac{\Delta p}{\rho w^2} = F\left(\frac{l}{d}, \frac{s}{d}, \frac{wd}{\nu}\right)$$

It means that the pressure drop along the pipe must be expressed by the following formula

$$\frac{\Delta p}{\rho w^2} = \frac{1}{2}\lambda\left(\frac{s}{d}, \frac{wd}{\nu}\right)\frac{l}{d} \quad \Rightarrow \Delta p = \lambda(\hat{s}, \mathrm{Re})\,\frac{1}{2}\rho w^2$$

The function $\lambda = \lambda(\hat{s}, \mathrm{Re})$ is called the **coefficient of distributed pressure losses**. Note that for the **Hagen-Poiseuille flow** in the smooth pipe we have $s = 0$ and

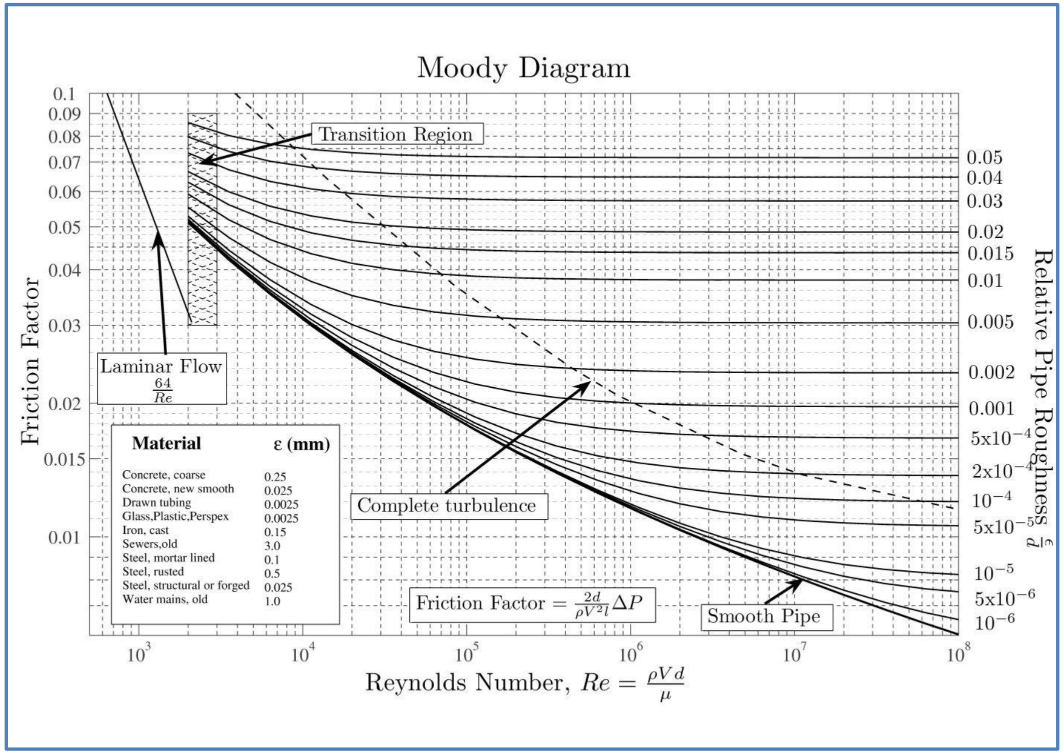
$$\lambda = \lambda(\mathrm{Re}) = \frac{64}{\mathrm{Re}}$$

This formula corresponds to the laminar flow described by the exact solution of the stationary Navier-Stokes equation (see **Lecture 8**). Such solution becomes unstable (and hence such flow is not observable in natural conditions) if *Re* exceeds the value of about 2300. The pipe flow with larger Reynolds number becomes **turbulent**.

For the fully developed turbulent flow inside a smooth pipe (no wall roughness) the coefficient of distributed pressure losses λ is well approximated (at least for Re not larger than – say – 10^5) by the Blasius formula

$$\lambda = \lambda(\mathrm{Re}) \approx \frac{0.316}{\sqrt[4]{\mathrm{Re}}}$$

For nonsmooth walls one can use the Moody diagram which summarizes the results of experimental investigations



Example 2: Aerodynamic drag of an obstacle

Let the quantities involved are: the reference surface S (says how large the obstacle is), air density ρ_∞, air pressure p_∞ and the velocity V_∞ of the free stream, dynamic viscosity μ and the magnitude of the aerodynamic drag force F_D. The Reader is recommended to follow the procedure as in the Example 1. The results is

$$\frac{F_D}{\rho_\infty V_\infty^2 S} = \phi\left(\frac{\mu}{\rho_\infty V_\infty \sqrt{S}}, \frac{p_\infty}{\rho_\infty V_\infty^2}\right)$$

In fact, it is customary to re-write the above formula in a following way

$$F_D = C_D\left(\frac{\rho_\infty V_\infty \sqrt{S}}{\mu}, \frac{p_\infty}{\rho_\infty V_\infty^2}\right) \frac{1}{2} \rho_\infty V_\infty^2 S$$

The nondimensional quantity C_D is called the **coefficient of aerodynamic drag**. It is the function of two dimensionless numbers. The first one can be easily recognized as the **Reynolds number** Re (where $\sqrt{S}$ plays the role of the characteristic length scale).

The meaning of the second one becomes more obvious when the speed of sound a_∞ in the free stream is used. We will see in the **Lecture 13** that the speed of sound in the Clapeyron gas is equal

$$a_\infty = \sqrt{\frac{\kappa p_\infty}{\rho_\infty}} \;, \quad \kappa = c_P / c_V$$

Thus

$$\frac{p_\infty}{\rho_\infty V_\infty^2} = \frac{a_\infty^2}{\kappa V_\infty^2} = \frac{1}{\kappa M_\infty^2}$$

The ratio $M_\infty = V_\infty / a_\infty$ is called the **Mach number** (of the free stream). Finally, the formula for the drag can be written as

$$F_D = \tfrac{1}{2} C_D(Re, Ma) \rho_\infty V_\infty^2 S$$

If the **Mach number is low** (say, lower that 0.3) then compressibility effects are negligible and the coefficient of the aerodynamic drag (as well as the lift force and other aerodynamic characteristics) depends only on the Reynolds number (and the shape of the obstacle!).
