PROGRAM ROZWOJOWY
POLITECHNIKI WARSZAWSKIEJ

FLUID MECHANICS 3 - LECTURE 7

BOUNDARY LAYER – PART 1

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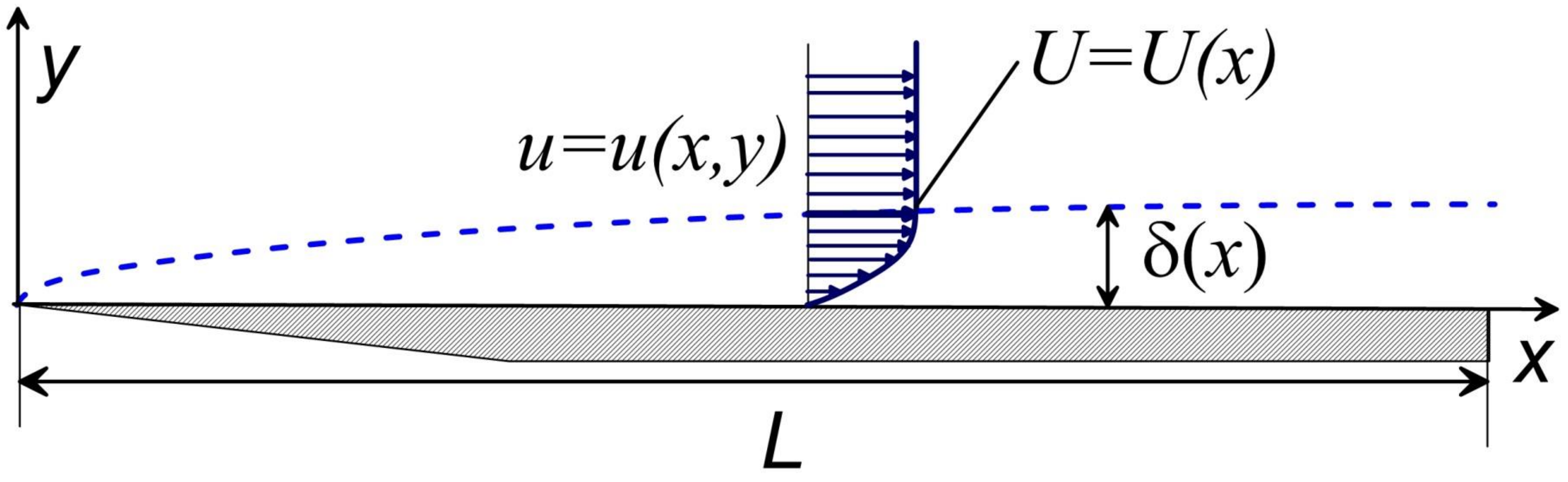
Concept of the BL and derivation of the Prandtl's Equation



The boundary layer – layer of fluid in the direct neighborhood of the solid wall, characterizing with very large transverse gradient of the velocity field. In this region viscous effect are of the same order as inertial effects.

General geometric assumption: $\delta \ll L$ or equivalently $\dfrac{\delta}{L} \ll 1$

We will derive the simplified equations of motion. The idea is to retain only those terms which are dominating in the BL region.

We begin with the equations governing 2D incompressible viscous fluid flow

$$u\partial_x u + \upsilon\partial_y u = -\tfrac{1}{\rho}\partial_x p + \nu(\partial_{xx}u + \partial_{yy}u)$$
$$u\partial_x \upsilon + \upsilon\partial_y \upsilon = -\tfrac{1}{\rho}\partial_y p + \nu(\partial_{xx}\upsilon + \partial_{yy}\upsilon)$$
$$\partial_x u + \partial_y \upsilon = 0$$

The following scaling of the velocity component parallel to the wall and its spatial derivatives can be accepted

$$u \sim U_\infty \quad , \quad \partial_x u \sim \frac{U_\infty}{L} \quad , \quad \partial_y u \sim \frac{U_\infty}{\delta} = \frac{U_\infty}{L}\frac{L}{\delta} \gg \partial_x u$$

Next, from the continuity equation we get

$$\partial_x u \sim \partial_y \upsilon \;\Rightarrow\; \partial_y \upsilon \sim \frac{U_\infty}{L} \;\Rightarrow\; \upsilon \sim \frac{\delta}{L}U_\infty \ll U_\infty$$

Thus the scaling of different terms in the linear momentum equation for x-direction reads

$$u\partial_x u + \upsilon\partial_y u = -\tfrac{1}{\rho}\partial_x p + \nu(\partial_{xx}u + \partial_{yy}u)$$

$$\sim\frac{U_\infty^2}{L} \qquad \sim\frac{U_\infty}{L}\frac{\delta U_\infty}{\delta} = \frac{U_\infty^2}{L} \qquad \sim\frac{U_\infty}{L^2} \qquad \sim\frac{U_\infty}{\delta^2}$$

Note that in the viscous term we have

$$\partial_{xx}u \sim (\tfrac{\delta}{L})^2\partial_{yy}u \ll \partial_{yy}u$$

Inside the boundary layer, the viscous term must be of the same order as inertial terms, hence

$$\nu\partial_{yy}u \frac{U_\infty^2}{L} \;\Rightarrow\; \nu\frac{U_\infty}{\delta^2} \sim \frac{U_\infty^2}{L} \;\Rightarrow\; \frac{\delta^2}{L^2} \sim \frac{\nu}{U_\infty L}$$

The following scaling follows

$$\frac{\delta}{L} \sim \frac{1}{\sqrt{\frac{U_\infty L}{\nu}}} = \frac{1}{\sqrt{\mathrm{Re}_L}}$$

Example: Estimate the (average) thickness of the BL knowing that
$L = 0.5\,m$, $U_\infty = 50\,\frac{m}{s}$, $\nu = 10^{-5}\,\frac{m^2}{s}$.

We have: $\mathrm{Re} = \frac{50 \cdot 0.5}{10^{-5}} = 2.5 \cdot 10^6 \quad \Rightarrow \; \delta \approx \frac{1}{\sqrt{2.5 \cdot 10^6}} L \approx 0.63\, mm$ (!)

As we see, the characteristic thickness of the BL at high Reynolds number is really small!

Consider now the linear momentum equation in *y*-direction (normal to the wall). All terms in this equation are smaller than the corresponding terms in the equation for x-direction by the factor of $\delta / L \ll 1$, namely

$$u\,\partial_x u + \upsilon\,\partial_y u = -\tfrac{1}{\rho}\partial_x p + \nu\,\partial_{yy} u \;\; , \;\; O(\tfrac{U_\infty^2}{L})$$

$$u\,\partial_x \upsilon + \upsilon\,\partial_y \upsilon \approx -\tfrac{1}{\rho}\partial_y p + \nu\,\partial_{xx} \upsilon \;\; , \;\; O(\tfrac{U_\infty^2}{L} \cdot \tfrac{\delta}{L})$$

The idea is to treat all kinematic terms in the *y*-equation as negligibly small, we leads to the conclusion that $\partial_y p \approx 0$ and hence $p = p(x)$. In other words, the pressure field is approximately constant across the boundary layer and practically it may change only in the streamwise (parallel to wall) direction *x*.

The pressure inside the boundary layer is identified with the pressure along the wall corresponding to the external flow, which is assumed inviscid (and potential). Hence, the Bernoulli equation can be written

$$p_\infty + \tfrac{1}{2}\rho U_\infty^2 = p(x) + \tfrac{1}{2}\rho U_0^2(x)$$

It follows that

$$-\tfrac{1}{\rho}\, p'(x) = U_0(x) U_0'(x)$$

After insertion, we obtain the central equation of the theory of the BL – the **Prandtl Equation**

$$\boxed{u\,\partial_x u + \upsilon\,\partial_y u = U_0(x) U_0'(x) + \nu\,\partial_{yy} u}$$

The Prandtl Equation is supplemented by the continuity equation

$$\partial_x u + \partial_y \upsilon = 0$$

Also, the boundary and far-field conditions must be defined

$$u,\upsilon\big|_{wall} \equiv u,\upsilon(x,0) = 0 \quad , \qquad \lim_{y\to\infty} u(x,y) = U_0(x)$$

The Prandtl Equation is of the parabolic time (the spatial coordinate *x* serves as the time-like variable). Thus the "initial" condition must be defined. This condition defines the velocity distribution (or profile) across the BL at a given section $x = x_0$

$$u(x_0, y) = u_0(y)$$

Hence, the solution of the initial/boundary value problem for the Prandtl Equation (usually obtained by numerical methods) consists in finding the velocity distribution in the region downstream from the initial section, i.e., for $x > x_0$. It should be noted that close vicinity of the leading edge (the beginning of the boundary layer) is a singular point which requires a "special treatment". Also, some attention is demanded for the far-field condition.

Self-similar solutions of the Prandtl Equation

The solution to the PE is called self-similar if it can be expressed as a function of one variable applied to the "self-similar" variable $\eta = \frac{y}{\delta(x)}$.

Since we consider only 2D case, the self-similar solution can be defined by means of the streamfunction, namely

$$\psi(x,y) = U_0(x)\delta(x) f[\tfrac{y}{\delta(x)}] = U_0(x)\delta(x) f[\eta(x,y)]$$

Thus, the velocity components are expresses as follows

$$u(x,y) = \partial_y \psi(x,y) = U_0(x)\delta(x) f'[\eta(x,y)]\partial_y \eta(x,y) =$$

$$= U_0(x)\delta(x) f'[\eta(x,y)]\frac{1}{\delta(x)} = U_0(x) f'[\eta(x,y)]$$

$$\upsilon(x,y) = -\partial_x \psi(x,y) = -U_0'(x)\delta(x) f[\eta(x,y)] - U_0(x)\delta'(x) f[\eta(x,y)] +$$

$$+ U_0(x)\delta(x) f'[\eta(x,y)]\frac{y\delta'(x)}{\delta^2(x)}$$

We need also to calculate their spatial derivatives:

$$\partial_x u(x,y) = U_0'(x) f'[\eta(x,y)] - U_0(x) f''[\eta(x,y)] \frac{y\delta'(x)}{\delta^2(x)}$$

$$\partial_y u(x,y) = \frac{U_0(x)}{\delta(x)} f''[\eta(x,y)] \quad , \quad \partial_{yy} u(x,y) = \frac{U_0(x)}{\delta^2(x)} f'''[\eta(x,y)]$$

Thus, the left-hand side of the Prandtl Equation reads

$$\mathcal{L} \equiv u\,\partial_x u + \upsilon\,\partial_y u = U_0(x) U_0'(x) f'^2[\eta(x,y)] -$$

$$- U_0(x) U_0'(x) f[\eta(x,y)] f''[\eta(x,y)] - U_0^2(x) \frac{\delta'(x)}{\delta(x)} f[\eta(x,y)] f''[\eta(x,y)]$$

The right-hand side of this equation takes the form

$$\mathcal{P} = U_0(x) U_0'(x) - \frac{\nu U_0(x)}{\delta^2(x)} f'''[\eta(x,y)]$$

Equating $\mathcal{L}$ and $\mathcal{P}$ we obtain

$$U_0(x)U_0'(x)f'^2[\eta(x,y)] - U_0(x)U_0'(x)f[\eta(x,y)]f''[\eta(x,y)] -$$
$$-U_0^2(x)\frac{\delta'(x)}{\delta(x)}f[\eta(x,y)]f''[\eta(x,y)] = U_0(x)U_0'(x) - \frac{\nu U_0(x)}{\delta^2(x)}f'''[\eta(x,y)]$$

After division by $\dfrac{\nu U_0(x)}{\delta^2(x)}$ and some algebra, the above equation assumes the following form

$$\frac{\delta U_0'}{\nu}(f'^2 - f f'' - 1) - \frac{U_0\delta\delta'}{\nu} f f'' = f'''$$

At this point, the left-hand side of the above equation refers explicitly to both spatial variables: x and η. A self-similar solution may exist only when the dependence with respect to x is removed, which imposes certain restrictions over the form of $U_0 = U_0(x)$.

Falkner and Skan noticed that self-similar solutions can be obtained assuming that

$$U_0(x) = U_\infty \left(\tfrac{x}{L}\right)^m$$

for $m \in R$.

Then

$$U_0'(x) = U_\infty m \frac{x^{m-1}}{L^m}$$

Assume also that the BL thickness is $\delta(x) = C\left(\tfrac{x}{L}\right)^\alpha L$, where constant C and exponent α are to be found. The coefficients of the differential equation for the function f are now equal

$$\frac{\delta^2 U_0'}{\nu} = \frac{C^2 U_\infty L}{\nu} m \left(\tfrac{x}{L}\right)^{2\alpha+m-1} \quad , \quad \frac{U_0 \delta \delta'}{\nu} = \frac{C^2 U_\infty L}{\nu} \alpha \left(\tfrac{x}{L}\right)^{2\alpha+m-1}$$

We see, that the x-dependence of these coefficient disappears if

$$2\alpha + m - 1 = 0 \quad \Rightarrow \quad \alpha = \tfrac{1-m}{2}$$

As a result, we have

$$\frac{\delta^2 U_0'}{\nu} = m\frac{C^2 U_\infty L}{\nu} \quad , \quad \frac{U_0 \delta\delta'}{\nu} = \frac{C^2 U_\infty L}{\nu}\alpha = \frac{1-m}{2}\frac{C^2 U_\infty L}{\nu}$$

The equation for the function f becomes now the **nonlinear ordinary differential equation of the 3rd order**

$$\frac{C^2 U_\infty L}{\nu} m(f'^2 - f f'' - 1) - \frac{C^2 U_\infty L}{\nu}\frac{1-m}{2} f f'' = f'''$$

or - after some algebra

$$\frac{C^2 U_\infty L}{\nu} m(f'^2 - 1) - \frac{C^2 U_\infty L}{\nu}\frac{1+m}{2} f f'' = f'''$$

The constant C can be chosen to "normalize" the coefficient at the second term, namely

$$\frac{C^2 U_\infty L}{\nu}\frac{1+m}{2} = 1 \quad \Rightarrow \quad C = \sqrt{\frac{2}{m+1}\frac{\nu}{U_\infty L}}$$

Corresponding formula for the BL thickness is

$$\delta(x) = C\left(\tfrac{x}{L}\right)^{\frac{1-m}{2}} L = \sqrt{C^2 \left(\tfrac{x}{L}\right)^{1-m}} L = \sqrt{\tfrac{2}{m+1} \tfrac{x}{L} \tfrac{\nu}{U_\infty L} \left(\tfrac{L}{x}\right)^m} L =$$

$$= \sqrt{\tfrac{2}{m+1} \tfrac{x}{L} \tfrac{\nu}{U_\infty L} \left(\tfrac{L}{x}\right)^m L^2} = \sqrt{\frac{2}{m+1} \frac{\nu x}{U_0(x)}}$$

The other coefficient in the equation becomes

$$\frac{C^2 U_\infty L}{\nu} m = \frac{2m}{1+m} \equiv \beta$$

which leads to the following final form of the differential (Falkner-Skan) equation

$$f''' + f f'' + \beta(1 - f'^2) = 0$$

where

$$f = f(\eta) \ , \ \eta = \frac{y}{\delta(x)} = \frac{y}{\sqrt{\tfrac{2}{m+1} \tfrac{\nu x}{U_0(x)}}}$$

The parameter β can be expressed by the exponent m

$$\beta = \frac{2m}{1+m} \quad \Rightarrow \quad m = \frac{\beta}{2-\beta} \Rightarrow \frac{2}{m+1} = 2-\beta$$

The formula for the self-similar variable η can be re-written as follows

$$\eta = \frac{y}{\sqrt{(2-\beta)\frac{\nu x}{U_0(x)}}}$$

The appropriate boundary/asymptotic conditions for the Falkner-Skan Equation are:

$$\begin{cases} u\big|_{wall} = 0 \quad \Rightarrow \quad f'(0) = 0 \\ \upsilon\big|_{wall} = 0 \quad \Rightarrow \quad f(0) = 0 \\ u \xrightarrow[y\to\infty]{} U_0(x) \Rightarrow \lim\limits_{\mu\to\infty} f'(\eta) = 1 \end{cases}$$

Interpretation of the Falkner-Skan self-similar solutions

We will consider the actual physical meaning of the FS self-similar solutions. To this aim, let us write the Laplace equation for the streamline of a potential 2D flow in polar coordinates.
This equation reads

$$\Delta\psi \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\psi\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\psi = 0$$

We claim that the (particular) solution to this equation is $\psi(r,\theta) = Kr^{m+1}\sin[(m+1)\theta]$.
Indeed, we have

$$\frac{\partial}{\partial r}\psi = K(m+1)r^m\sin[(m+1)\theta]$$

$$r\frac{\partial}{\partial r}\psi = K(m+1)r^{m+1}\sin[(m+1)\theta]$$

$$\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\psi\right) = K(m+1)^2 r^m\sin[(m+1)\theta]$$

$$\left.\begin{aligned}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\psi\right) &= K(m+1)^2 r^{m-1}\sin[(m+1)\theta]\\ \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\psi &= -K(m+1)^2 r^{m-1}\sin[(m+1)\theta]\end{aligned}\right\} \Rightarrow \Delta\psi = 0$$

The corresponding polar components of the velocity field are

$$\begin{cases} \upsilon_r = \frac{1}{r}\frac{\partial}{\partial\theta}\psi = K(m+1)r^m \cos[(m+1)\theta] \\ \upsilon_\theta = -\frac{\partial}{\partial r}\psi = -K(m+1)r^m \sin[(m+1)\theta] \end{cases}$$

We ask the question: which radial lines (i.e., lines $\theta = const$) are the streamlines of this flow? Obviously, the answer is: the lines $\theta = \theta_*$, such that $\sin[(m+1)\theta_*] = 0$, meaning that

$$\theta_{*,1} = 0 \quad , \quad \theta_{*,2} = \frac{\pi}{m+1}$$

Assume that $m \geq 0$. Then $\beta \in [0,2)$. We also have

$$\frac{1}{m+1} = 1 - \frac{1}{2}\beta \quad \Rightarrow \quad \theta_{*,2} = \pi - \frac{1}{2}\beta\pi$$

The radial velocity along the line $\theta_{*,1} = 0$ is

$$\upsilon_r(r,\theta=0) = \underbrace{K(m+1)}_{\frac{U_\infty}{L^m}} \underbrace{r^m}_{x^m} = U_\infty (x/L)^m \quad , \quad m = \frac{\beta}{2-\beta}$$

Hence, in such flow we have exactly the velocity assumed by Falkner and Skan

$$U_0(x) \equiv \upsilon_r(x,0) = U_\infty (x/L)^m$$

Consider the particular case $\beta = 0$. Then $m = 0$ and $U_0(x) \equiv U_\infty$ (zero pressure gradient along the BL). The Falkner-Skan Equation reduces to the Blasius equation

$$f''' + f\, f'' = 0$$

This special case describes the laminar boundary layer at the flat plate (at zero incidence angle).

The self-similar variable is defined as $\eta = \dfrac{y}{\sqrt{2\frac{\nu x}{U_\infty}}}$.

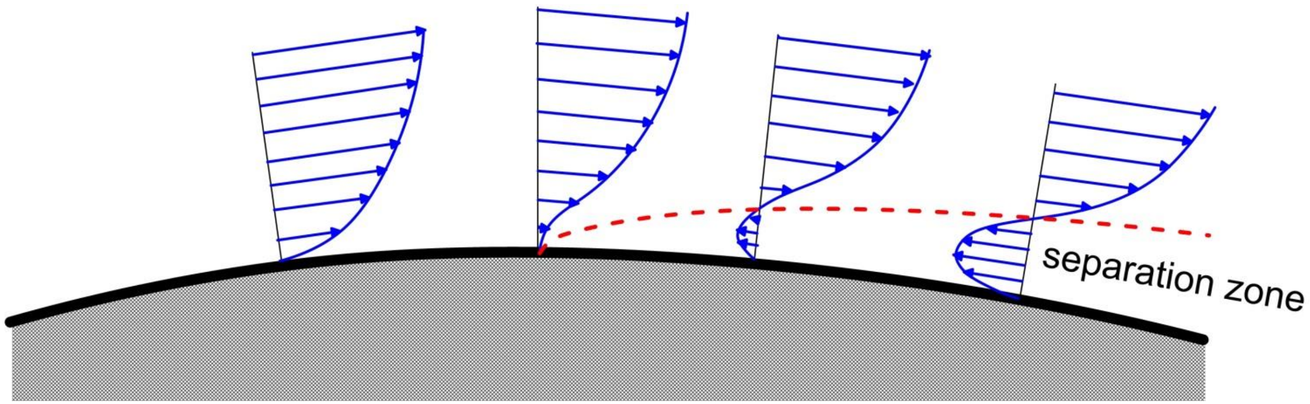
<u>Summarizing ….</u>

- Wedge flow $\begin{cases} m > 0 \;\;,\;\; \beta \in (0,2) \;\;,\;\; U_0(x) = U_\infty (\frac{x}{L})^m \\ U_0'(x) > 0 \quad and \quad p'(x) < 0 \end{cases}$

- Flow around the concave corner $\begin{cases} -\frac{1}{2} \le m < 0 \;\;,\;\; \beta \in [-2,0) \;\;,\;\; U_0(x) = U_\infty (\frac{x}{L})^m \\ U_0'(x) < 0 \quad and \quad p'(x) > 0 \end{cases}$

Separation of the BL

In certain circumstances the boundary layer can separate from the wall. The actual meaning of this statement is that the inverse flow can appear at the wall, creating the separation region. Such region can have only finite spatial extent, followed by re-attachment – we say that we have a separation bubble. Otherwise, the separation region extends to the trailing edge of the body (an airfoil, a wing) and we have global (massive) separation. From the point of view of aerodynamics of lifting surfaces, the separation of the boundary layer is unfavorable (and potentially dangerous) phenomenon, as it leads to sudden increase of the aerodynamic drag and drop of the lifting force.



It can be shown within the framework of the Prandtl's theory that the BL separation may occur only when the pressure rises along the wall (positive pressure gradient is present).

To see why, we write the Prandtl Eq. at the point arbitrarily close to the boundary. Due to flow continuity we obtain ($y=0$ corresponds to the wall)

$$0=U_0(x)U_0'(x)+\nu\partial_{yy}u(x,y=0)$$

From the obtained equation we can easily conclude that

$$p'(x)<0 \;\Rightarrow\; U_0'(x)>0 \;\Rightarrow\; \partial_{yy}u\big|_{wall}<0$$
$$p'(x)>0 \;\Rightarrow\; U_0'(x)<0 \;\Rightarrow\; \partial_{yy}u\big|_{wall}>0$$

Since the necessary condition for the separation is that the velocity profile at and near the boundary is described by the convex function (i.e., $\partial_{yy}u\big|_{wall}>0$), the conclusion follows as stated.