PROGRAM ROZWOJOWY
POLITECHNIKI WARSZAWSKIEJ

LECTURE 5

EULER EQUATION AND ITS FIRST INTEGRALS

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THE EULER EQUATION

The **Euler equation** is the equation of motion of an ideal fluid with zero viscosity. We have already derived its standard form which is

$$\rho\left[\partial_t \boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\right] = -\nabla p + \rho \boldsymbol{f}$$

Using the **Lamb-Gromeko form** of the convective acceleration

$$(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = \nabla\left(\tfrac{1}{2}\upsilon^2\right) + \boldsymbol{\omega}\times\boldsymbol{v}$$

we can write the Euler equation as follows

$$\partial_t \boldsymbol{v} + \nabla\left(\tfrac{1}{2}\upsilon^2\right) + \boldsymbol{\omega}\times\boldsymbol{v} = -\frac{1}{\rho}\nabla p + \boldsymbol{f}$$

Let us assume that:

(1) the flow is steady (or **stationary**), i.e., all flow quantities do not depend explicitly on time

(1) volumetric force field is **potential**, i.e. there exist such scalar field Φ_f that $\boldsymbol{f} = \nabla \Phi_f$,

(2) fluid is **barotropic**, i.e. its density is uniquely determined by pressure $\rho = \rho(p)$. We know that in such case the pressure function *P* can be defined by the formula

$$P(\rho) := \int \frac{1}{\rho} p'(\rho)\, d\rho .$$

Then $\dfrac{\partial}{\partial x_i} P[\rho(\boldsymbol{x})] = \dfrac{1}{\rho(\boldsymbol{x})} p'[\rho(\boldsymbol{x})] \dfrac{\partial \rho}{\partial x_i} = \dfrac{1}{\rho(\boldsymbol{x})} \dfrac{\partial}{\partial x_i} p[\rho(\boldsymbol{x})] \quad \Rightarrow \quad \nabla P = \dfrac{1}{\rho} \nabla p$

Also, let us recall (see the lecture about fluid statics) that for the **incompressible fluid** (ρ constant) we simply have $P = p / \rho$, while for the **ideal gas in isentropic conditions** we have

$$P = \frac{\kappa\, p}{(\kappa - 1)\,\rho} = c_p T = i \quad \text{(specific enthalpy)} \quad , \quad \kappa = \frac{c_p}{c_v}$$

With the above assumption the Euler Equation can be written in the following form

$$\nabla\left(\tfrac{1}{2}\upsilon^2+P-\Phi_f\right)=\boldsymbol{\upsilon}\times\boldsymbol{\omega}$$

Choose any streamline and define the tangent unary vector $\boldsymbol{\tau}=\boldsymbol{\upsilon}/\upsilon$. Next, multiply the above equation (in the sense of the inner product) by $\boldsymbol{\tau}$. The result is

$$\frac{d}{d\boldsymbol{\tau}}\left(\tfrac{1}{2}\upsilon^2+P-\Phi_f\right):=\nabla\left(\tfrac{1}{2}\upsilon^2+P-\Phi_f\right)\cdot\boldsymbol{\tau}=\frac{1}{\upsilon}\boldsymbol{\upsilon}\cdot(\boldsymbol{\upsilon}\times\boldsymbol{\omega})=0$$

Hence, the function under the gradient operator is **constant along the streamline**:

$$\boxed{\tfrac{1}{2}\upsilon^2+P-\Phi_f=C_B}$$

The above equality is called the **Bernoulli Integral** of the Euler Equation. The Bernoulli constant C_B can be – in general – different for different streamlines.

If the cross product $\boldsymbol{\upsilon} \times \boldsymbol{\omega} = 0$ then

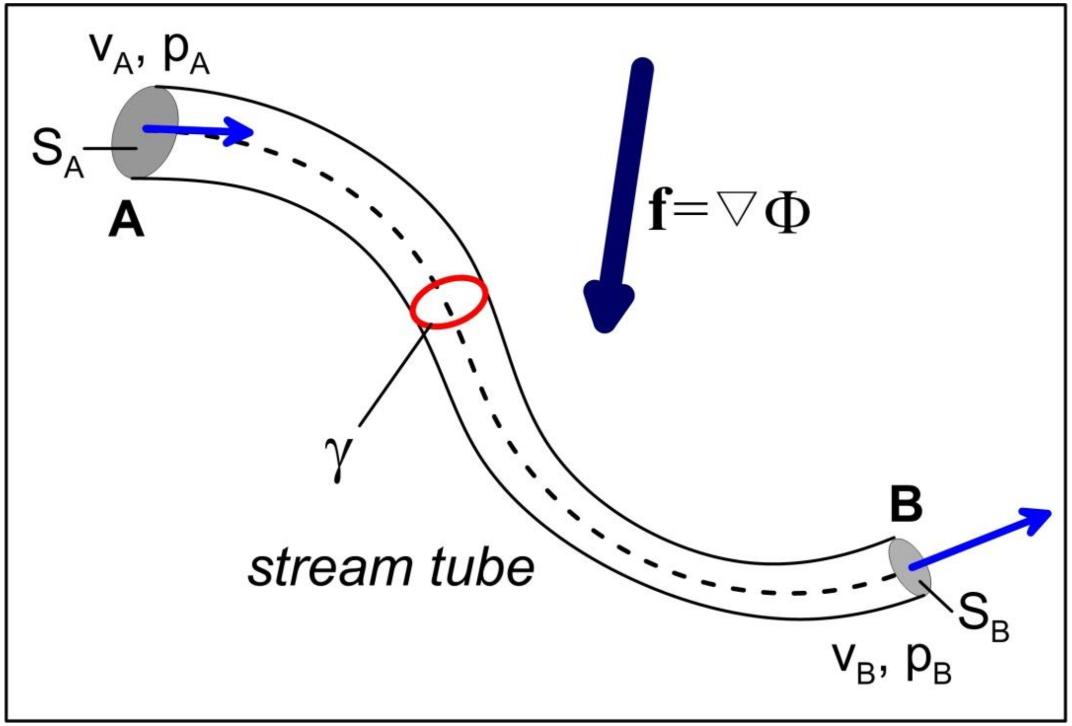
$$\nabla\left(\tfrac{1}{2}\upsilon^2 + P - \Phi_f\right) = 0 \implies \tfrac{1}{2}\upsilon^2 + P - \Phi_f = const$$

i.e., the Bernoulli constant is the same for all streamlines. In particular, this happens if the vorticity $\boldsymbol{\omega}$ vanishes identically in the whole flow domain ($\boldsymbol{\omega} \equiv 0$). Then, the velocity is the potential vector field, i.e., there exists such scalar field Φ_V such, that $\boldsymbol{\upsilon} = \nabla \Phi_V$.

In practice, we use the **Bernoulli Equation** which is obtained by equating the values of the Bernoulli Integral calculated for two different points A and B located in the same streamline

$$\left(\tfrac{1}{2}v^2 + P - \Phi_f\right)_A = \left(\tfrac{1}{2}v^2 + P - \Phi_f\right)_B$$

RELATION OF THE BERNOULLI EQUATION TO ENERGY CONSERVATION



In many engineering handbooks one can find a different "derivation" of the Bernoulli Equation, suitable for incompressible flows only. Let us demonstrate this approach.

Consider the stream tube, i.e., the volume of fluid bounded by the surface made of all streamlines passing through the closed loop γ (see Figure). The general assumption is that the loop is very small so is the cross-section of the tube and hence uniform distribution of pressure and velocity can be assumed across the tube.

Let us consider the change of the kinetic energy of the fluid contained between sections A and B of the stream tube, which occurs in the short time interval Δt. First of all note that the mass of fluid portions which cross both sections of the tube in this interval is exactly the same and equal (we assume that the velocity vector is perpendicular to the section plane)

$$\Delta m \equiv \underbrace{\rho S_A \upsilon_A \Delta t}_{\Delta m_A} = \underbrace{\rho S_B \upsilon_B \Delta t}_{\Delta m_B}$$

The change of the kinetic energy is equal to the work performed by pressure and volumetric force fields

$$\Delta E_{mech} \equiv \tfrac{1}{2}\Delta m \upsilon_{\mathrm{B}}^2 - \tfrac{1}{2}\Delta m \upsilon_{\mathrm{A}}^2 = \underbrace{p_{\mathrm{A}} S_{\mathrm{A}} \upsilon_{\mathrm{A}} \Delta t - p_{\mathrm{B}} S_{\mathrm{B}} \upsilon_{\mathrm{B}} \Delta t}_{\text{work performed by pressure force}} + \underbrace{\Delta m (\Phi_{\mathrm{B}} - \Phi_{\mathrm{A}})}_{\substack{\text{work performed by} \\ \text{potential force field}}}$$

Division of the above equality by Δm and simple rearrangement of terms yields the Bernoulli Equation for the points A and B

$$\tfrac{1}{2}\upsilon_{\mathrm{A}}^2 + \tfrac{1}{\rho} p_{\mathrm{A}} - \Phi_{\mathrm{A}} = \tfrac{1}{2}\upsilon_{\mathrm{B}}^2 + \tfrac{1}{\rho} p_{\mathrm{B}} - \Phi_{\mathrm{B}}$$

In particular, when the volumetric force field is the uniform gravity field

$$\boldsymbol{f} = -g\,\boldsymbol{e}_z \quad \Rightarrow \quad \Phi = -gz$$

then the Bernoulli Equation is often written as (after multiplication by density)

$$\tfrac{1}{2}\rho\upsilon_{\mathrm{A}}^2 + p_{\mathrm{A}} + \rho g z_{\mathrm{A}} = \tfrac{1}{2}\rho\upsilon_{\mathrm{B}}^2 + p_{\mathrm{B}} + \rho g z_{\mathrm{B}}$$

For the combined gravity and centrifugal forces (see Example 3 in the Lecture 1)

$$\boldsymbol{f}(r,z)=\Omega^2 r\,\boldsymbol{e}_r - g\,\boldsymbol{e}_z \;\Rightarrow\; \Phi=\tfrac{1}{2}\Omega^2 r^2 - gz$$

The Bernoulli Equation has the following for

$$\tfrac{1}{2}\rho\upsilon_A^2 + p_A - \tfrac{1}{2}\rho\Omega^2 r_A^2 + \rho g z_A = \tfrac{1}{2}\rho\upsilon_B^2 + p_B - \tfrac{1}{2}\rho\Omega^2 r_B^2 + \rho g z_B$$

Note that **in general, the Bernoulli Equation is derived without assumption of incompressibility**! For instance, we can write BE for the adiabatic continuous motion of the ideal Clapeyron gas. We know from thermodynamics that in such conditions pressure and density are related by the isentropic formula

$$p=C\rho^{\kappa} \quad , \quad \kappa=\frac{c_p}{c_v}$$

Thus the flow is barotropic and the pressure function P can be computed as follows

$$P=\int\frac{dp}{\rho(p)}=\frac{1}{C^{1/\kappa}}\int p^{-1/\kappa}dp=\frac{1}{1-(1/\kappa)}\frac{1}{C^{1/\kappa}}\,p^{1-1/\kappa}=\frac{\kappa}{\kappa-1}\frac{p}{\rho}$$

and the B. Eq. can be written $\quad \tfrac{1}{2}\upsilon_A^2+\frac{\kappa}{\kappa-1}\frac{p_A}{\rho_A}-\Phi_A=\tfrac{1}{2}\upsilon_B^2+\frac{\kappa}{\kappa-1}\frac{p_B}{\rho_B}-\Phi_B$

Note that due to low density of gases the volumetric force field is often neglected. We will show later that the above Bernoulli equation is in fact equivalent to the first integral of the energy equation. However, **in general the Bernoulli Equation may have nothing to do with the energy conservation**. For instance, this equation can be written for the isothermal motion of the Clapeyron gas. Indeed, in such circumstances the flow is barotropic since from the equation of state we immediately have

$$\rho = \frac{1}{RT} p = Cp$$

The pressure function for such motion is equal

$$P = \int \frac{dp}{\rho(p)} = \frac{1}{C}\int p^{-1} dp = \frac{1}{C}\ln(p / p_{ref}) = RT\ln(p / p_{ref})$$

where p_{ref} is some reference pressure. The corresponding form of the Bernoulli equation is

$$\tfrac{1}{2}\upsilon_{\mathrm{A}}^2 + RT\ln(p_{\mathrm{A}} / p_{ref}) - \Phi_{\mathrm{A}} = \tfrac{1}{2}\upsilon_{\mathrm{B}}^2 + RT\ln(p_{\mathrm{B}} / p_{ref}) - \Phi_{\mathrm{B}}$$

Clearly, the total energy (sum of the mechanical and internal energy) cannot be conserved during isothermal motion: the internal energy remains fixed while the mechanical one changes.

CAUCHY-LAGRANGE EQUATION

The Bernoulli Integral is not the only first integral of the Euler Equation. We may get another one if we change the assumption about the flow properties.

Now, let us assume that the **flow is nonstationary** but the **velocity field is potential**. It means that there exist the scalar field (the velocity potential) Φ_v such that

$$\boldsymbol{v} = \nabla \Phi_v$$

Then, the following equalities hold

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{v} = 0 \quad , \quad \frac{\partial \boldsymbol{v}}{\partial t} = \nabla \left(\frac{\partial}{\partial t} \Phi_v \right) \quad , \quad \upsilon^2 = |\nabla \Phi_v|^2 .$$

As we see, the flow is irrotational (the vorticity vanishes identically). The Euler equation takes the form of

$$\nabla \left(\tfrac{\partial}{\partial t} \Phi_v + \tfrac{1}{2} |\nabla \Phi_v|^2 + P - \Phi_f \right) = \boldsymbol{0}$$

Thus, the expression under the gradient operator can depend on be time only, i.e.

$$\frac{\partial \Phi_v}{\partial t} + \tfrac{1}{2}\left|\nabla \Phi_v\right|^2 + P - \Phi_f = C(t)$$

In the above formula, the function C is an arbitrary chosen function of time. Typically, we can conveniently assume that $C(t) \equiv 0$.