PROGRAM ROZWOJOWY
POLITECHNIKI WARSZAWSKIEJ

FLUID MECHANICS 3 - LECTURE 3

ONE-DIMENSIONAL STEADY GAS FLOWS WITH HEATING/COOLING OR FRICTION

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In the previous lecture, we considered quasi one-dimensional theory of steady and adiabatic gas flows through the ducts with variable cross section (nozzles). Here, we will developed an elementary theory on “really” one-dimensional flows, trying to account for effect invokes by the heat delivery (heating) or removal (cooling) .

Next, we develop a simple one –dimensional model of gas flow which mimics the presence of internal and wall friction (viscosity).

One-dimensional steady flow of a gas with heating or cooling (the Rayleigh model)

Let us remind that in case of one-dimensional, steady and thermally isolated flow, the following three equations hold (due to conservation of mass, linear momentum and energy)

$$\rho u = const$$
$$p + \rho u^2 = const$$
$$\tfrac{1}{2} u^2 + \tfrac{1}{\kappa - 1} a^2 = const$$

where $a = \sqrt{\kappa R T} = \sqrt{\dfrac{\kappa p}{\rho}}$ stands for the speed of sound.

If the flow exchanges heat with external environment then it is not adiabatic and the above form of the energy equation is not valid. Two first conservations laws remain unchanged, though.

Let us see what relations can be derived from the mass and linear momentum conservation. Such relations will be also applicable to the flows with heating or cooling.

We begin with the mass conservation equations. In can be written and transformed as follows

$$\rho u = \frac{p}{RT}u = \frac{\kappa p}{\kappa RT}u = \frac{\kappa p}{\sqrt{\kappa RT}}\underbrace{\frac{u}{\sqrt{\kappa RT}}}_{a} = \frac{\kappa}{\sqrt{\kappa R}}\frac{p}{\sqrt{T}}M = const$$

Hence

$$\frac{p}{\sqrt{T}}M = pMT^{-\frac{1}{2}} = const$$

Let us apply (formally) the logarithm to the above equality

$$\ln p + \ln M - \tfrac{1}{2}\ln T = const$$

Next, let us differentiate obtained formula. The result id the following relation between differentials of pressure, temperature and the Mach number

$$\frac{dp}{p} - \frac{1}{2}\frac{dT}{T} + \frac{dM}{M} = 0$$

Concern now the equation of the linear momentum. Using the Clapeyron relation, the momentum equation can be written as follows

$$p+\rho u^2 = p(1+\frac{\rho u^2}{p}) \underset{\substack{\Uparrow \\ \text{Clapeyron} \\ \text{Equation}}}{=} p(1+\frac{u^2}{RT}) = p(1+\kappa\frac{u^2}{\underset{a^2}{\kappa RT}}) = p(1+\kappa M^2)$$

Hence

$$p(1+\kappa M^2) = const$$

Again, we apply the logarithm

$$\ln p + \ln(1+\kappa M^2) = const$$

… and differentiate

$$\frac{dp}{p} + 2\frac{\kappa M\, dM}{1+\kappa M^2} = 0$$

We have obtained another differential relation, this time involving only pressure and Mach number.

The next move is to use the above relation to express (logarithmic) differential of temperature by the differential of the Mach number. The result reads

$$\frac{dT}{T} = 2\frac{(1-\kappa M^2)\,dM}{M(1+\kappa M^2)}$$

For the Clapeyron gas, the mass-specific enthalpy is related to the temperature by the formula $i = c_p T$. Then

$$\frac{di}{i} = \frac{dT}{T}$$

Therefore

$$\frac{di}{i} = 2\frac{(1-\kappa M^2)\,dM}{M(1+\kappa M^2)}$$

This formula can be integrated (even analytically) to obtain the explicit relation $i = i(M)$.

Let us now consider the mass-specific entropy. From the 1st Principle of Thermodynamics and the Clapeyron equation, the following relation can be derived

$$ds = \frac{dq}{T} \underset{\substack{\text{1st Principle} \\ \text{of Thermodynamics}}}{=} c_P \frac{dT}{T} - \frac{dp}{\rho T} \underset{\substack{\text{Clapeyron} \\ \text{Equation}}}{=} c_P \frac{dT}{T} - R \frac{dp}{p}$$

Hence

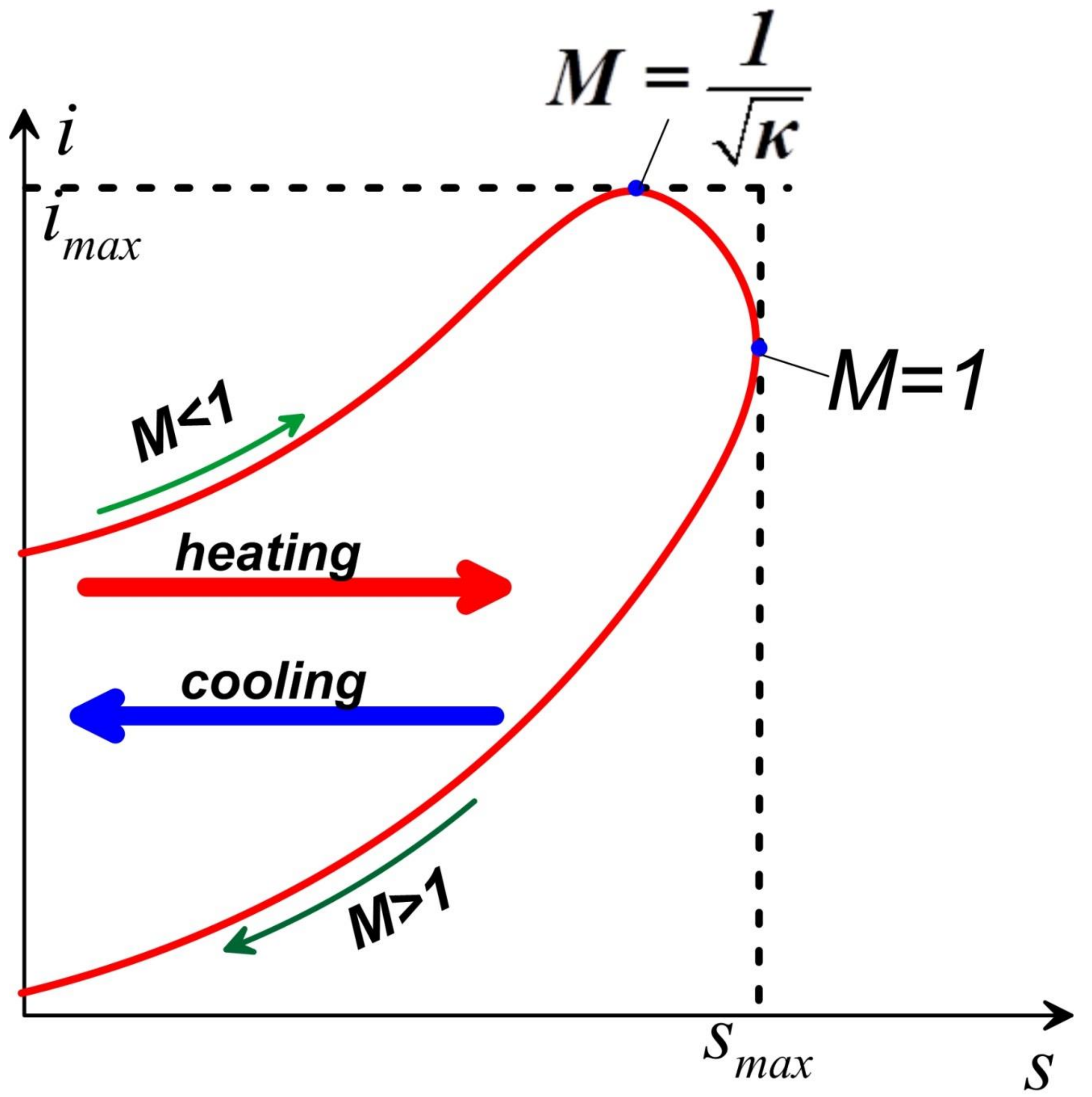
$$\frac{ds}{c_V} = \frac{c_P}{c_V} \frac{dT}{T} - \frac{R}{c_V} \frac{dp}{p} = \kappa \frac{dT}{T} - (\kappa - 1) \frac{dp}{p}$$

The logarithmic differentials of temperature and pressure can be plugged into the above expression. As a results, the differential of the mass-specific entropy can be expressed by means of the Mach number and its differential

$$\frac{ds}{c_V} = 2\kappa \frac{(1 - M^2)\, dM}{M(1 + \kappa M^2)}$$

Again, the above formula can be integrated (even analytically) to obtain the explicit relation $s = s(M)$.

The obtained relations can be illustrated in the form of the M-parametric plot in the entropy/enthalpy plane (the momentum line)



The obtained relations allows to find how heating/cooling affects the flow.

Note that if the heat is delivered along the duct, the entropy increases downstream. According to the above relation, heating will cause the subsonic flow to accelerate ($dM > 0$), while it will decelerate the supersonic flow ($dM < 0$). Clearly, cooling of the flow along the duct will have exactly opposite effect: it slows down the subsonic flow and accelerates the supersonic flow.

Interesting conclusion can be also drawn from the relation for the mass-specific enthalpy

$$\frac{di}{i} = 2\frac{(1-\kappa M^2)\,dM}{M(1+\kappa M^2)}$$

Note that the sign of di is the same as the sign of dM if $1-\kappa M^2 > 0$ and opposite if $1-\kappa M^2 < 0$. The "switching" value $M = \frac{1}{\sqrt{\kappa}} < 1$ lays within the subsonic flow region.

Thus, if the subsonic flow is accelerated downstream by heat delivery then its enthalpy rises along the duct only to such section where the Mach number reaches the value of $M = 1/\sqrt{\kappa}$. At this section the enthalpy reaches its maximal value and further downstream it actually drop despite of heating! Since the enthalpy is proportional to local temperature – the same is true for temperature as well. It means that in the duct segment where the Mach number assumes values between $M = 1/\sqrt{\kappa}$ and $M = 1$ the gas is getting colder despite absorption of heat. In other words, the "effective" specific heat of the gas becomes negative!

Effective calculation of flow parameter in the presence of heating/cooling requires derivation of some new relations.

Calculation of temperatures

Note that the flow with heating/cooling is not adiabatic, hence its total temperature T_0 is not globally constant anymore – it actually changes continuously along the heated/cooled segment of a duct. It means that the relation between temperature and total temperature derived previously, i.e.

$$T\left(1+\tfrac{\kappa-1}{2}M^2\right)=T_0$$

can be still used locally.

Using two relations derived from the mass and the linear momentum conservation principles

$$\frac{p}{\sqrt{T}}M=const \quad , \quad p(1+\kappa M^2)=const$$

we can easily conclude that

$$\frac{1+\kappa M^2}{M}\sqrt{T}=const$$

Taking into account the relation between actual and total temperatures (the latter defined locally), one can re-write the above "conservation" law in terms of the total temperature, namely

$$\frac{1+\kappa M^2}{M\sqrt{1+\frac{\kappa-1}{2}M^2}}\sqrt{T_0}=const$$

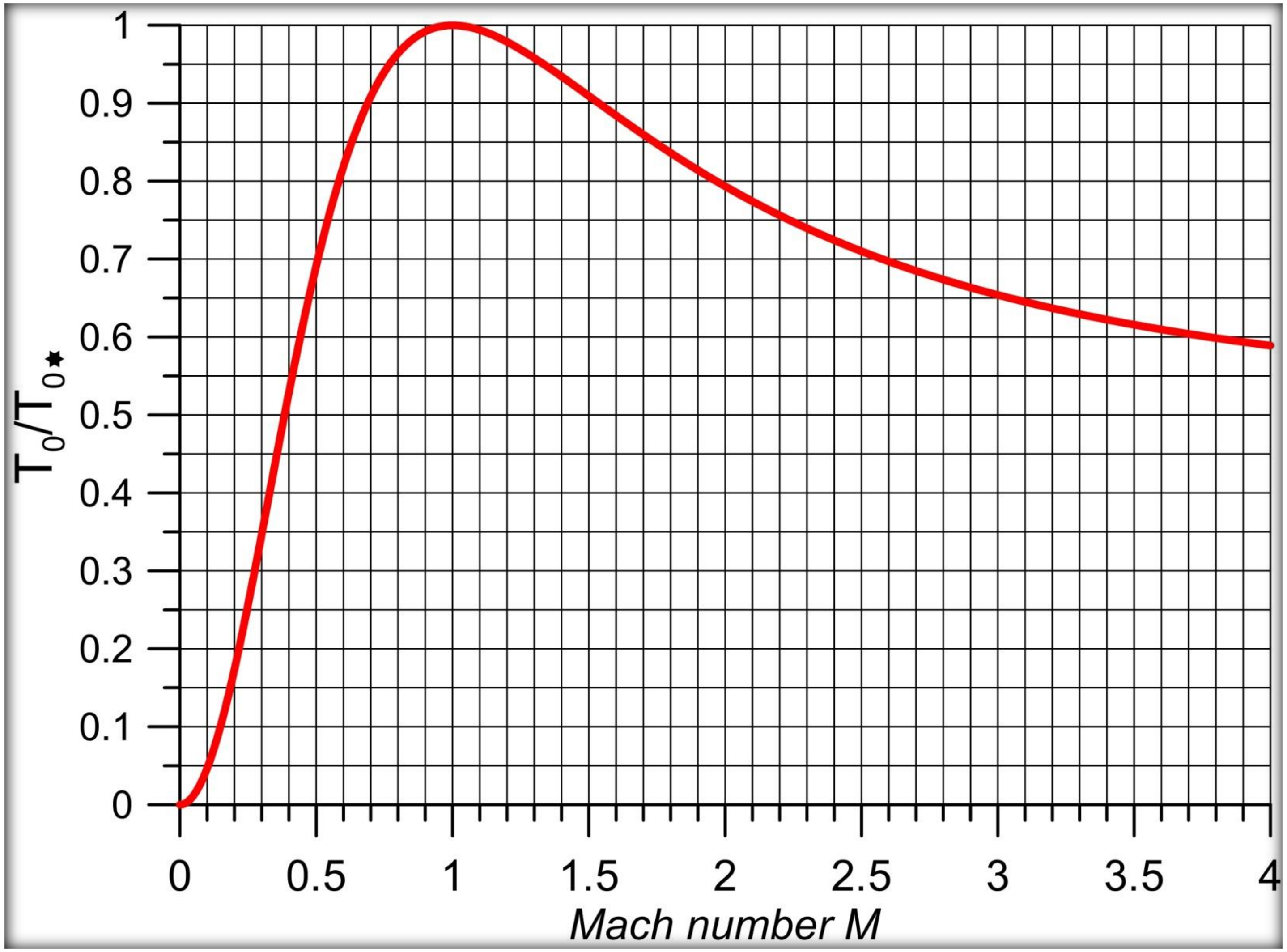
It is practical to define the constant in the above formula by evaluating it at the special section of a duct. This special section (real or only hypothetical) is such that the Mach number reaches the value of 1 (the critical section). As usual, the parameter at such section (critical parameters) are denoted by the symbols marked by the star in the lower index.

Thus, above formula implies that

$$\frac{1+\kappa M^2}{M\sqrt{1+\frac{\kappa-1}{2}M^2}}\sqrt{T_0}=\frac{1+\kappa}{\sqrt{1+\frac{\kappa-1}{2}}}\sqrt{T_{0*}}$$

or

$$\frac{T_0}{T_{0*}}=2(\kappa+1)\frac{(1+\frac{\kappa-1}{2}M^2)M^2}{(1+\kappa M^2)^2}$$



It can be seen that $\dfrac{T_0(M)}{T_{0*}} \leq 1$. This is consistent with our previous observation that heat delivery always pushes the flow towards the critical conditions.

The just derived formula is also very useful for determination of the amount of heat delivered to or removed from the flow.

In particular, the amount of heat related with the change of the Mach number from the local value M to the critical conditions ($M = 1$) can be computed as follows

$$q(M) = i_{0*} - i_0(M) = c_P[T_{0*} - T_0(M)]$$

Note also that the heat obtained this way expressed in the physical units [J/kg]. Multiplication of this quantity by the mass flux brings the amount of heat delivered/removed from the flow per the unit time, i.e., it brings the heating/cooling power.

Calculation of pressures

The calculation of pressures is based on the formula derived from the linear momentum balance, namely

$$p(1+\kappa M^2) = const$$

Written for two different duct's sections, the above "conservation" law brings the equation

$$p_1(1+\kappa M_1^2) = p_2(1+\kappa M_2^2)$$

It is again convenient to derive an equivalent relation expressed in terms of the stagnation pressures. Using the well-known isentropic relation

$$\frac{p}{p_0}(M) = \left(1+\frac{\kappa-1}{2}M^2\right)^{\frac{\kappa}{1-\kappa}} \Rightarrow p = \frac{p_0}{\left(1+\frac{\kappa-1}{2}M^2\right)^{\frac{\kappa}{\kappa-1}}}$$

one obtains

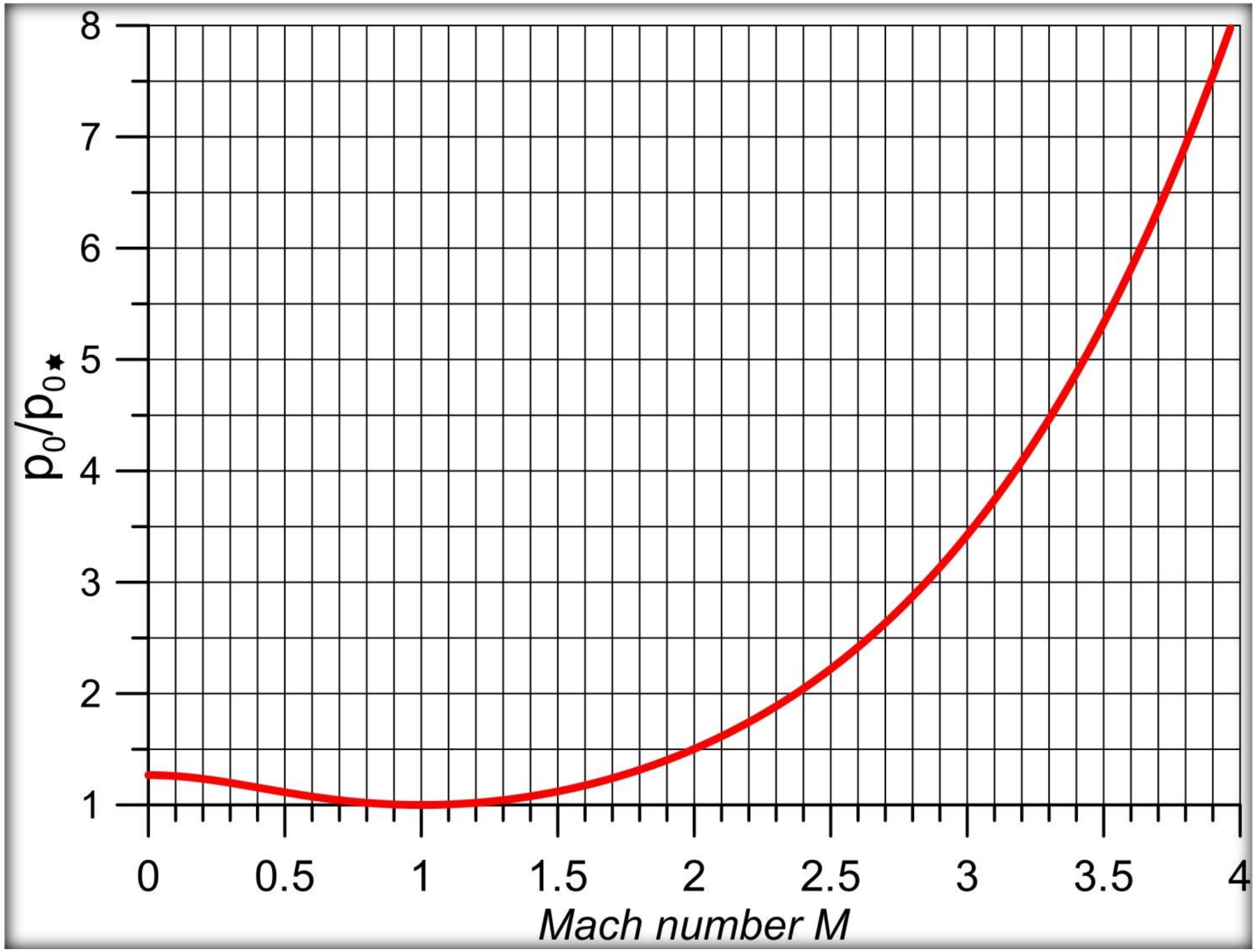
$$\frac{p_{01}(1+\kappa M_1^2)}{\left(1+\frac{\kappa-1}{2}M_1^2\right)^{\frac{\kappa}{\kappa-1}}} = \frac{p_{02}(1+\kappa M_2^2)}{\left(1+\frac{\kappa-1}{2}M_2^2\right)^{\frac{\kappa}{\kappa-1}}}$$

Again, let's assume that one of the sections is a special one – the critical section (say $M_1 = M$ and $M_2 = 1$). Then, we get the equality

$$\frac{p_0(1+\kappa M^2)}{(1+\frac{\kappa-1}{2}M^2)^{\frac{\kappa}{\kappa-1}}} = \frac{p_{0*}(1+\kappa)}{(1+\frac{\kappa-1}{2})^{\frac{\kappa}{\kappa-1}}}$$

This formula can be transformed to the ratio

$$\frac{p_0}{p_{0*}}(M) = \frac{[2+(\kappa-1)M^2]^{\frac{\kappa}{\kappa-1}}}{(1+\kappa M^2)(1+\kappa)^{\frac{2\kappa-1}{\kappa-1}}}$$



Note that the ratio $\frac{p_0}{p_{0*}}(M) \geq 1$. This result is consistent with the observation that heating always pushes the flow towards the critical condition, while at the same time increases the entropy and - thus – reducing the value of the total (stagnation) pressure.

One-dimensional steady flow of a gas with friction (the Fanno model)

Here, we will derive the simple model of a stationary gas flow in a duct with friction.

Again, let's recall the original set of conservation equations valid for 1D steady and adiabatic flow.

$$\rho u = const$$
$$p + \rho u^2 = const$$
$$\tfrac{1}{2} u^2 + \tfrac{1}{\kappa - 1} a^2 = const$$

where $a = \sqrt{\kappa R T} = \sqrt{\dfrac{\kappa p}{\rho}}$ stands for the speed of sound.

In the presence of friction, the linear momentum in the above form is no more valid. Yet, two remaining equations are still OK. In particular, it is assumed that the work performed by the friction forces affect only the amount of different forms of energy (kinetic, internal) but the total energy is conserved.

We have already derived from the mass conservation equation that

$$\frac{p}{\sqrt{T}}M = pMT^{-\frac{1}{2}} = const$$

Using the technique of logarithmic differentials we have obtained the relation

$$\frac{dp}{p} - \frac{1}{2}\frac{dT}{T} + \frac{dM}{M} = 0$$

The relation between actual and total temperature is

$$T(1+\tfrac{\kappa-1}{2}M^2) = T_0$$

This time, the total energy is conserved, hence the total temperature T_0 is a global constant.

Thus

$$\ln T + \ln(1+\tfrac{\kappa-1}{2}M^2) = const$$

and

$$\frac{dT}{T} = -\frac{(\kappa-1)M\,dM}{1+\tfrac{\kappa-1}{2}M^2} = -\frac{(\kappa-1)d(M^2)}{2(1+\tfrac{\kappa-1}{2}M^2)}$$

Two obtained above differential relations can be solved with respect to the logarithmic differential of pressure. One obtains

$$\frac{dp}{p} = -\frac{1+(\kappa-1)M^2}{M\left(1+\frac{\kappa-1}{2}M^2\right)}dM$$

Earlier, we have derived the formula for the entropy

$$\frac{ds}{c_V} = (\kappa-1)\frac{1-M^2}{M\left(1+\frac{\kappa-1}{2}M^2\right)}dM$$

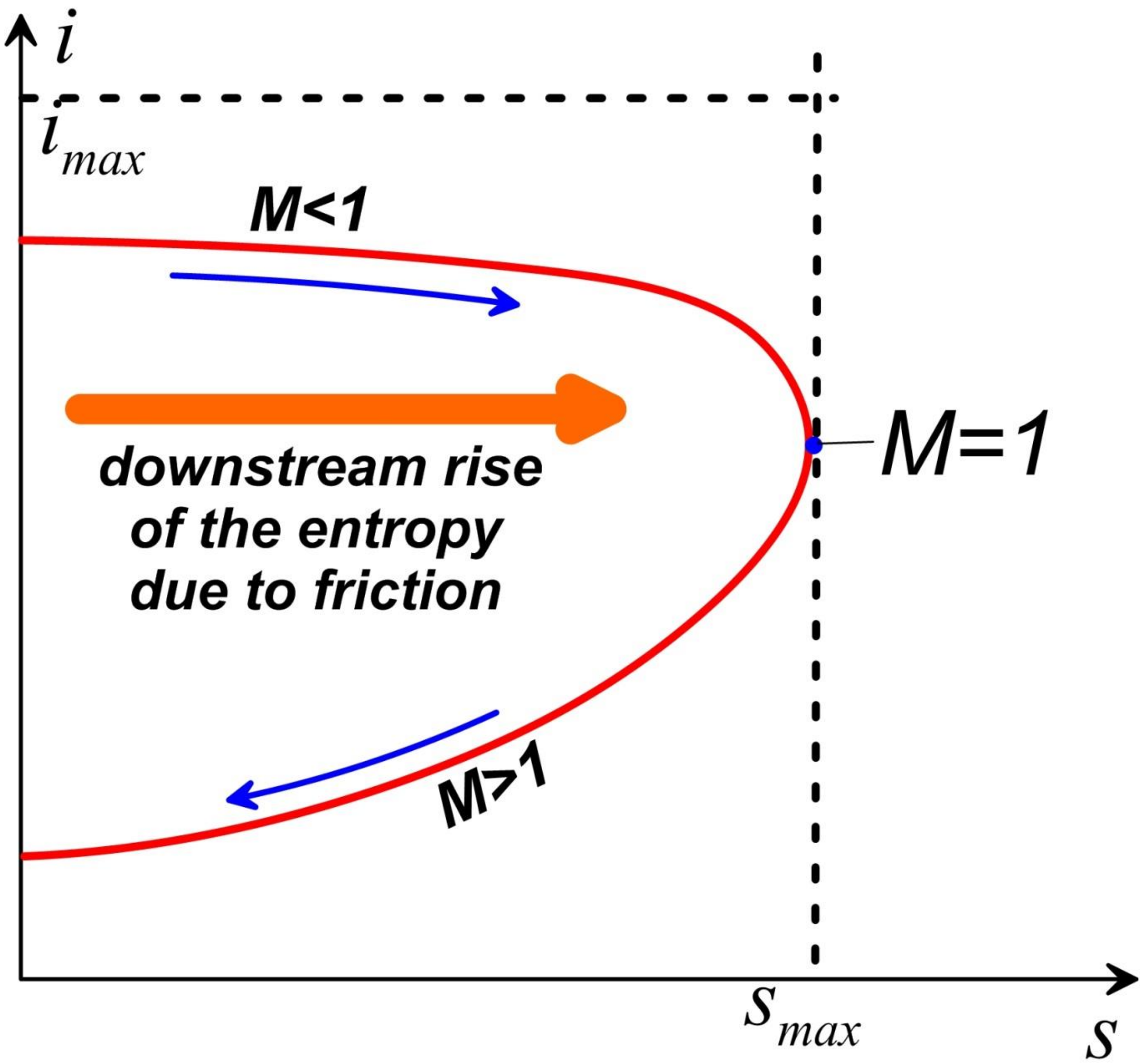
which can be integrated analytically to an explicit (algebraic) form $s = s(M)$

The logarithmic differential of the mass-specific enthalpy is

$$\frac{di}{i} = \frac{dT}{T} = \frac{(\kappa-1)M\,dM}{1+\frac{\kappa-1}{2}M^2}$$

By integrating the above relation, one can obtain the formula $i = i(M)$.

The obtained relations can be illustrated in the form of the M-parametric plot in the entropy/enthalpy plane (the energy line)



Note that the formula for the differential of the mass-specific entropy implies that the sign of ds matches the sign of dM for a subsonic flow ($M < 1$), while for the supersonic flow $M > 1$) the signs of these differentials are opposite. However, the friction works as a thermodynamically irreversible “converter” of the mechanical energy into the internal one – clearly, a systematic downstream increase of entropy must occur! This implies further that in the **subsonic flow the Mach number must increase along the duct**. As usual (the principle of reverse action), the supersonic flow reacts in an opposite way, i.e., the **presence of friction decelerates such flow**.

Summarizing – **the friction always pushes the flow in the direction of the critical conditions**.

We will derive the quantitative description of the above effects. The idea is to modify the equation of motion (the 1D Euler equation) by an additional term mimicking the presence of friction.

Following the ideas developed for incompressible flows in hydraulic systems, we will assume that the pressure drop due to friction can be expressed by a Darcy-Wiesbach formula

$$\Delta p = -\lambda \tfrac{1}{2} \rho u^2 \tfrac{\Delta x}{D}$$

In order to incorporate this formula into the equation of motion, we define an equivalent volumetric force

$$f_x = \frac{A \Delta p}{\Delta m} = \frac{-\lambda \tfrac{1}{2} \rho u^2 \tfrac{\Delta x}{D} A}{\rho A \Delta x} = -\frac{\lambda}{D} \frac{u^2}{2}$$

After insertion to the Euler equation

$$\rho u \tfrac{d}{dx} u = -\tfrac{d}{dx} p + \rho f_x$$

one obtains

$$u \tfrac{d}{dx} u = -\tfrac{1}{\rho} \tfrac{d}{dx} p - \tfrac{\lambda}{2D} u^2$$

or

$$\tfrac{1}{u} \tfrac{d}{dx} u = -\tfrac{1}{\rho u^2} \tfrac{d}{dx} p - \tfrac{\lambda}{2D}$$

The further transformation of this equation are

$$\frac{1}{\rho u^2}\frac{d}{dx}p = \frac{p}{\rho u^2}\frac{1}{p}\frac{d}{dx}p = \frac{1}{\kappa}\frac{\kappa p}{\rho u^2}\frac{1}{p}\frac{d}{dx}p = \frac{1}{\kappa M^2}\frac{1}{p}\frac{d}{dx}p$$

$$\frac{du}{u} + \frac{1}{\kappa M^2}\frac{dp}{p} = -\frac{\lambda\, dx}{2D}$$

The final goal is to derive the relation between the differentials dM and dx. To this aim, we eliminate the velocity using the formula

$$u = Ma = M\sqrt{\kappa RT}$$

Logarithmic differential of this formula reads

$$\frac{du}{u} = \frac{dM}{M} + \frac{1}{2}\frac{dT}{T} = \left(\frac{1}{M} - \frac{\kappa-1}{2}\frac{M}{1+\frac{\kappa-1}{2}M^2}\right)dM$$

After insertion to the equation of motion, one arrives at

$$\left(\frac{1}{M}-\frac{(\kappa-1)M}{2+(\kappa-1)M^2}-\frac{1}{\kappa M^2}\frac{1+(\kappa-1)M^2}{M(1+\frac{\kappa-1}{2}M^2)}\right)dM=-\frac{\lambda}{2D}dx$$

After some algebra we finally get the differential equality

$$\frac{\lambda}{D}dx=\frac{(1-M^2)}{\kappa(M^2)^2(1+\frac{k-1}{2}M^2)}d(M^2)$$

This equality can be integrated in the interval $[x,x_*]$

$$\int_x^{x_*}\frac{\lambda}{D}dx=\int_{M^2}^{1}\frac{(1-\eta)}{\kappa\eta^2(1+\frac{k-1}{2}\eta)}d\eta$$

where x_* denotes the x-location of the (actual or hypothetical) section of a duct where critical conditions ($M=1$) are achieved.

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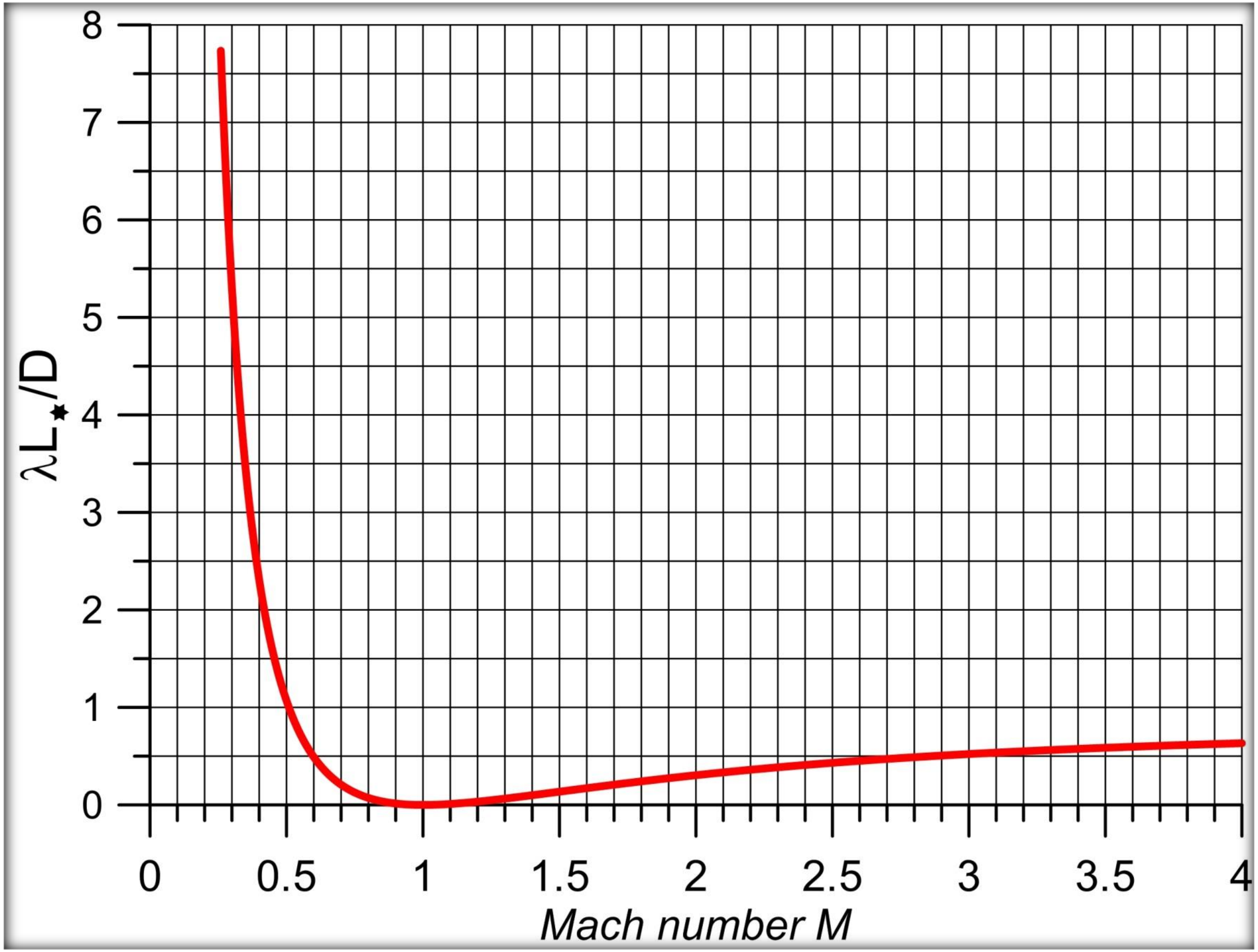
Using the mean value of the coefficient of pressure lose λ, the left-hand side of the above integral equality can be written as follows

$$\int_{x}^{x_*} \frac{\lambda}{D} dx = \frac{\bar{\lambda}(x_* - x)}{D} = \frac{\bar{\lambda} L_*}{D}(M)$$

The integral in the right-hand side can be computed analytically in a closed form. The final formula reads

$$\frac{\bar{\lambda} L_*}{D}(M) = \frac{1 - M^2}{\kappa M^2} + \frac{\kappa + 1}{2\kappa} \ln \frac{(\kappa + 1) M^2}{2 + (\kappa - 1) M^2}$$

The interpretation of the above formula is straightforward: assuming that the average value of the pressure lose coefficient is known, this formula allows for determination of a distance between a given duct's section (where the Mach number is M) and the (real or hypothetical) section downstream when the gas reaches the critical state.



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Mach number M

As in the case of the flow with heating/cooling, we have to know how to calculate local pressure and temperature.

In the case of friction, temperature calculation is very easy since the total temperature is globally constant (the flow is still adiabatic). Hence, the knowledge of s local Mach number immediately translated to the knowledge of a local temperature via the formula

$$T(M) = \frac{T_0}{1 + \frac{\kappa - 1}{2} M^2}$$

As concern the pressure calculations, we have derived the differential relation

$$\frac{dp}{p} = -\frac{1 + (\kappa - 1) M^2}{M\left(1 + \frac{\kappa - 1}{2} M^2\right)} dM$$

Which can be effectively integrated as follows

$$\int_p^{p_*} \frac{dp}{p} = -\int_M^1 \frac{1 + (\kappa - 1) M^2}{M\left(1 + \frac{\kappa - 1}{2} M^2\right)} dM$$

In the above, the symbol p_* denotes the pressure of the gas in (real or hypothetical) critical section of the flow.

The integral in the right-hand side can be calculated analytically. The final results is

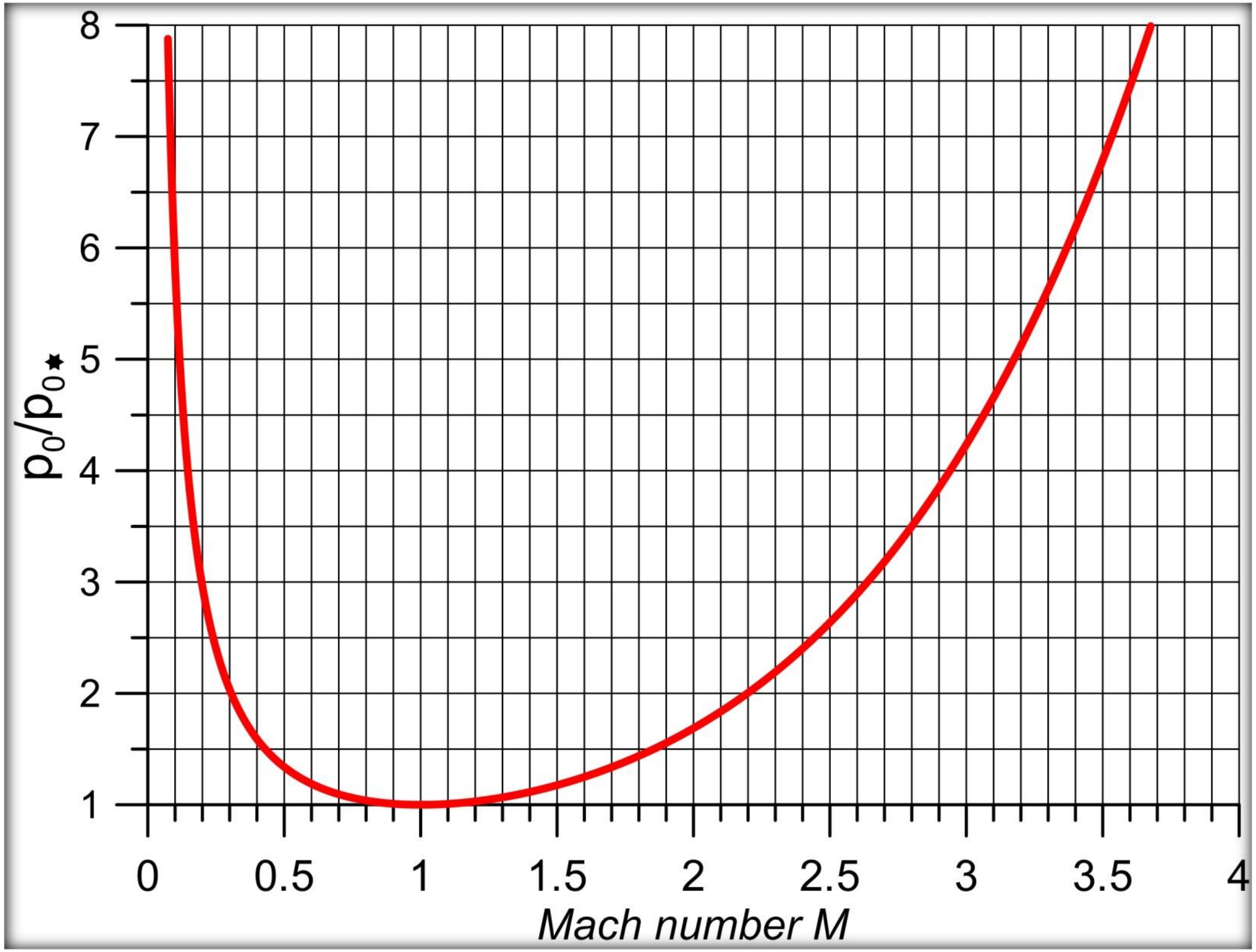
$$\frac{p}{p_*}(M) = \frac{1}{M}\sqrt{\frac{\kappa+1}{2+(\kappa-1)M^2}} \qquad (*)$$

Again, it is a common "tradition" to use the ratio of stagnation pressures rather than local pressures. The following operation brings a desirable result

$$\frac{p_0}{p_{0*}}(M) = \underbrace{\frac{p}{p_0}(M)}_{\substack{\text{isentropic}\\ \text{for } M}} \underbrace{\frac{p}{p_*}(M)}_{\substack{\text{formula } (*)\\ \text{for } M}} \underbrace{\frac{p_*}{p_{0*}}}_{\substack{\text{isentropic}\\ \text{for } M_* \equiv 1}}$$

In the explicit form

$$\frac{p_0}{p_{0*}}(M) = \frac{1}{M}\left[\frac{2+(\kappa-1)M^2}{\kappa+1}\right]^{\frac{\kappa+1}{2(\kappa-1)}}$$



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Mach number M