PROGRAM ROZWOJOWY
POLITECHNIKI WARSZAWSKIEJ

LECTURE 9

DETERMINATION OF REACTION FORCES USING INTEGRAL FORM OF THE LINEAR MOMENTUM PRINCIPLE

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The integral form of the momentum equation (steady motion)

In the Lecture 3 we derived the integral form of the Linear Momentum Principle. Let us write it again in the form

$$\int_{\Omega} \frac{\partial}{\partial t}(\rho \boldsymbol{v})\, dV + \int_{\partial\Omega} (\rho \boldsymbol{v}) \underbrace{(\boldsymbol{v} \cdot \boldsymbol{n})}_{\upsilon_n} dS = \int_{\partial\Omega} \boldsymbol{\sigma}\, dS + \int_{\Omega} \rho \boldsymbol{f}\, dV$$

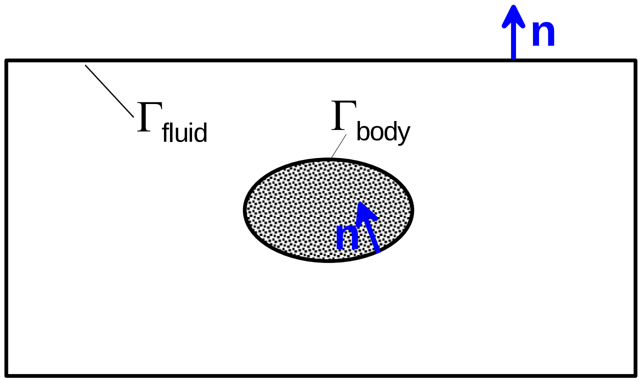
Consider a stationary flow and assume that the external force field can be neglected. The above equality simplifies to

$$\int_{\partial\Omega} \boldsymbol{\sigma}\, dS = \underbrace{\int_{\partial\Omega} (\rho \boldsymbol{v}) \upsilon_n\, dS}_{\substack{\text{momentum flux} \\ \text{through the boundary}}}$$

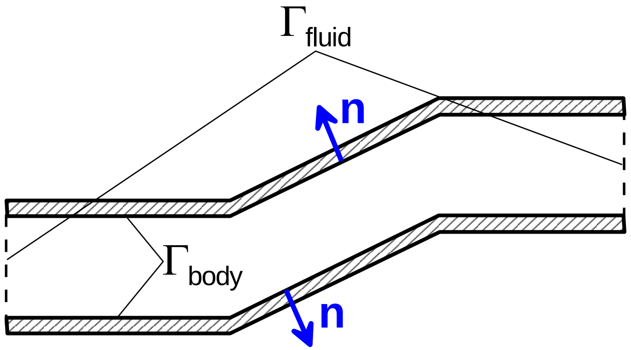
The obtained relation is nothing else like the integral form of the momentum principle written for a stationary flow. **Note that it contains exclusively the integrals over the boundary of the control volume (no volume integral are present).**

Assume next that the boundary of the control volume Ω can be divided into two parts: surface of the body and the fluid boundary. Typical examples of such configurations for external and internal flows are depicted in the figures. Thus, we can write

$$\underbrace{\int_{\Gamma_{body}} \boldsymbol{\sigma}\, dS}_{-\boldsymbol{F}} = \int_{\partial\Omega} (\rho \boldsymbol{v}) v_n \, dS - \int_{\Gamma_{fluid}} \boldsymbol{\sigma}\, dS$$



where the vector $\boldsymbol{F}$ is the **reaction on the immersed body from the fluid** contained in the control volume.



If we assume that the body is **impermeable** then $\left. v_n \right|_{\Gamma_{body}} = 0$ and we arrive at the formula which contains the surface integrals over the fluid part of the boundary.

$$-\boldsymbol{F} = \int_{\Gamma_{fluid}} (\rho \boldsymbol{v}) v_n \, dS - \int_{\Gamma_{fluid}} \boldsymbol{\sigma}\, dS$$

Note that the obtained formula is valid for both incompressible and compressible flows.

Consider now an incompressible flow. As we know, the surface stress vector is equal

$$\boldsymbol{\sigma} = -p\,\boldsymbol{n} + 2\mu\,\boldsymbol{D}\boldsymbol{n}$$

and the formula for the reaction force $\boldsymbol{F}$ can be written as follows

$$\boldsymbol{F} = -\int_{\Gamma_{fluid}} (\rho\boldsymbol{v})\upsilon_n\,dS - \int_{\Gamma_{fluid}} p\,\boldsymbol{n}\,dS + 2\mu\int_{\Gamma_{fluid}} \boldsymbol{D}\boldsymbol{n}\,dS$$

Quite often, we can choose Γ_{fluid} in such way that **the viscous term is relatively small and can be neglected**. Then

$$\boldsymbol{F} = -\int_{\Gamma_{fluid}} (\rho\boldsymbol{v})\upsilon_n\,dS - \int_{\Gamma_{fluid}} p\,\boldsymbol{n}\,dS\,.$$

Sometimes the part of the body surface is in the contact with some other motionless fluid (typically, the ambient air) having a uniform pressure p_a.

Note that for the closed surface Γ_{fluid} we have

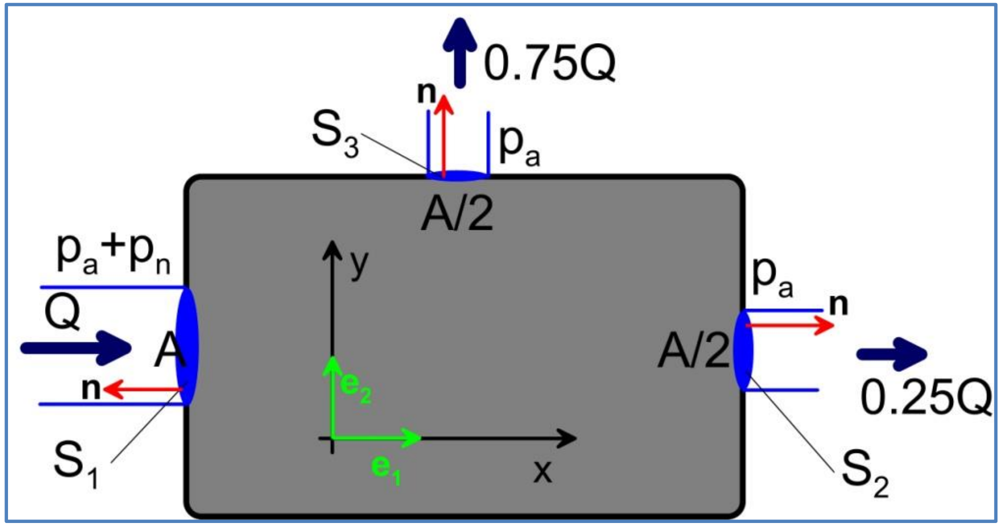
$$\int_{\Gamma_{fluid}} p_a\,\boldsymbol{n}\,dS = p_a\int_{\Gamma_{fluid}} \boldsymbol{n}\,dS = \boldsymbol{0}$$ (why?)

The formula for the actual (net) force can be then written as follows

$$\boldsymbol{F}_{net} = -\int_{\Gamma_{fluid}} (\rho \boldsymbol{v}) \upsilon_n \, dS - \int_{\Gamma_{fluid}} (p - p_a) \boldsymbol{n} \, dS$$

During the tutorial part we will see that the formula in the above form is particularly useful to calculate the reaction force exerted by a free stream colliding with the solid body.

Example: Certain fluid machinery device hidden in the control volume splits the incoming uniform stream of liquid into three outflows – see figure. Calculate the reaction force exerted by the liquid on this device.



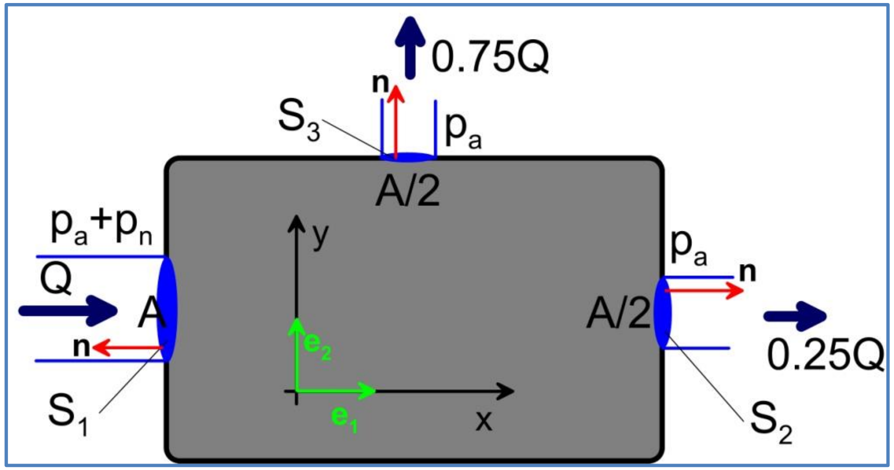
Since the flow is only through the inlet and outlets and we assume that everywhere except the inlet the pressure is p_a, we can assume that $\Gamma = S_1 \cup S_2 \cup S_3$.

Inlet S_1:

$$\boldsymbol{n} = -\boldsymbol{e}_1 = [-1, 0] \ , \ \boldsymbol{v} = \tfrac{Q}{A} \boldsymbol{e}_1 = [\tfrac{Q}{A}, 0]$$

$$\upsilon_n = \boldsymbol{v} \cdot \boldsymbol{n} = -\tfrac{Q}{A}$$

$$\int_{S_1} (\rho \boldsymbol{v}) \upsilon_n \, dS + \int_{S_1} \underbrace{(p - p_a)}_{p_n} \boldsymbol{n} \, dS = [-\rho (\tfrac{Q}{A})^2 A - p_n A] \boldsymbol{e}_1 = -(\rho \tfrac{Q^2}{A} + p_n A) \boldsymbol{e}_1$$



Outlet S_2:

$$\boldsymbol{n}=\boldsymbol{e}_1=[1,0],\ \boldsymbol{\upsilon}=\frac{\frac{1}{4}Q}{\frac{1}{2}A}\boldsymbol{e}_1=[\frac{Q}{2A},0],\ \upsilon_n=\boldsymbol{\upsilon}\cdot\boldsymbol{n}=\frac{Q}{2A}$$

$$\int_{S_2}(\rho\boldsymbol{\upsilon})\upsilon_n\,dS+\int_{S_2}\underbrace{(p-p_a)}_{0}\boldsymbol{n}\,dS=\rho(\frac{Q}{2A})^2\frac{1}{2}A\boldsymbol{e}_1=$$

$$=\frac{1}{8}\rho\frac{Q^2}{A}\boldsymbol{e}_1$$

Outlet S_3:

$$\boldsymbol{n}=\boldsymbol{e}_2=[0,1]\ ,\ \boldsymbol{\upsilon}=\frac{\frac{3}{4}Q}{\frac{1}{2}A}\boldsymbol{e}_2=[0,\frac{3Q}{2A}]\ \ ,\ \ \upsilon_n=\boldsymbol{\upsilon}\cdot\boldsymbol{n}=\frac{3Q}{2A}$$

$$\int_{S_3}(\rho\boldsymbol{\upsilon})\upsilon_n\,dS+\int_{S_3}\underbrace{(p-p_a)}_{0}\boldsymbol{n}\,dS=\rho(\frac{3Q}{2A})^2\frac{1}{2}A\boldsymbol{e}_2=\frac{9}{8}\rho\frac{Q^2}{A}\boldsymbol{e}_2$$

Finally, the reaction force is

$$\boldsymbol{F}=(\rho\frac{Q^2}{A}+p_nA)\boldsymbol{e}_1-\frac{1}{8}\rho\frac{Q^2}{A}\boldsymbol{e}_1-\frac{9}{8}\rho\frac{Q^2}{A}\boldsymbol{e}_2=$$

$$=(\underbrace{\frac{7}{8}\rho\frac{Q^2}{A}+p_nA}_{F_1})\boldsymbol{e}_1+(\underbrace{-\frac{9}{8}\rho\frac{Q^2}{A}}_{F_2})\boldsymbol{e}_2$$

STRESS AND REACTION FORCE EXERTED AT AN IMMERSED SURFACE

We will derive the pretty general formula for the wall stress and reaction the force which shows the relation between wall tangent stress and wall distribution of vorticity.

We again begin with the most general formula

$$\boldsymbol{F} = \int_{\partial\Omega} \boldsymbol{\sigma}\, dS = \int_{\partial\Omega} \boldsymbol{\Xi}\,\boldsymbol{n}\, dS.$$

The constitutive relation for an incompressible Newtonian fluid can be written as follows

$$\boldsymbol{\Xi} = -p\boldsymbol{I} + 2\mu\boldsymbol{D} = -p\boldsymbol{I} + 2\mu \underbrace{\nabla\boldsymbol{v}}_{\text{gradient of velocity}} - 2\mu \underbrace{\boldsymbol{R}}_{\text{rotation tensor}} .$$

Since

$$\boldsymbol{R}\boldsymbol{n} = -\tfrac{1}{2}\boldsymbol{n}\times \mathrm{rot}\,\boldsymbol{v} = -\tfrac{1}{2}\boldsymbol{n}\times\boldsymbol{\omega}$$

we can also write

$$\boldsymbol{\sigma} = \boldsymbol{\Xi}\,\boldsymbol{n} = -p\boldsymbol{n} + 2\mu\nabla\boldsymbol{v}\cdot\boldsymbol{n} + \mu\boldsymbol{n}\times\boldsymbol{\omega}.$$

The following theorem holds:

If $div\,\boldsymbol{v} = 0$ (incompressible flow) and $\boldsymbol{v}\big|_{\partial\Omega} = \boldsymbol{0}$ then $\nabla\boldsymbol{v}\cdot\boldsymbol{n} + \boldsymbol{n}\times\boldsymbol{\omega} = \boldsymbol{0}$.

Proof:

Since $\boldsymbol{v}\big|_{\partial\Omega} = \boldsymbol{0}$ then the boundary $\partial\Omega$ is the **izosurface** for all components of the velocity field and the **gradients** of these components **must be perpendicular** (normal) to $\partial\Omega$.

Thus, we can write

$$\nabla\upsilon_j\big|_{\partial\Omega}\times\boldsymbol{n} = \boldsymbol{0} \;\Rightarrow\; \frac{\partial\upsilon_j}{\partial x_k} = \lambda_j\, n_k \;,\; k = 1,2,3$$

for some real numbers λ_j (j = 1, 2, 3).

Next, in the index notation we have

$$\boldsymbol{n}\times\boldsymbol{\omega} = \in_{ijk} n_j\, \omega_k\, \boldsymbol{e}_i \quad , \quad \nabla\boldsymbol{v}\cdot\boldsymbol{n} = \frac{\partial\upsilon_i}{\partial x_j} n_j\, \boldsymbol{e}_i$$

After insertion we get

$$\nabla \boldsymbol{\upsilon} \cdot \boldsymbol{n} + \boldsymbol{n} \times \boldsymbol{\omega} = (\tfrac{\partial}{\partial x_j} \upsilon_i + \in_{ijk} \omega_k) n_j \boldsymbol{e}_i = (\tfrac{\partial}{\partial x_j} \upsilon_i + \in_{ijk} \in_{k\alpha\beta} \tfrac{\partial}{\partial x_\alpha} \upsilon_\beta) n_j \boldsymbol{e}_i =$$

$$= \left[\tfrac{\partial}{\partial x_j} \upsilon_i + (\delta_{i\alpha}\delta_{j\beta} - \delta_{i\beta}\delta_{j\alpha}) \tfrac{\partial}{\partial x_\alpha} \upsilon_\beta \right] n_j \boldsymbol{e}_i = (\tfrac{\partial}{\partial x_j} \upsilon_i + \tfrac{\partial}{\partial x_i} \upsilon_j - \tfrac{\partial}{\partial x_j} \upsilon_i) n_j \boldsymbol{e}_i =$$

$$= \tfrac{\partial}{\partial x_i} \upsilon_j n_j \boldsymbol{e}_i = (\tfrac{\partial}{\partial x_i} \upsilon_j n_j - \underbrace{\tfrac{\partial}{\partial x_j} \upsilon_j}_{= div\, \boldsymbol{\upsilon} = 0} n_i) \boldsymbol{e}_i = (\lambda_j n_i n_j - \lambda_j n_j n_i) \boldsymbol{e}_i = \boldsymbol{0}$$

Using the above result in the formula for the stress vector, we finally obtain the formula

$$\boldsymbol{\sigma} = -p\boldsymbol{n} - \mu \boldsymbol{n} \times \boldsymbol{\omega}.$$

Note that $p\boldsymbol{n} \times \boldsymbol{n} = 0$ and $(\boldsymbol{n} \times \boldsymbol{\omega}) \cdot \boldsymbol{n} = 0$ so the **first term corresponds to the normal stress** while **the second one – to the tangent stress** at the boundary surface $\partial\Omega$.

The **total aerodynamic force** can be calculated from the integral formula

$$\boldsymbol{F} = -\int_{\partial\Omega} (p\boldsymbol{n} + \mu \boldsymbol{n} \times \boldsymbol{\omega}) dS$$

Interestingly enough, the **above formula for the force F can be derived without the assumption that the velocity is zero at the boundary** (however in such case the formula for the stress vector is not generally true!). Indeed, we have

$$\boldsymbol{F} = \int_{\partial\Omega} \boldsymbol{\sigma}\, dS = -\int_{\partial\Omega} p\boldsymbol{n}\, dS + 2\mu \int_{\partial\Omega} \nabla\boldsymbol{v}\cdot\boldsymbol{n}\, dS + \mu \int_{\partial\Omega} \boldsymbol{n}\times\boldsymbol{\omega}\, dS\,.$$

But

$$\int_{\partial\Omega} \nabla\boldsymbol{v}\cdot\boldsymbol{n}\, dS \underset{\substack{\Uparrow\\ \text{tensor version}\\ \text{of GGO}}}{=} \int_{\Omega} Div(\nabla\boldsymbol{v})\, d\boldsymbol{x} = \int_{\Omega} \underset{\substack{\text{Laplacian of}\\ \text{the velocity}}}{\Delta\boldsymbol{v}}\, d\boldsymbol{x} =$$

$$= \int_{\Omega} \nabla(\underbrace{\nabla\cdot\boldsymbol{v}}_{=0})\, d\boldsymbol{x} - \int_{\Omega} \nabla\times(\underbrace{\nabla\times\boldsymbol{v}}_{=\boldsymbol{\omega}})\, d\boldsymbol{x} = -\int_{\Omega} \nabla\times\boldsymbol{\omega}\, d\boldsymbol{x} \underset{\substack{\Uparrow\\ \text{GGO for}\\ \text{the cross}\\ \text{product}}}{=} -\int_{\partial\Omega} \boldsymbol{n}\times\boldsymbol{\omega}\, dS$$

Thus

$$\boldsymbol{F} = -\int_{\partial\Omega} p\boldsymbol{n}\, dS - 2\mu \int_{\partial\Omega} \boldsymbol{n}\times\boldsymbol{\omega}\, dS + \mu \int_{\partial\Omega} \boldsymbol{n}\times\boldsymbol{\omega}\, dS = -\int_{\partial\Omega} (p\boldsymbol{n} + \mu\boldsymbol{n}\times\boldsymbol{\omega})\, dS$$

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