PROGRAM ROZWOJOWY
POLITECHNIKI WARSZAWSKIEJ

LECTURE 15

INTRODUCTION TO HYDRODYNAMIC INSTABILITY AND TURBULENCE

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General rule: when the Reynolds number corresponding to a given laminar flow increases, the flow gets more complicated and finally undergoes a transition to a turbulent state. The transition problem is initiated by external disturbances, which are usually quite small and uncontrollable.

The crucial question is about flow stability: how large disturbances can be absorbed by a given flow without changing its long-term form?

General mathematical approach

$\boldsymbol{v}^{(1)}$ - velocity of the basic flow;

$\boldsymbol{v}^{(2)}$ - velocity of the flow in the same domain but with different initial conditions.

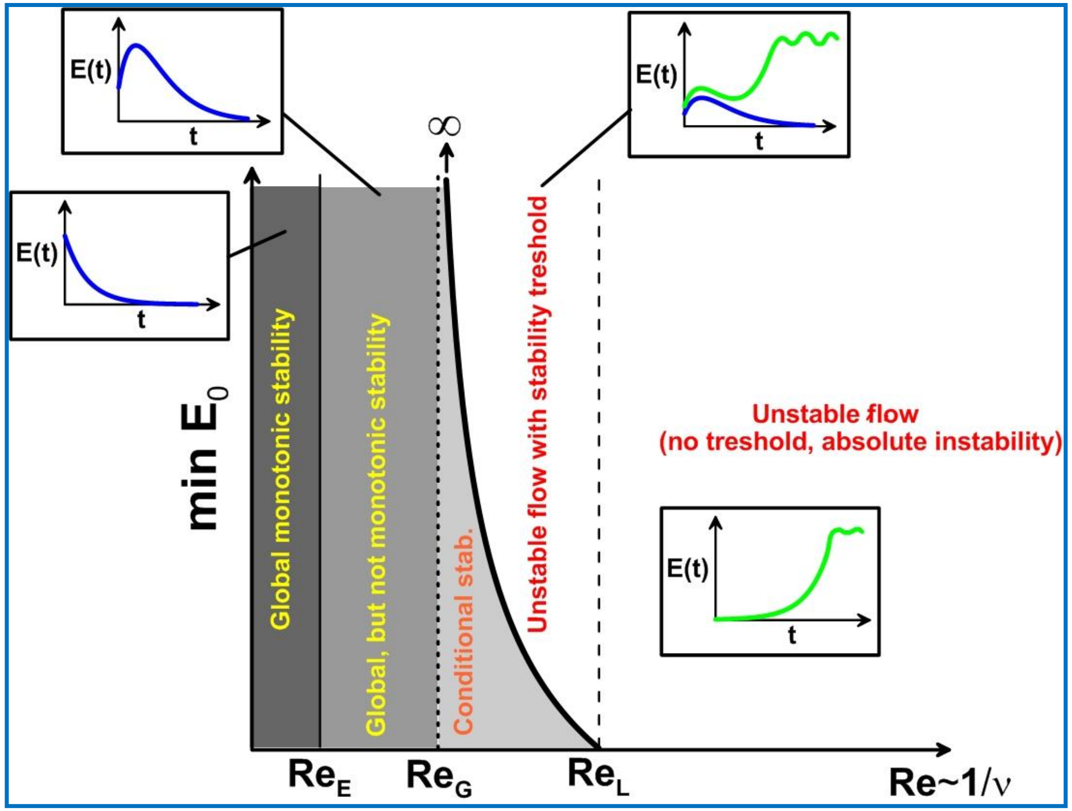
Convenient measure of the global discrepancy between two flows

$$E(t) = \frac{1}{2|\Omega|} \int_{\Omega} \| \boldsymbol{v}^{(1)} - \boldsymbol{v}^{(2)} \|^2 dV$$

Note that $E(0)$ is known; it can be computed since we know $\boldsymbol{v}^{(1)}(t=0)$ and $\boldsymbol{v}^{(2)}(t=0)$

The flow is (asymptotically) stable if $\lim_{t\to\infty} E(t) = 0$

Scheme of different scenarios



For Re > Re_L there exists a disturbance field with a shape such that – no matter how small – it always leads to a permanent change of the flow.

Let $\boldsymbol{\upsilon} = \boldsymbol{\upsilon}_b + \boldsymbol{\upsilon}'$ and $p = p_b + p'$, where $(\boldsymbol{\upsilon}_b, p_b)$ is the basic flow and $(\boldsymbol{\upsilon}', p')$ is the disturbance field. We insert the formulae to the continuum and Navier-Stokes equations. The terms involving only basic flow parameters will cancel out (the basic flow is itself the NS solution); the remaining nonlinear terms involve only the disturbances. Since they are assumed small, the nonlinear terms can be dropped. **This procedure is called the linearization.**

Continuity equation for disturbances

$$\frac{\partial \upsilon'_k}{\partial x_k} = 0$$

Linearized N-S equations

$$\frac{\partial \upsilon'_k}{\partial t} + \upsilon_{b,i}\frac{\partial \upsilon'_k}{\partial x_i} + \upsilon'_i \frac{\partial \upsilon_{b,k}}{\partial x_i} = -\frac{1}{\rho}\frac{\partial p'}{\partial x_k} + \nu\,\Delta \upsilon'_k$$

Velocity disturbances vanish at the boundary

$$\upsilon'_k\big|_{\partial\Omega} = 0$$

We have obtained the linear differential system. The functional coefficients depend on the known basic flow solution.

The general approach to linear stability equations is to seek their solutions in the following:

$$\upsilon'_k = \sum_r q_k^{(r)} e^{i\lambda_r t}, \;\; p' = \sum_r \beta^{(r)} e^{i\lambda_r t}$$

Since the equations are linear, it is sufficient to see what happens with a single Fourier mode. We insert it in the equations and get

$$\frac{\partial q_k^{(r)}}{\partial x_k} = 0$$

$$i\lambda_r q_k^{(r)} + \upsilon_{b,i} \frac{\partial q_k^{(r)}}{\partial x_i} + q_k^{(r)} \frac{\partial \upsilon_{b,k}}{\partial x_i} = -\frac{1}{\rho} \frac{\partial \beta^{(r)}}{\partial x_k} + \nu \Delta q_k^{(r)}$$

$$q_k^{(r)} \Big|_{\partial\Omega} = 0$$

Nontrivial (nonzero) solutions exist only for certain values (eigenvalues) of the complex frequency λ_r

$$\lambda_r = \alpha_r + i\gamma_r$$

Note that the exponential factor can be written as follows :

$$e^{i\lambda_r t} = \underbrace{\left(e^{i\alpha_r t}\right)}_{\textit{harmonic oscillation}} \underbrace{e^{-\gamma_r t}}_{\textit{amplification factor}}$$

Sufficient condition for (absolute) instability is that at least one number γ_r is negative!

Particular stability theories have been developed for different classes of flows (parallel and nearly-parallel flow, wake flows, mixing layers, rotating flows, etc.).

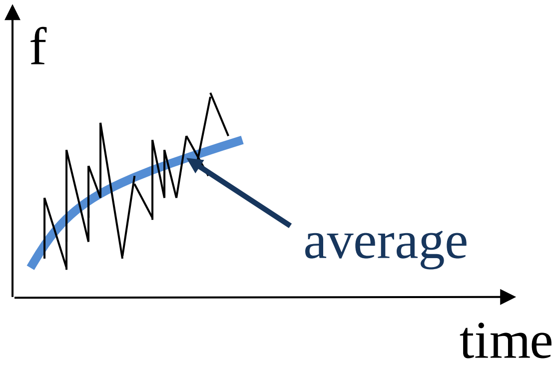
FOUNDATIONS OF TURBULENT FLOWS

Main features of turbulent flow fields:

- All flow parameters (velocity, pressure, etc.) exhibit highly irregular variations in time and space (spatial-temporal chaos). Although in principle turbulent flows are believed to be governed by entirely deterministic physical laws, the most adequate approach is to treat them as random phenomena and apply to them appropriate tools from the theory of stochastic processes.
- The turbulent flows exhibit very strong mixing and transport properties - effectiveness of turbulent mixing and heat transfer is usually by several orders of magnitude larger than effectiveness of analogous molecular processes.
- Turbulence in fluids is a strongly nonlinear phenomenon very closely related to the dynamics (generation, advection and diffusion) of vorticity. “Real” turbulence can exists in the 3D flows, where the process of vortex tubes stretching and tilting is possible. This process is mostly responsible of the generation of time scales of vortex motion.
- Turbulent flow are characterized by a unidirectional “cascade” of energy: large vortex structures fed on the mean flow, the smaller-scale motion sucks energy from larger scales, finally the smallest-scale “ripples” dissipate due to molecular viscosity. This process has some universal properties described by simple power laws (e.g. Kolmogorov law).

Reynolds averaging procedure

Any dynamic quantity in the turbulent flow description can be expressed as a sum of its averaged value (constant or changing slowly) plus fluctuation in the form of rapid irregular oscillation around the averaged value. The fluctuations are assumed completely random.



$$f = \overline{f} + f'$$

average ($\overline{f}$), *fluctuation* (f')

$$\overline{f'} = 0$$

average of the fluctuation

Definition of the time average

$$\overline{f} = \frac{1}{2T}\int_{t-T}^{t+T} f(\tau)\,d\tau$$

Averaging period 2T should be much longer than characteristic time of the fluctuations yet short enough to maintain the information of the time variation of the averaged flow.

Average quantity $\overline{f}$ depends of time and position. Its spatial derivatives are computes as follows

$$\frac{\partial \overline{f}}{\partial x} = \frac{1}{2T}\frac{\partial}{\partial x}\int_{t-T}^{t+T} f(\tau)\,d\tau = \frac{1}{2T}\frac{\partial}{\partial x}\int_{t-T}^{t+T}\frac{\partial f(\tau)}{\partial x}\,d\tau = \overline{\frac{\partial f}{\partial x}}$$

Hence, Spatial derivative of the average is the average of the spatial derivative – time averaging and spatial differentiation commute!

Time derivative is a bit more tricky …

$$\frac{\partial \overline{f}}{\partial t} = \frac{1}{2T}\frac{\partial}{\partial t}\int_{t-T}^{t+T} f(\tau)\,d\tau = \frac{1}{2T}\left[f(t+T) - f(t-T)\right] = \frac{1}{2T}\int_{t-T}^{t+T}\frac{\partial f}{\partial \tau}\,d\tau$$

… but the conclusion is similar.

Instantaneous flow field is the sum of the average and the fluctuating parts

$$\upsilon_k = \overline{\upsilon}_k + \upsilon'_k \qquad p = \overline{p} + p'$$

We insert these expressions to Navier-Stokes equations …

$$\frac{\partial \upsilon_k}{\partial t} + \frac{\partial}{\partial x_i}\left(\upsilon_k \upsilon_i\right) = -\frac{1}{\rho}\frac{\partial p}{\partial x_k} + \nu \Delta \upsilon_k$$

… and apply the time-averaging procedure. The result **the RANS (Reynolds-Averaged Navier-Stokes) equations**

$$\frac{\partial \bar{\upsilon}_k}{\partial t} + \frac{\partial}{\partial x_i}\left(\bar{\upsilon}_k \cdot \bar{\upsilon}_i\right) = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_k} + \nu \Delta \bar{\upsilon}_k + \frac{\partial}{\partial x_i}\left(-\overline{\upsilon_i' \upsilon_k'}\right)$$

Note that the **RANS eqs. differ from the NS eqs.** by an additional term in the right-hand side, i.e. $\partial_{x_i}\overline{\left(-\upsilon_i'\upsilon_k'\right)}$.

We introduce a symmetric tensor called the **Reynolds tensor**

$$\overline{\upsilon_i' \upsilon_k'} = \overline{\upsilon_k' \upsilon_i'} = -R_{ik} = -R_{ki}$$

The averaged velocity satisfies the continuity equation $div\,\bar{\boldsymbol{\upsilon}} = 0$.

In contrast to the **NS eqs**, the **RANS system is not closed**! We need to determine the components of the **RT**. The concept is to express these components by the averaged values …

$$\boldsymbol{R} = \boldsymbol{R}\left(\bar{\upsilon}_1, \bar{\upsilon}_2, \bar{\upsilon}_3, \frac{\partial \bar{\upsilon}_1}{\partial x_1}, \frac{\partial \bar{\upsilon}_1}{\partial x_2}, \cdots \right)$$

Closure hypothesis: to formulate some explicit form of the above formula. Such formula would express the features of the flow rather than physical properties of the fluid!

We can write **RT** in the following way:

$$R_{ik} = \frac{1}{3} Tr(\boldsymbol{R})\delta_{ik} + \left(R_{ik} - \frac{1}{3} Tr(\boldsymbol{R})\delta_{ik} \right)$$

where the operator **Tr** denotes the trace of the tensor.

We have

$$-Tr(\mathbf{R}) = \overline{\upsilon_1'\upsilon_1'} + \overline{\upsilon_2'\upsilon_2'} + \overline{\upsilon_3'\upsilon_3'} = 2\frac{\overline{\upsilon_1'\upsilon_1'} + \overline{\upsilon_2'\upsilon_2'} + \overline{\upsilon_3'\upsilon_3'}}{2} = 2\kappa$$

where the symbol κ denotes the turbulent kinetic energy. Thus, we have

$$Tr(\mathbf{R}) = -2\kappa \quad \Longrightarrow \quad R_{ik} = -\frac{2}{3}\kappa\delta_{ik} + \left(R_{ik} + \frac{2}{3}\kappa\delta_{ik}\right)$$

Next, we can write the “turbulent” term in the RANS eqs. as follows:

$$\frac{\partial}{\partial x_i}\left(-\overline{\upsilon_i'\upsilon'}_k\right) = \frac{\partial R_{ik}}{\partial x_i} = -\frac{2}{3}\frac{\partial \kappa}{\partial x_k} + \frac{\partial}{\partial x_i}T_{ik}^t$$

Once inserted into the RANS eqs., the result is

$$\frac{\partial \bar{\upsilon}_k}{\partial t} + \frac{\partial}{\partial x_i}(\bar{\upsilon}_k\bar{\upsilon}_i) = -\frac{1}{\rho}\frac{\partial}{\partial x_k}\left(\bar{p} + \frac{2}{3}\rho\kappa\right) + \nu\,\Delta\bar{\upsilon}_k + \frac{\partial T_{ik}^t}{\partial x_i}$$

The following objects are present in the RANS eqs.:

$$p_t = \bar{p} + \frac{2}{3}\rho\kappa$$ ⇐ turbulent pressure

$$T^t_{ik} = R_{ik} + \frac{2}{3}\kappa\delta_{ik}$$ ⇐ turbulent stress tensor (TST)

Note that both TST and the averaged deformation rate tensor

$$\bar{D}_{ij} = \frac{1}{2}\left(\frac{\partial \bar{\upsilon}_i}{\partial x_k} + \frac{\partial \bar{\upsilon}_k}{\partial x_i}\right)$$

have the zero trace.

Thus, the following (and commonly accepted in engineering applications) hypothesis can be formulated: there exists a **scalar field of a turbulent viscosity** μ_{turb} such that

$$\boldsymbol{T}^t = 2\mu_{turb} \cdot \bar{\boldsymbol{D}}$$

or, in components …

$$T_{ik}^t = \mu_{turb}\left(\frac{\partial \bar{\upsilon}_i}{\partial x_k} + \frac{\partial \bar{\upsilon}_k}{\partial x_i}\right)$$

μ_{turb} – the property of the flow, not the physical property of the fluid (like the molecular viscosity μ).

Imagine, we can determine μ_{turb} **in terms of the averaged quantities. It so, the turbulent motion is described by the following set of equations:**

$$\begin{cases} \dfrac{\partial \bar{\upsilon}_i}{\partial x_i} = 0 \\ \dfrac{\partial \bar{\upsilon}_k}{\partial t} + \dfrac{\partial}{\partial x_i}\left(\bar{\upsilon}_k \bar{\upsilon}_i\right) = -\dfrac{1}{\rho}\dfrac{\partial}{\partial x_k} p_{turb} + \nu \Delta \bar{\upsilon}_k + \dfrac{\partial}{\partial x_i}\left(\mu_{turb} \bar{D}_{ik}\right) \end{cases}$$

If μ_{turb} is known as a function of averaged quantities then the above equations (subject to appropriate boundary and initial conditions) can be solved for $\bar{\upsilon}_k, k=1,2,3$ and p_{turb}.

Determination of μ_{turb}

Classification is based on the number of additional differential equations which are added to the RANS system is order to make it closed. We have:

- Zero-equation models: no differential equations, just some algebraic relations
- One-equation models: one differential equation (transport equations for turbulent viscosity) plus some additional algebraic relations (eg. Spalart-Allmaras model)
- Two-equations models: two additional transport equations are added to RANS eqs.

1. Mixing length theory

$$\mu_{turb} \sim l^2 \left| \frac{\partial \bar{\upsilon}}{\partial n} \right|$$

$\frac{\partial \bar{\upsilon}}{\partial n}$ - normal derivative of the dominant velocity component

l - mixing length (determined experimentally)

2. κ – ε method

$$\mu_{turb} \sim \frac{\kappa^2}{\varepsilon}$$

κ - turbulent kinetic energy
ε – turbulent dissipation field (mass-specific power dissipated to heat by turbulent fluctuations)

The fields of κ and ε are computed from two additional partial differential equations which are solved (numerically) simultaneously with the RANS and averaged continuity equations.