PROGRAM ROZWOJOWY
POLITECHNIKI WARSZAWSKIEJ

FLUID MECHANICS 3 - LECTURE 2

STEADY GAS FLOWS IN DUCTS WITH VARIABLE CROSS-SECTIONS

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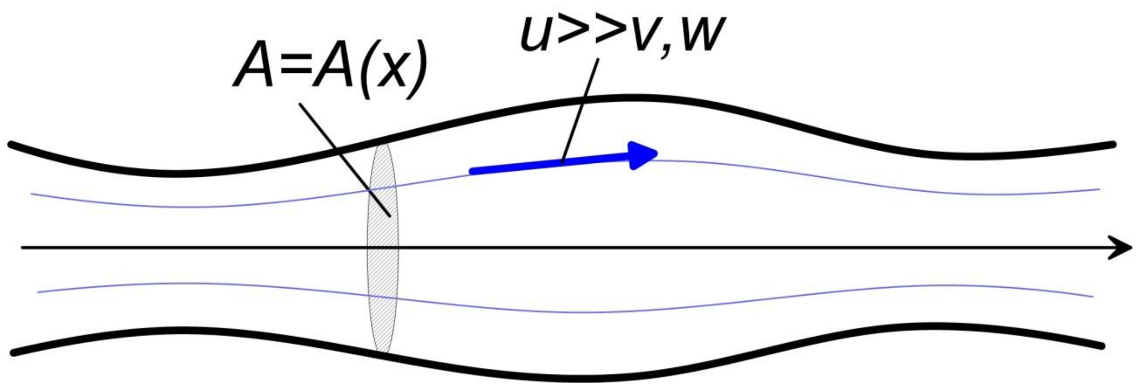
Consider the stationary gas flow through the variable-section duct (see figure)



We assume that the relative rate of change of the cross-section area along the duct is small

$$\frac{1}{A(x)}\frac{dA(x)}{dx} \ll 1$$

Consequently, the flow predominantly unidirectional. It means that

$$u(x) \gg \upsilon(x), w(x) \quad , \quad \boldsymbol{\upsilon}(x) \approx u(x)\,\boldsymbol{e}_x$$

Flow is assumed adiabatic and continuous, hence it is isentropic.

The mass flow rate is constant

$$Q_m = \rho(x)u(x)A(x) = const$$

The mass balance can be written by referencing (real or hypothetical) critical cross-section.

$$\rho u A = \rho_* u_* A_* \qquad M = M_* \equiv 1$$

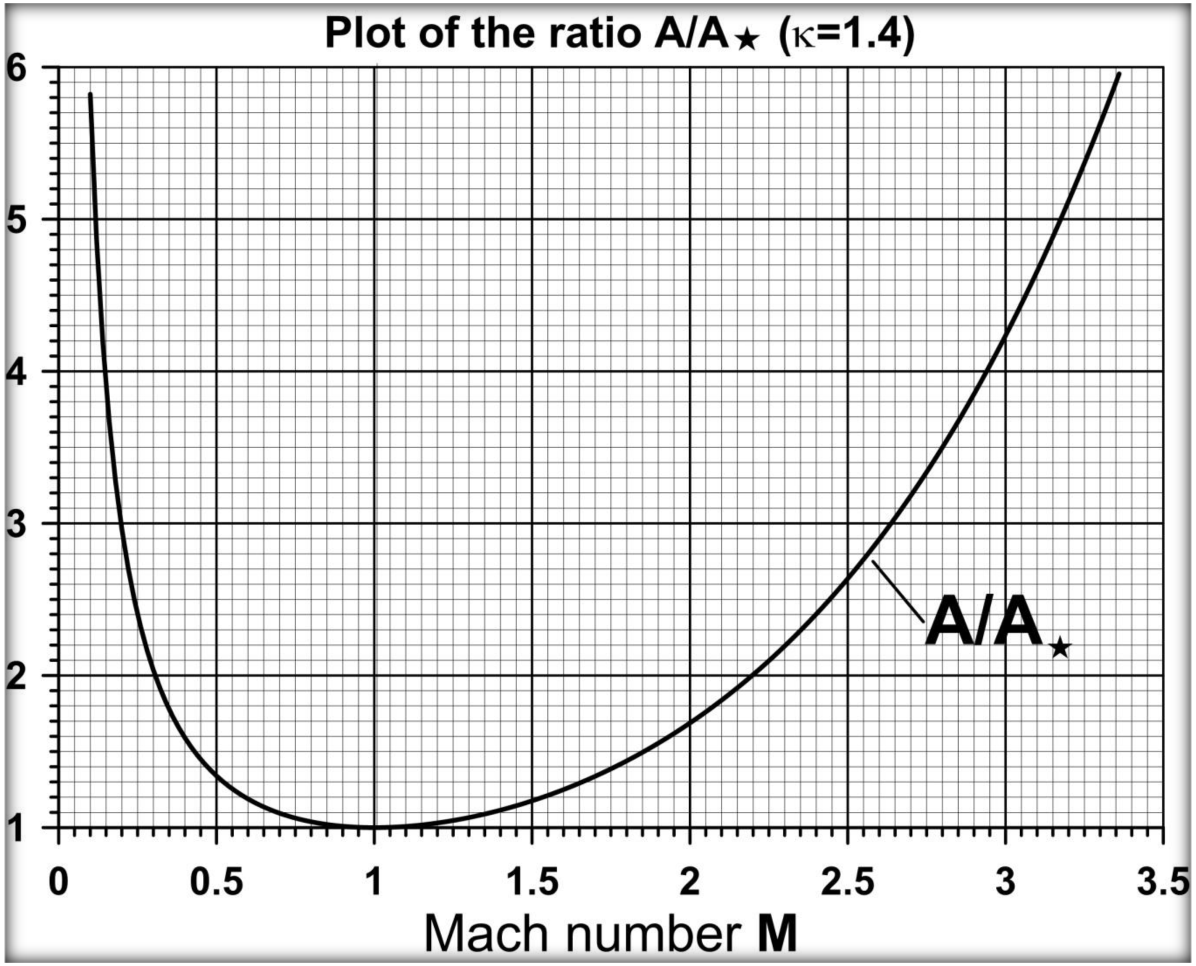
The ration between local section area $A = A(x)$ and the area of a critical section A_* can be expressed as follows

$$\frac{A}{A_*} = \frac{\rho_*}{\rho}\frac{u_*}{u} = \frac{\rho_*}{\rho_0}\left[\frac{\rho}{\rho_0}(M)\right]^{-1}\frac{a_*}{a_0}\left[\frac{a}{a_0}(M)\right]^{-1}\frac{a}{u} =$$

$$= \frac{\rho_*}{\rho_0}\frac{a_*}{a_0}\left[\frac{\rho}{\rho_0}(M)\right]^{-1}\left[\frac{a}{a_0}(M)\right]^{-1}\frac{1}{M} = \mathfrak{F}(M)$$

The explicit form of this relation is

$$\frac{A}{A_*} = \left(\frac{2}{\kappa+1}\right)^{\frac{\kappa+1}{2(\kappa-1)}} \left(1+\frac{\kappa-1}{2}M^2\right)^{\frac{\kappa+1}{2(\kappa-1)}} M^{-1}$$



Analysis of the above relation leads to the following observation

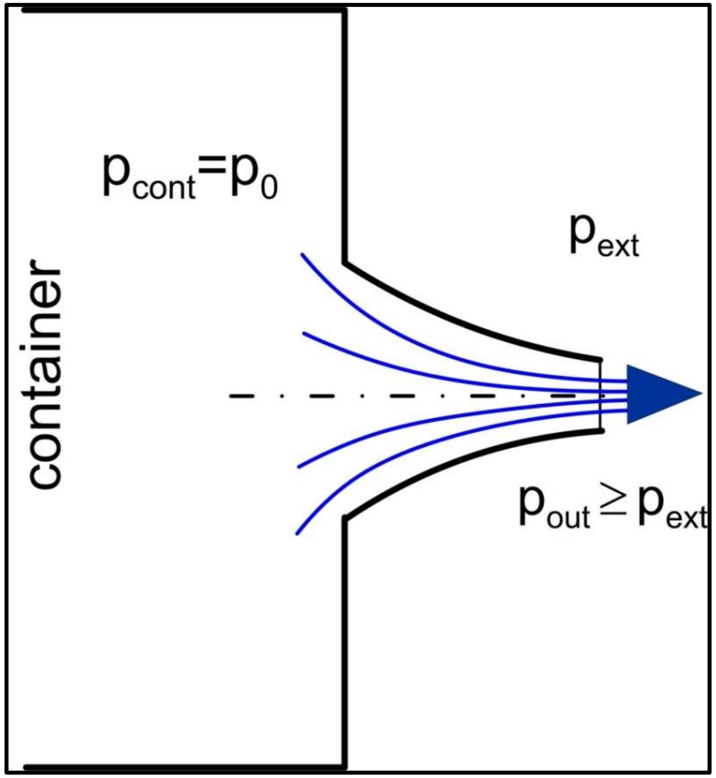
$$M < 1 \quad \Rightarrow \quad \begin{cases} A(x) \searrow \quad then \quad M(x) \nearrow \\ A(x) \nearrow \quad then \quad M(x) \searrow \end{cases}$$

$$M > 1 \quad \Rightarrow \quad \begin{cases} A(x) \searrow \quad then \quad M(x) \searrow \\ A(x) \nearrow \quad then \quad M(x) \nearrow \end{cases}$$

In words:

- subsonic flow speeds up in the convergent channel and slows down in a divergent channel
- supersonic flow slows down in the convergent channel and speeds up in a divergent channel

Flow through convergent nozzle



The outlet pressure:

$$p_{out} = \begin{cases} p_{ext} & if \;\; M_{out} < 1 \\ p_* \geq p_{ext} & if \;\; M_{out} = 1 \end{cases}$$

Critical parameters are achieved at the outlet section if

$$\frac{p_{cont}}{p_{ext}} \geq \frac{p_0}{p_*} = \left(\frac{\kappa+1}{2}\right)^{-\frac{\kappa}{\kappa-1}} \underset{\kappa=1.4}{\approx} \frac{1}{0.528} = 1.894$$

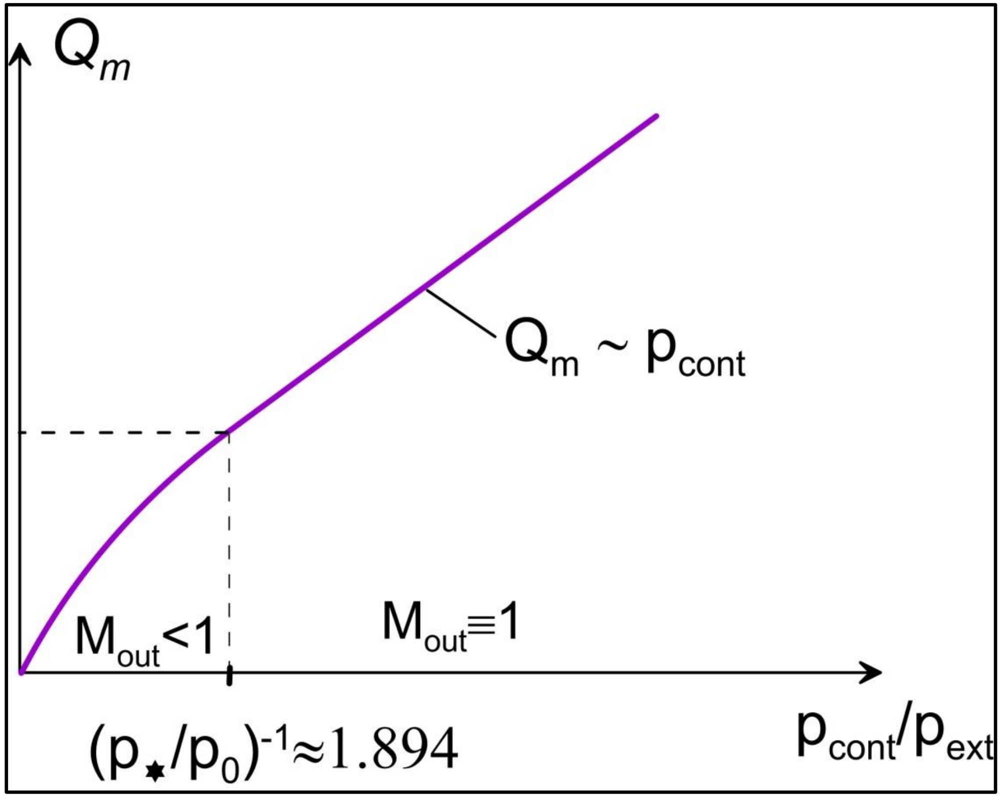
Consider the mass flow $Q_m = (\rho u A)_{out}$. It can be expressed as follows $(T_{cont} \equiv T_0)$

$$Q_m = \rho_0 a_0 A_{out} \frac{\rho}{\rho_0}(M_{out}) \frac{a}{a_0}(M_{out}) \frac{u_{out}}{a_{out}} = \rho_0 a_0 A_{out} \frac{\rho}{\rho_0}(M_{out}) \frac{a}{a_0}(M_{out}) M_{out} =$$

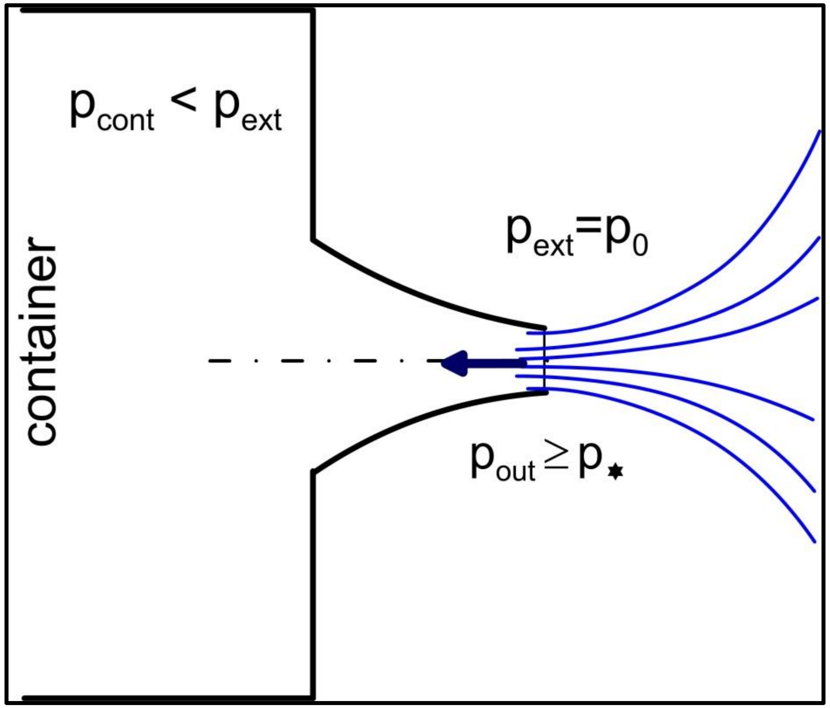
$$= \frac{p_{cont}}{RT_{cont}} \sqrt{\kappa R T_{cont}} A_{out} \frac{\rho}{\rho_0}(M_{out}) \frac{a}{a_0}(M_{out}) M_{out}$$

$$Q_m = A_{out}\sqrt{\frac{\kappa}{R}}\frac{p_{cont}}{\sqrt{T_{cont}}}\frac{\rho}{\rho_0}(M_{out})\frac{a}{a_0}(M_{out})M_{out} \sim \frac{p_{cont}}{\sqrt{T_{cont}}}\frac{\rho}{\rho_0}(M_{out})\frac{a}{a_0}(M_{out})M_{out}$$

Note that if $M_{out} = 1$ then $Q_m \sim \dfrac{p_{cont}}{\sqrt{T_{cont}}}$

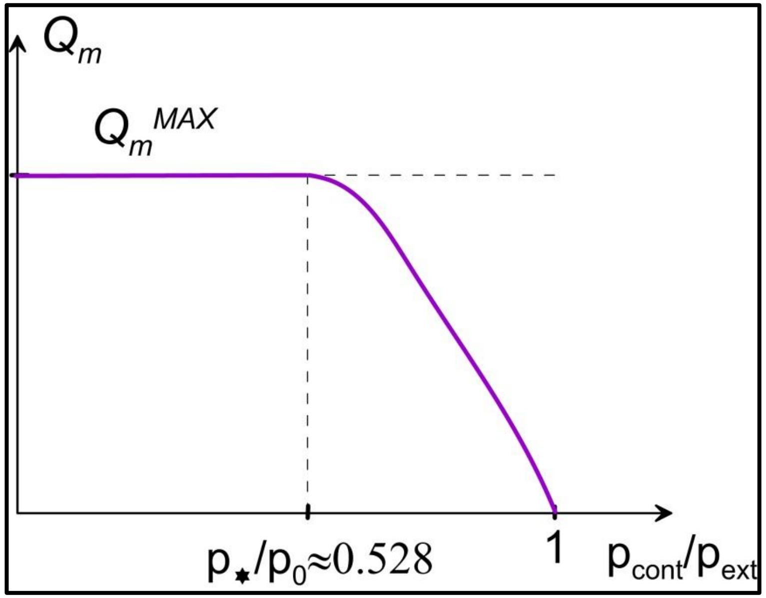


Consider the suction of the gas from free atmosphere to a low-pressure container. The gas flows into the container via the converging channel.

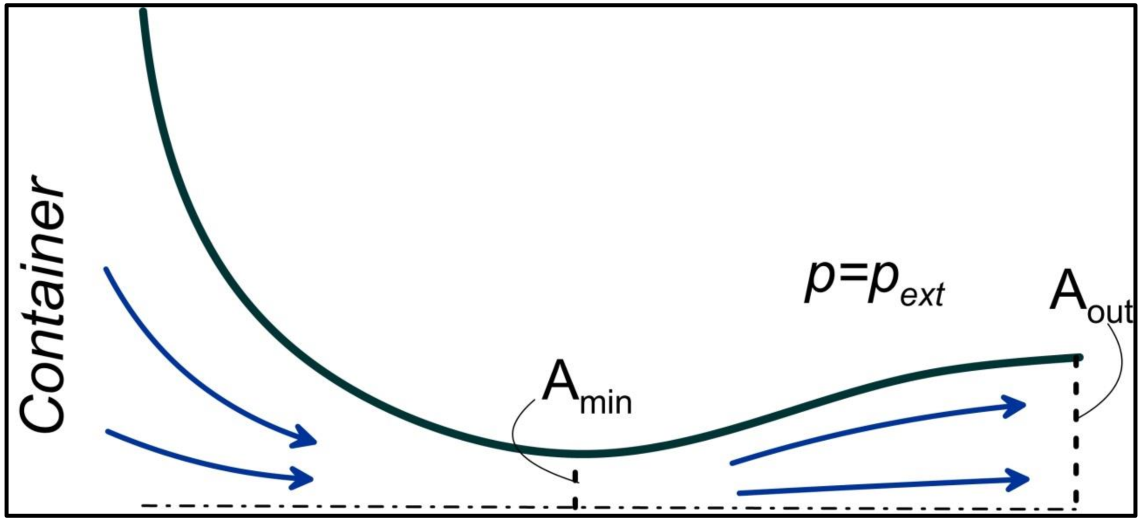


This time, the external pressure plays the role of the total (stagnation) pressure p_0. While the pressure in the container diminishes, the flow rate rises until the critical conditions are achieved in the outlet (or rather inlet) section. Further decrease of the contained pressure cannot affect further the external part of the flow (no information about what happens inside the container can reach the external flow)

$$Q_m^{MAX} = \rho_0 a_0 A_{out} \frac{\rho_*}{\rho_0} \frac{a_*}{a_0}$$



Flows through Laval (converging-diverging) nozzle



Two "extreme" cases of flow:

Case 1

Flow accelerates in the converging part (confusor) and reaches critical conditions at the throat (where $A = A_{\min}$) . Then, inside the diverging part (diffusor) the flow slows down to subsonic conditions. At the outlet section the Mach number is smaller than unity and the outlet pressure matches exactly the external pressure.

This may happen only when the $\left(p_{ext} / p_{cont} \right)_1$ ratio has precisely selected value (typically only slightly smaller then 1).

The procedure to find the value of $\left(p_{ext}/p_{cont}\right)_1$:

1. Calculate the geometric ratio $A_{out}/A_{\min}$. Since the critical Mach number ($M=1$) is attained at the throat of the nozzle, this ratio is equivalent to A_{out}/A_* .

2. Use plot $A/A_* = f(M)$ to read the subsonic value of the Mach number corresponding to $A/A_* = A_{out}/A_{\min}$

3. Use plot $p/p_0 = g(M)$ to find the ratio $(p_{ext}/p_{cont})_1 = g(M_{out})$. Here, we use the fact that $p_{ext} \equiv p_{out}$

Case 2:

Flow accelerates in the converging part (confusor) and reaches critical conditions at the throat (where $A = A_{\min}$) . Then, inside the diverging part (diffusor) the flow continues to accelerate reaching supersonic condition. At the outlet section the Mach number reaches its maximal value (larger than 1).

Outlet pressure does not necessarily matches the external pressure. If it does, we say that the nozzle works in the design mode. This may happen only when the $\left(p_{ext}/p_{cont}\right)_2$ ratio has precisely selected value (typically much smaller then 1).

The procedure to find the value of $\left(p_{ext}/p_{cont}\right)_2$:

1. Calculate the geometric ratio $A_{out}/A_{\min}$. Since the critical Mach number ($M = 1$) ia attained at the throat of the nozzle, this ratio is equivalent to A_{out}/A_* .

2. Use plot $A/A_* = f(M)$ to read the supersonic value of the Mach number corresponding to $A/A_* = A_{out}/A_{\min}$

3. Use plot $p/p_0 = g(M)$ to find the ratio $(p_{ext}/p_{cont})_2 = g(M_{out})$. Here, we use the fact that – in the design mode - $p_{ext} \equiv p_{out}$.

Note that typically $$(p_{ext}/p_{cont})_1 \gg (p_{ext}/p_{cont})_2$$

What happens if $1 > p_{ext}/p_{cont} > (p_{ext}/p_{cont})_1$ **?**

The answer is easy – the **flow in the whole nozzle is entirely subsonic**, i.e., the Mach number does not reach the value of 1 even at the throat!
Assume that an actual value of p_{ext}/p_{cont} is given. How to calculate maximal value of the Mach number in such conditions?

1. First, knowing this pressure ratio and using isentropic pressure relation $p/p_0 = g(M)$ for $p = p_{ext}$ and $p_0 = p_{cont}$ we find the outlet Mach number M_{out}.
2. Next, knowing $M_{out} < 1$ we determine the value of $A_{out}/A_* = f(M_{out})$. Note that here the symbol A_* refers to "hypothetical" (meaning, non-existing in the actual flow conditions) cross section where the critical conditions are achieved. Obviously, $A_* < A_{\min}$!
3. Then, we calculate the ratio

$$\frac{A_{\min}}{A_*} = \underbrace{\frac{A_{\min}}{A_{out}}}_{given} \quad \underbrace{\frac{A_{out}}{A_*}}_{(A/A_*)(M_{out})}$$

and find the value of the throat Mach number using again the relation $A_{\min}/A_* = f(M_{throat})$.
For typical geometries the interval $1 > p_{ext}/p_{cont} > (p_{ext}/p_{cont})_1$ is actually very small.

What happens if the pressure ratio is very small, i.e. smaller than $\left(p_{ext} / p_{cont} \right)_2$?

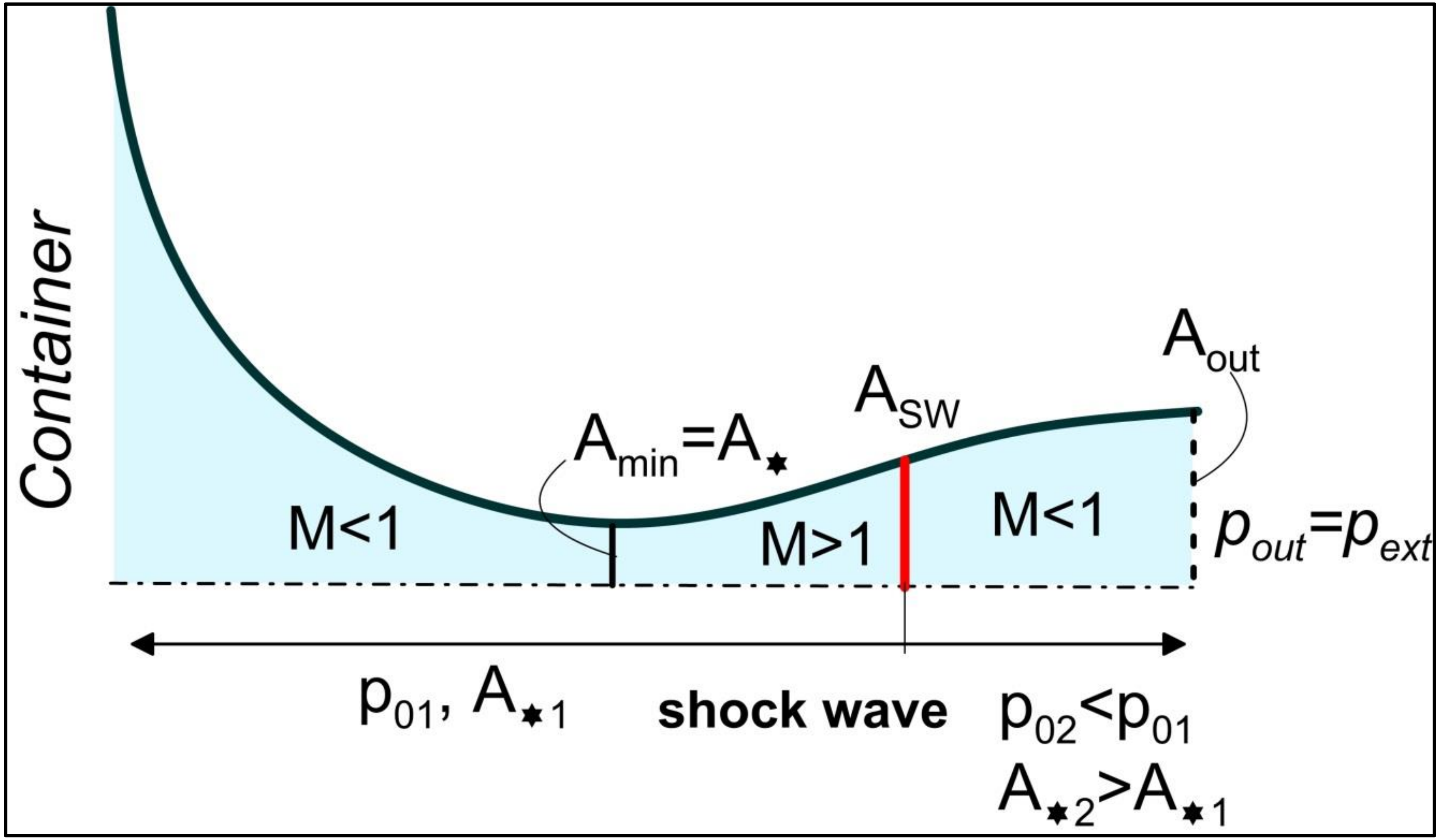
In such circumstances the supersonic flow in the diffusor does not have sufficient space to decompress to external pressure ! It means that $p_{out} > p_{ext}$ and further decompression takes place in the open space beyond the nozzle's exhaust. We say that the nozzle is "too short". The calculations of any gas dynamic parameters inside the "too-short" nozzle do not differ from the case of the "design mode".

What happens if the pressure ratio is $\left(p_{ext}/p_{cont}\right)_1 > p_{ext}/p_{cont} > \left(p_{ext}/p_{cont}\right)_2$?

Let us finally consider the situation when the pressure ratios lay within the wide range of pressure ration between $\left(p_{ext}/p_{cont}\right)_2$ and $\left(p_{ext}/p_{cont}\right)_1$. Since at the borders of this range the flows in diffusor are quite different something interesting must happen!

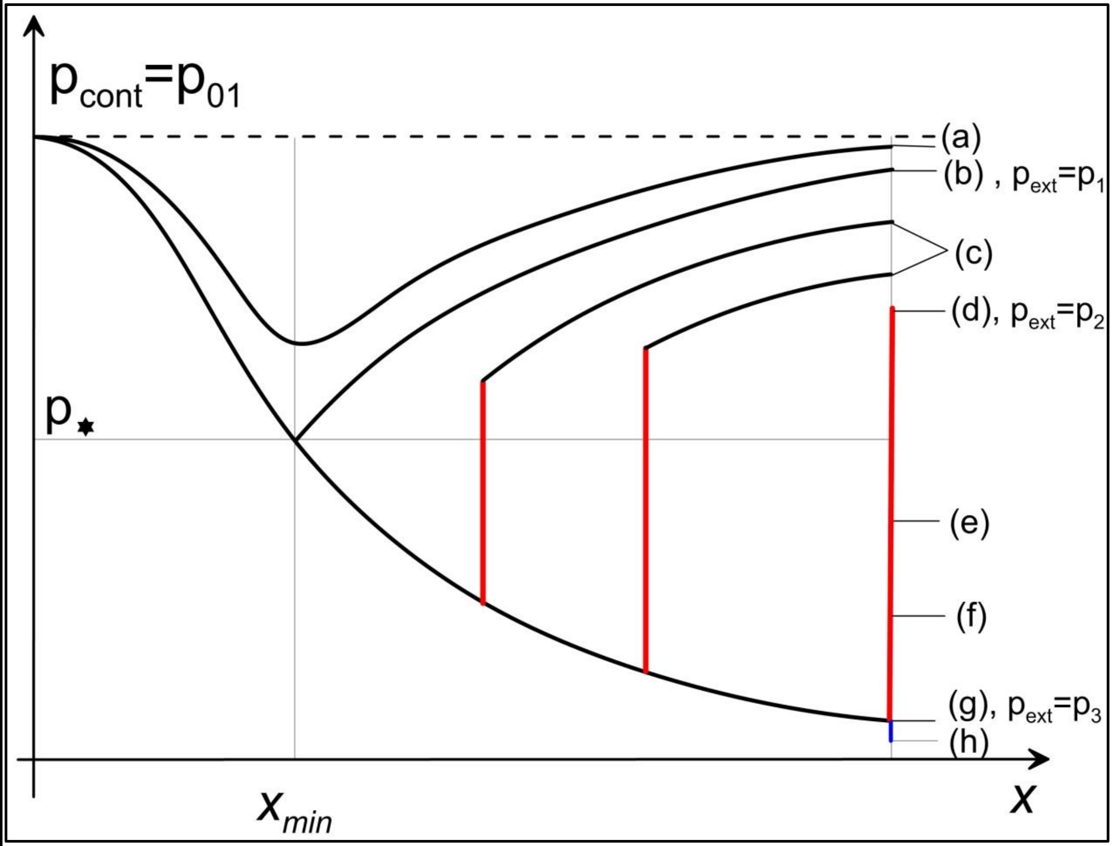
Consider first the situation when the ratio p_{ext}/p_{cont} is only slightly larger than $\left(p_{ext}/p_{cont}\right)_2$. Now, the outlet pressure p_{out} becomes slightly smaller than the external pressure p_{ext}. Thus, the gas must be compressed a little in the stream outside the exhaust.

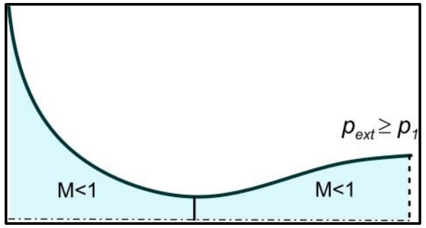
Since the outflowing stream is supersonic such compression cannot be achieved “smoothly” – some pattern of shock waves must appear! We will leave a more detailed analysis of such patterns of external shock wave systems for later. The key point is that at precisely determined pressure ratio the normal shock will appear at the outlet section. With a further reduction of the pressure ratios, the NSW moves inside the diffusor (see the figure).



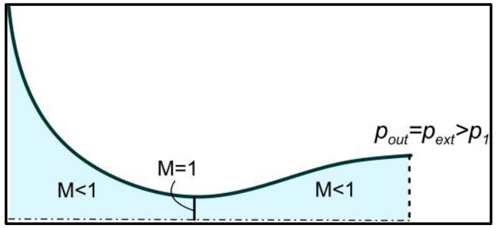
Container
A_out
A_SW
A_min=A_*
M<1
M>1
M<1
p_out=p_ext
p_01, A_*1
shock wave
p_02<p_01
A_*2>A_*1

Different patterns of flow through the Laval nozzle





(a)



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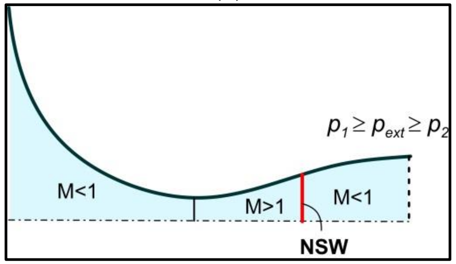
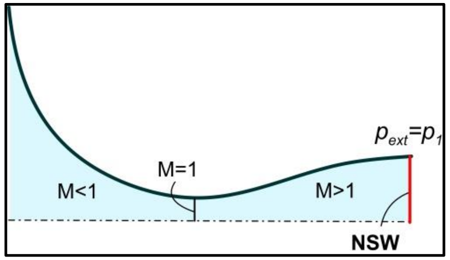
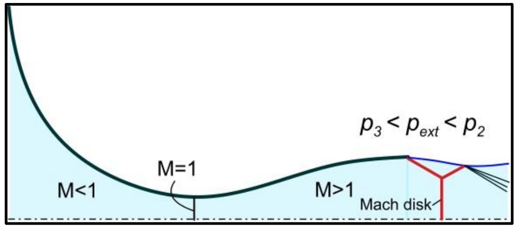


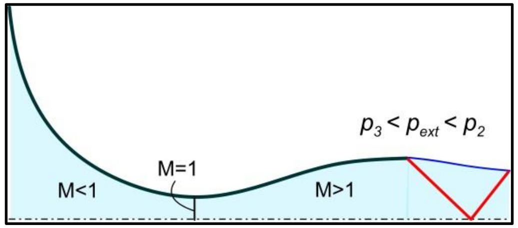
(c)



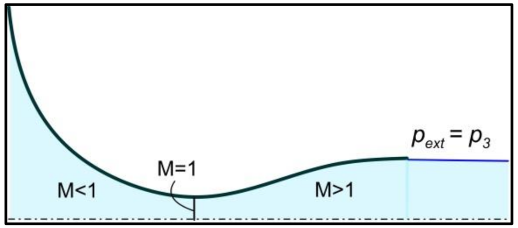
(d)



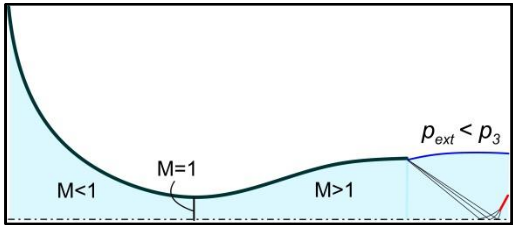
(e)



(f)



(g)



(h)

Distribution of selected parameters along the Laval nozzle flow with the internal NSW