PROGRAM ROZWOJOWY
POLITECHNIKI WARSZAWSKIEJ

FLUID MECHANICS 3 - LECTURE 1

BASIC EQUATIONS AND THEOREM IN THE THEORY OF IDEAL FLUID FLOWS

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In this lecture, we recapitulate main equations and theorems of Fluid Mechanics, we have learnt in the course of Fluid Mechanics I.

Differential equation of the mass conservation

Basic (conservative) form

$$\frac{\partial}{\partial t}\rho + \nabla\cdot(\rho\boldsymbol{v}) = 0$$

Other (equivalent) forms

$$0 = \frac{\partial}{\partial t}\rho + \nabla\cdot(\rho\boldsymbol{v}) = \underbrace{\frac{\partial}{\partial t}\rho + \boldsymbol{v}\cdot\nabla\rho}_{\frac{D}{Dt}\rho} + \rho\nabla\cdot\boldsymbol{v} = \frac{D}{Dt}\rho + \rho\nabla\cdot\boldsymbol{v}$$

For a stationary (steady) flow …

$$\nabla\cdot(\rho\boldsymbol{v}) = \boldsymbol{v}\cdot\nabla\rho + \rho\nabla\cdot\boldsymbol{v} = 0$$

<u>Equation of motion of and ideal fluid (Euler Equation)</u>

Basic form

$$\rho\left[\partial_t \boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\right] = -\nabla p + \rho \boldsymbol{f}$$

Conservative form

$$\partial_t(\rho\boldsymbol{v}) + \nabla\cdot(\rho\boldsymbol{v}\otimes\boldsymbol{v} + p) = \rho \boldsymbol{f}$$

The Lamb-Gromeko form

$$(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = \nabla\left(\tfrac{1}{2}\upsilon^2\right) + \boldsymbol{\omega}\times\boldsymbol{v}$$

$$\partial_t\boldsymbol{v} + \nabla\left(\tfrac{1}{2}\upsilon^2\right) + \boldsymbol{\omega}\times\boldsymbol{v} = -\frac{1}{\rho}\nabla p + \boldsymbol{f}$$

<u>Bernoulli integral of the Euler Equation</u>

Assumptions:

- Flow is stationary
- Fluid is in the barotropic state, hence the pressure potential can be defined

$$P(\rho) := \int \frac{1}{\rho} p'(\rho)\, d\rho \, .$$

$$\frac{\partial}{\partial x_i} P[\rho(\boldsymbol{x})] = \frac{1}{\rho(\boldsymbol{x})} p'[\rho(\boldsymbol{x})] \frac{\partial \rho}{\partial x_i} = \frac{1}{\rho(\boldsymbol{x})} \frac{\partial}{\partial x_i} p[\rho(\boldsymbol{x})] \quad \Rightarrow \quad \nabla P = \frac{1}{\rho} \nabla p$$

Example – adiabatic flow of the Clapeyron gas …

$$P = \frac{\kappa\, p}{(\kappa - 1)\, \rho} = c_P T = i \quad \text{(specific enthalpy)}$$

- The volumetric force field is potential, i.e. $\boldsymbol{f} = \nabla \Phi$ for some scalar field Φ

With the above assumption the Euler Equation can be written in the following form

$$\nabla\left(\tfrac{1}{2}\upsilon^2 + P - \Phi_f\right) = \boldsymbol{\upsilon}\times\boldsymbol{\omega}$$

We choose arbitrary streamline and write ($\boldsymbol{\tau}$ - unary vector tangent to this streamline)

$$\frac{d}{d\boldsymbol{\tau}}\left(\tfrac{1}{2}\upsilon^2 + P - \Phi_f\right) := \nabla\left(\tfrac{1}{2}\upsilon^2 + P - \Phi_f\right)\cdot\boldsymbol{\tau} = \frac{1}{\upsilon}\boldsymbol{\upsilon}\cdot(\boldsymbol{\upsilon}\times\boldsymbol{\omega}) = 0$$

Hence, the function under the gradient operator is **constant along the streamline**:

$$\tfrac{1}{2}\upsilon^2 + P - \Phi_f = C_B$$

If the cross product $\boldsymbol{\upsilon}\times\boldsymbol{\omega} = 0$ then

$$\nabla\left(\tfrac{1}{2}\upsilon^2 + P - \Phi_f\right) = 0 \implies \tfrac{1}{2}\upsilon^2 + P - \Phi_f = const$$

i.e., the Bernoulli constant is global (the same for all streamlines)

Equation of energy conservation

We begin with the differential energy equation, which in the case of an ideal fluid reduces to (u - mass-specific internal energy)

$$\rho \frac{D}{Dt}\left(u+\tfrac{1}{2}\upsilon^2\right) = -\nabla\cdot(p\boldsymbol{v}) + \rho\boldsymbol{f}\cdot\boldsymbol{v}$$

By expanding the pressure term, this equations can be re-written equivalently as

$$\rho \frac{D}{Dt}\left(u+\tfrac{1}{2}\upsilon^2\right) = -p\nabla\cdot\boldsymbol{v} - \boldsymbol{v}\cdot\nabla p + \rho\boldsymbol{f}\cdot\boldsymbol{v}$$

Assume now:

- Flow steadiness
- Potentiality of the volumetric force field

We do not assume that the flow is barotropic!

Since the volume force is potential, the corresponding term in the right-hand side can be transformed as follows

$$\rho \boldsymbol{f} \cdot \boldsymbol{v} = \rho \boldsymbol{v} \cdot \nabla \Phi = \rho(\underbrace{\partial_t \Phi}_{=0} + \boldsymbol{v} \cdot \nabla \Phi) = \rho \frac{D}{Dt} \Phi$$

Moreover, due to flow steadiness we have

$$\boldsymbol{v} \cdot \nabla p = \underbrace{\partial_t p}_{=0} + \boldsymbol{v} \cdot \nabla p = \frac{D}{Dt} p$$

Next, from the mass conservation equation

$$\frac{D}{Dt} \rho + \rho \nabla \cdot \boldsymbol{v} = 0$$

we get the following expression for divergence of the velocity field

$$\nabla \cdot \boldsymbol{v} = -\frac{1}{\rho} \frac{D}{Dt} \rho$$

The energy equation can be now written in the following form

$$\frac{D}{Dt}\left(u+\frac{1}{2}\upsilon^2\right)=\underbrace{\frac{p}{\rho^2}\frac{D}{Dt}\rho-\frac{1}{\rho}\frac{D}{Dt}p}_{=-\frac{D}{Dt}(p/\rho)}+\frac{D}{Dt}\Phi$$

or

$$\frac{D}{Dt}\left(\underbrace{u+p/\rho}_{=i}+\frac{1}{2}\upsilon^2-\Phi\right)=0$$

where $i=u+p/\rho$ denotes the **mass-specific enthalpy** of the fluid.

Thus, the energy equation can be written as

$$\frac{D}{Dt}\left(i+\frac{1}{2}\upsilon^2-\Phi\right)=0$$

Since the flow is **stationary**, the above equation is equivalent to

$$\boldsymbol{\upsilon}\cdot\nabla\left(i+\frac{1}{2}\upsilon^2-\Phi\right)=0$$

Using the same arguments as in the case of the Bernoulli Eq., we conclude that **along each individual streamline**

$$i + \frac{1}{2}\upsilon^2 - \Phi = C_e = const$$

In particular, for the Clapeyron gas $i = c_p T$ and we get

$$c_p T + \frac{1}{2}\upsilon^2 - \Phi = const \ , \quad c_p = \frac{\kappa}{\kappa - 1} R$$

In general the energy constant C_e can be different for each streamline**.**
If C_e is the same for all streamlines then the flow is called **homoenergetic**.

Let us recall that if the flow is **barotropic** then along each streamline we have

$$P + \frac{1}{2}\upsilon^2 - \Phi = C_B = const$$

Thus, when the flow is barotropic then

$$i - P = C_e - C_B = const$$

i.e., the enthalpy i and the pressure potential P differ only by an additive constant.

Internal energy equation. Entropy of a smooth flow of ideal fluid

The equation of internal energy of an ideal fluid reads

$$\rho \frac{D}{Dt} u = -p \nabla \cdot \boldsymbol{v}$$

We know that

$$\nabla \cdot \boldsymbol{v} = -\frac{1}{\rho} \frac{D}{Dt} \rho$$

Thus, the equation for the internal energy u can be written as follows

$$\frac{D}{Dt} u = -\frac{p}{\rho^2} \frac{D}{Dt} \rho = -p \frac{D}{Dt} (1/\rho) = -p \frac{D}{Dt} \vartheta$$

Let us remind that the 1st Principle of Thermodynamics can be expressed in terms of complete differentials of three parameters of thermodynamic state: entropy s, internal energy u and specific volume $\vartheta = 1/\rho$.

The corresponding form of this principle reads

$$Tds = du + pd\vartheta$$

For the thermodynamic process inside individual fluid element one can write

$$T\frac{D}{Dt}s = \frac{D}{Dt}u + p\frac{D}{Dt}\vartheta = -p\frac{D}{Dt}\vartheta + p\frac{D}{Dt}\vartheta = 0$$

Conclusion: If the flow is smooth (i.e., all kinematic and thermodynamic fields are sufficiently regular) then the entropy of the fluid is conserved along trajectories of fluid elements.

We have already introduced the concept of **homoenergetic** flows. In such flows we have

$$i + \tfrac{1}{2}\upsilon^2 - \Phi = C_e^{global}$$

or equivalently

$$\nabla\left(i + \tfrac{1}{2}\upsilon^2 - \Phi\right) = 0.$$

Similarly, we call the flow **homoentropic** if $\nabla s \equiv 0$. Thus, when the flow is homoentropic then the entropy is uniformly distributed in the flow domain.

Since the 1st Principle of Thermodynamics can be written in the following form

$$T\,ds = di - (1/\rho)dp$$

then for any stationary flow one has

$$T\nabla s = \nabla i - (1/\rho)\nabla p$$

In the case of a homoentropic flow we get

$$\nabla i = (1/\rho)\nabla p = \nabla P.$$

Hence, **any homoenergetic and homoentropic flow is automatically barotropic and the Bernoulli constant C_B is global**. Note that in the case of 2D flows, it implies that the velocity field is potential (its vorticity vanishes identically in the whole flow domain).

The Crocco Equation

Consider again the Euler equation in the Lamb-Gromeko form

$$\nabla\left(\tfrac{1}{2}\upsilon^2\right)+\boldsymbol{\upsilon}\times\boldsymbol{\omega}=-\tfrac{1}{\rho}\nabla p+\nabla\Phi$$

Using the entropy/enthalpy form of the 1st thermodynamic principle, we can re-write the above equation in the following form called the **Crocco Equation**

$$T\nabla s=\nabla\left(\tfrac{1}{2}\upsilon^2+i-\Phi\right)+\boldsymbol{\upsilon}\times\boldsymbol{\omega}$$

According to the **Crocco Equation**, **any inhomogeneity in the spatial distribution of entropy in the homoenergetic flow immediately leads to vorticity generation**.

Mechanics and thermodynamics of small disturbances

Consider again the **First Principle of Thermodynamics** …

$$TdS = \underbrace{c_v dT}_{dU - \text{int.}energ.} + p \underbrace{d(1/\rho)}_{d\vartheta - spec.vol.}$$

The differential of (mass-specific) entropy can be expressed as follows

$$dS = \frac{c_v}{T} dT - \frac{p}{T\rho^2} d\rho \underset{\substack{Clapeyron\\equation}}{=} \frac{c_v}{T} dT - \frac{R}{\rho} d\rho = \frac{c_v}{T} dT - \frac{(\kappa - 1)c_v}{\rho} d\rho$$

Using the **Clapeyron equation** can write

$$\frac{1}{T} dT = \frac{1}{T} d\left(\frac{p}{R\rho}\right) = \frac{1}{TR}\left(\frac{dp}{\rho} - \frac{p}{\rho^2} d\rho\right) = \frac{dp}{p} - \frac{d\rho}{\rho}$$

Thus

$$dS = \frac{c_v}{p} dp - \frac{\kappa c_v}{\rho} d\rho = \frac{c_v}{p} dp - \frac{c_p}{\rho} d\rho$$

Flow is isentropic, hence $dS = 0$ and

$$\frac{dp}{p} = \kappa \frac{d\rho}{\rho} \Rightarrow \left.\frac{dp}{d\rho}\right|_{S=const} = \kappa \frac{p}{\rho} = \kappa RT \geq 0$$

Thus, the **flow is barotropic** and the derivative of the pressure as the function of density is always **nonnegative function**.

We can introduce the quantity a defined as $a = \sqrt{\kappa RT}$.

Then

$$\left.\frac{dp}{d\rho}\right|_{S=const} = a^2.$$

The physical unit of a is [m/s]. In has been demonstrated in the course of FM 1, that this quantity is the velocity of small (acoustic) disturbances measured with respect to the gas.

If an external force field is absent, then the energy integral can be written as

$$i + \tfrac{1}{2}\upsilon^2 = const$$

The mass-specific enthalpy can be expressed in several forms

$$i = c_p T = \tfrac{\kappa}{\kappa-1} RT = \tfrac{\kappa}{\kappa-1}\, p/\rho = \tfrac{1}{\kappa-1} a^2$$

Mach number: $M = \dfrac{V}{a}$

We define:

- **Stagnation parameter**: the parameter's value at such point where $\upsilon = 0$; e.g. T_0
- **Critical parameter:** the parameter's value at such point where $\upsilon = a \ (M = 1)$; e.g. T_*

In gas dynamics we often use three equivalent forms of the energy equation

$$c_p T + \tfrac{1}{2}\upsilon^2 = c_p T_0$$

$$\frac{\kappa p}{(\kappa-1)\rho} + \tfrac{1}{2}\upsilon^2 = \frac{\kappa p_0}{(\kappa-1)\rho_0}$$

$$\frac{a^2}{\kappa-1} + \tfrac{1}{2}\upsilon^2 = \frac{a_0^2}{\kappa-1} = \frac{\kappa+1}{2(\kappa-1)} a_*^2$$

Maximal velocity which can be achieved in any stationary flows is ($T \to 0$)

$$c_p T + \tfrac{1}{2}\upsilon^2 = c_p T_0 = \tfrac{1}{2}\upsilon_{\max}^2 \;\Rightarrow\; \upsilon_{\max} = \sqrt{2 c_p T_0}$$

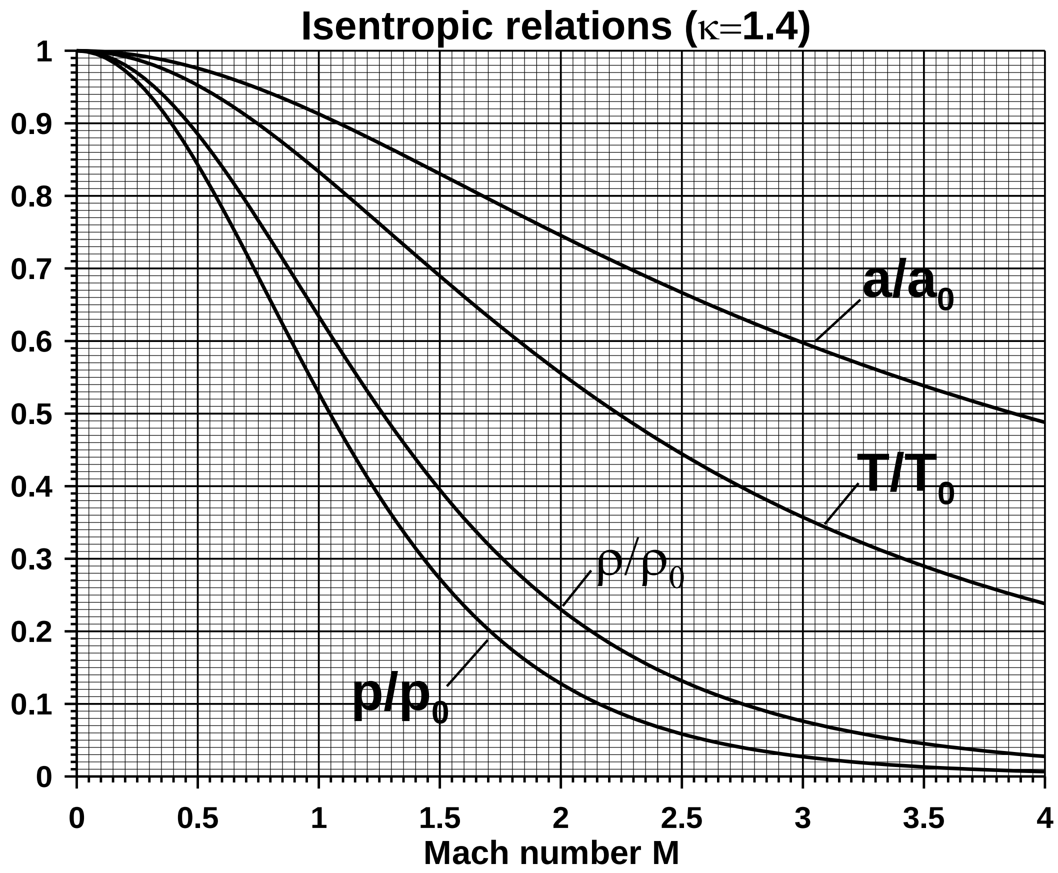
If the flow is adiabatic, the temperature of the gas is a simple function of a local Mach number

$$1 + \frac{\upsilon^2}{2 c_p T} = \frac{T_0}{T} \;\Rightarrow\; 1 + \frac{\upsilon^2}{\frac{2}{\kappa-1} a^2} = \frac{T_0}{T} \;\Rightarrow 1 + \frac{\kappa-1}{2} M^2 = \frac{T_0}{T}$$

We have $\dfrac{T}{T_0}(M) = \left(1 + \dfrac{\kappa-1}{2} M^2\right)^{-1}$ and $\dfrac{a}{a_0}(M) = \left(1 + \dfrac{\kappa-1}{2} M^2\right)^{-\frac{1}{2}}$

If the **flow is also isentropic**, we have $p/\rho^{\kappa} = const$ and $p = \rho RT$. Then

$$\frac{\rho}{\rho_0}(M) = \left(1 + \frac{\kappa - 1}{2} M^2\right)^{\frac{1}{1-\kappa}} \quad , \quad \frac{p}{p_0}(M) = \left(1 + \frac{\kappa - 1}{2} M^2\right)^{\frac{\kappa}{1-\kappa}}$$



Normal shock wave – summary of main formulae and results

Equations for conserved quantities:

(1) Mass

$$\int_{\partial\Omega} \rho\, \upsilon_n \, ds = 0 \quad \underset{\text{1D case}}{\Rightarrow} \quad \rho_1 u_1 = \rho_2 u_2$$

(2) Linear momentum

$$\int_{\partial\Omega} (\rho\, \upsilon_n \boldsymbol{\upsilon} + p\boldsymbol{n}) ds = \boldsymbol{0} \quad \underset{\text{1D case}}{\Rightarrow} \quad \rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2$$

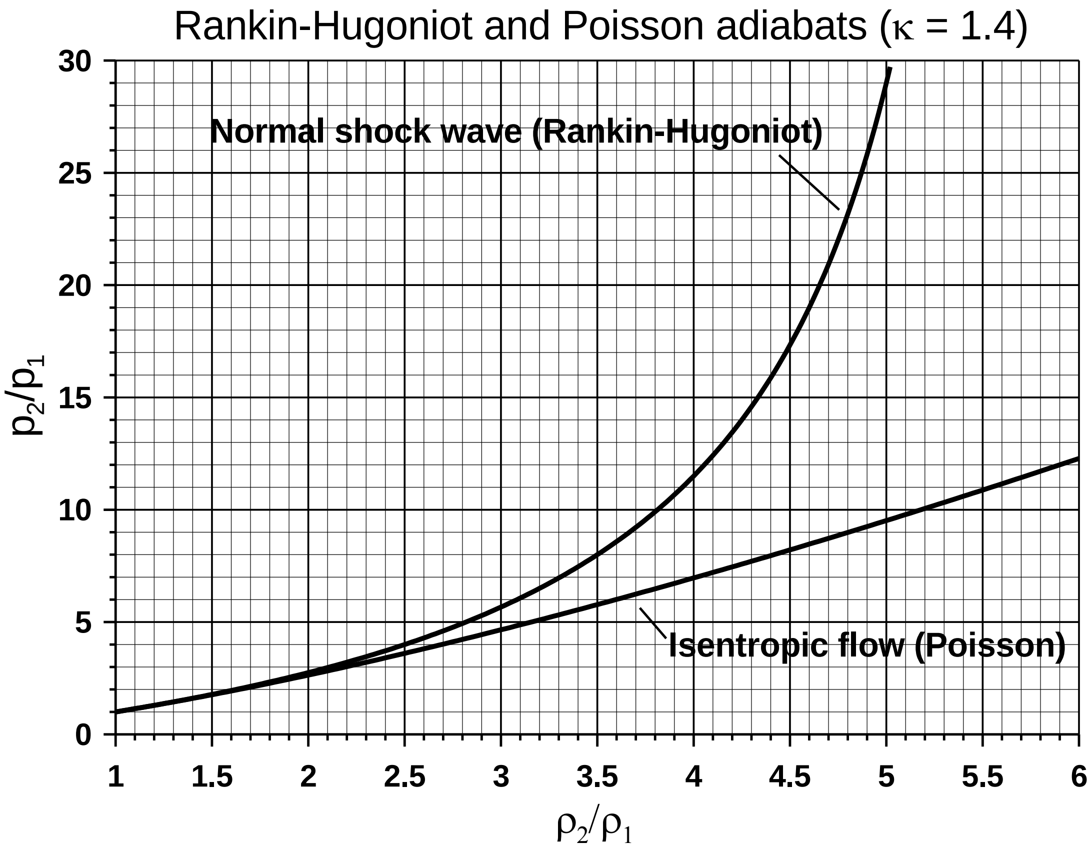
(3) Energy

$$\frac{\kappa\, p_1}{(\kappa - 1)\rho_1} + \tfrac{1}{2} u_1^2 = \frac{\kappa\, p_2}{(\kappa - 1)\rho_2} + \tfrac{1}{2} u_2^2$$

After a lengthy algebra, we have shown in the Fluid Mechanics I course that the nontrivial relation between density and pressure ratios reads

$$\frac{p_2}{p_1} = \frac{\frac{\kappa+1}{\kappa-1} - \frac{\rho_1}{\rho_2}}{\frac{\kappa+1}{\kappa-1}\frac{\rho_1}{\rho_2} - 1} = \frac{\frac{\kappa+1}{\kappa-1}\frac{\rho_2}{\rho_1} - 1}{\frac{\kappa+1}{\kappa-1} - \frac{\rho_2}{\rho_1}}$$

This formula describes the Rankin-Hugoniot adiabat which is different from the (isentropic) Poisson adiabat, see figure below.



Yet, for the density ration only slightly larger than unity, the difference is really small (as RH and P lines are strictly tangent for the argument equal 1)

Physically it means that **weak shock waves are nearly isentropic** and

$$\left.\frac{p_2}{p_1}\right|_{Hugoniot} - \left.\frac{p_2}{p_1}\right|_{isentropic} = C\left(\frac{\rho_2}{\rho_1} - 1\right)^3 + h.o.t$$

Since

$$s = c_v \ln p - \underbrace{c_p}_{=\kappa c_v} \ln \rho + const = c_v \ln\left(p/\rho^\kappa\right) + const$$

the change of entropy between two thermodynamic states can be expressed as follows

$$\frac{\Delta s}{c_v} \equiv \frac{s_2 - s_1}{c_v} = \ln\left(\frac{p_2}{\rho_2^\kappa}\right) - \ln\left(\frac{p_1}{\rho_1^\kappa}\right) = \ln\left(\frac{p_2}{p_1}\right) - \ln\left(\frac{\rho_2^\kappa}{\rho_1^\kappa}\right)$$

From the 2nd Principle of Thermodynamics we conclude that **physically admissible shock waves must be compressing shocks** (the entropy cannot diminish while crossing the shock).

The most important relations concerning the normal shock wave are:

- The Prandtl's relation

$$u_1 u_2 = a_*^2$$

- Relation between Mach numbers

$$M_2 = \sqrt{\frac{2+(\kappa-1)M_1^2}{2\kappa M_1^2-\kappa+1}} < 1$$

Other important relations can be derived. We usually use either plot or tabularized values.

For instance

$$\frac{\rho_2}{\rho_1}(M_1) = \frac{u_1}{u_2}(M_1) = \frac{M_1}{M_2(M_1)}\frac{a_1}{a_2}(M_1) = \frac{M_1}{M_2(M_1)}\left(\frac{a}{a_0}\right)_{is}(M_1)\left(\frac{a}{a_0}\right)_{is}^{-1}[M_2(M_1)] > 1$$

To evaluate the pressure ratio (as the function of M_1) we rewrite the momentum equation in the following way

$$p+\rho u^2 = p\left(1+\frac{u^2}{p/\rho}\right) = p\left(1+\frac{\kappa u^2}{a^2}\right) = p(1+\kappa M^2) = const$$

Since the above expression has the same value at both sides of the shock wave, we get

$$\frac{p_2}{p_1}(M_1) = \frac{1+\kappa M_1^2}{1+\kappa M_2^2(M_1)} > 1$$

We can also write

$$\frac{T_2}{T_1}(M_1) = \frac{\left(T/T_0\right)(M_2)}{\left(T/T_0\right)[M_2(M_1)]} > 1$$

where

$$\frac{T}{T_0}(M) = \left(1+\frac{\kappa-1}{2}M^2\right)^{-1}$$

Entropy of the gas increases while crossing the shock wave. The formula derived earlier can be written for stagnation parameters, namely

$$0 < \frac{s_2 - s_1}{c_v} = \ln\left(p_{02}/p_{01}\right) - \kappa \ln\left(\rho_{02}/\rho_{01}\right)$$

The energy is conserved, hence the total temperature at both sides is the same and

$$T_{01} = T_{02} \equiv T_0 \quad \underset{\text{Clapeyron Equation}}{\Longrightarrow} \quad \frac{\rho_{02}}{\rho_{01}} = \frac{p_{02}}{p_{01}}$$

We conclude that the stagnation pressure drops while crossing the SW …

$$0 < \frac{\Delta s}{c_v} = (1-\kappa)\ln\left(p_{02}/p_{01}\right) \;\Rightarrow\; p_{02}/p_{01} < 1$$