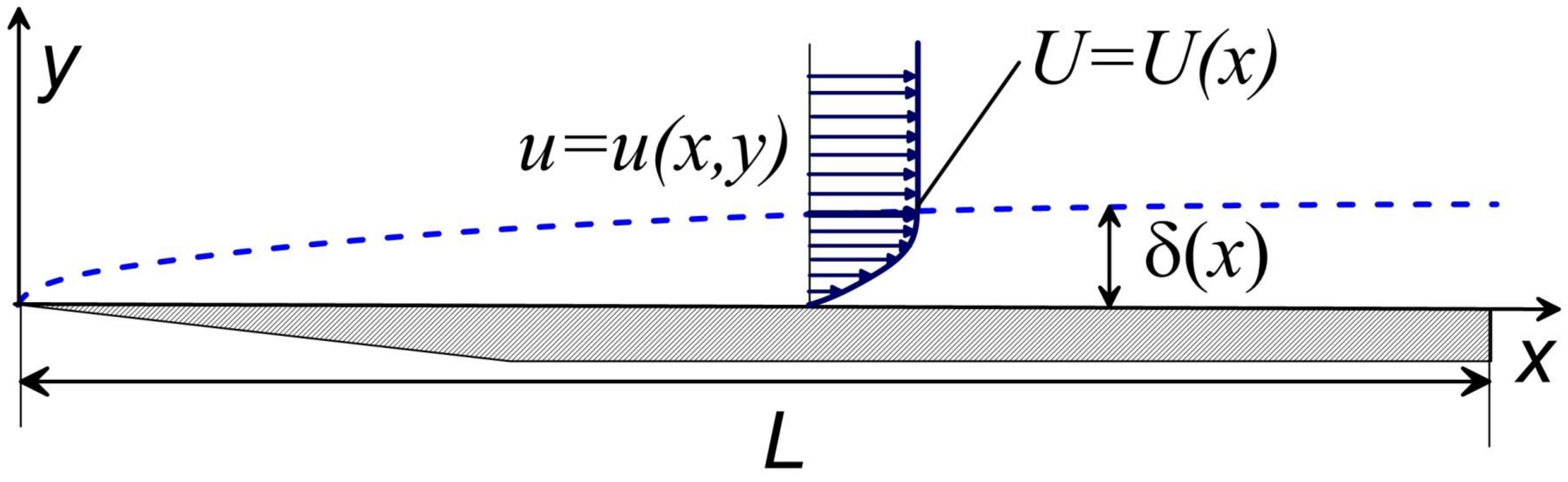
LECTURE 4

ELEMENTS OF THE BOUNDARY LAYER THEORY – PART 1

Concept of a boundary layer. Prandtl's Equation



Boundary layer (BL) – thin layer of a fluid adjacent to the surface of an immersed body. The BL features huge transversal gradient of the flow velocity. Within the BL, the dynamical effect due to fluid viscosity are comparable in magnitude to inertial effects.

General assumption – the BL is thin in comparison to a characteristic longitudinal extension of the flow. Hence, we have

$$\delta \ll L \quad \text{or} \quad \frac{\delta}{L} \ll 1$$

We will present rudiments of the BL theory as formulated by Ludwig Prandtl in the beginning of the XX century. The idea is to introduce a simplification of the Navier-Stokes Equations by careful estimation of magnitude of different terms.

The departure point is the set of equations governing steady 2D motion of a viscous incompressible fluid

$$u\partial_x u + \upsilon\partial_y u = -\tfrac{1}{\rho}\partial_x p + \nu(\partial_{xx}u + \partial_{yy}u)$$
$$u\partial_x \upsilon + \upsilon\partial_y \upsilon = -\tfrac{1}{\rho}\partial_y p + \nu(\partial_{xx}\upsilon + \partial_{yy}\upsilon)$$
$$\partial_x u + \partial_y \upsilon = 0$$

One can assume that the flow inside the (wall-attached) BL is **nearly parallel**. We expect that the velocity component parallel to the wall is much larger than the component normal to the wall.

Let $U = U(x)$ denote the velocity distribution (along the wall) at the external "edge" of the BL. Roughly speaking, the velocity U can be identified with the velocity of an external (potential) flow computed at the boundary of the immersed body. For slender bodies, we typically have $U \sim U_\infty$, where U_∞ is the free-stream velocity.

Hence, the following scaling holds true

$$u \sim U_\infty \quad , \quad \partial_x u \sim \frac{U_\infty}{L} \quad , \quad \partial_y u \sim \frac{U_\infty}{\delta} = \frac{U_\infty}{L}\frac{L}{\delta} \gg \partial_x u$$

Next, from the continuity equation we have

$$\partial_x u \sim \partial_y \upsilon \;\Rightarrow\; \partial_y \upsilon \sim \frac{U_\infty}{L} \;\Rightarrow\; \upsilon \sim \frac{\delta}{L} U_\infty \ll U_\infty$$

Thus, the terms in the equation of motion in x-direction can be scaled as follows

$$\underbrace{u\,\partial_x u}_{\sim \frac{U_\infty^2}{L}} + \underbrace{\upsilon\,\partial_y u}_{\sim \frac{U_\infty \delta}{L}\frac{U_\infty}{\delta} = \frac{U_\infty^2}{L}} = -\tfrac{1}{\rho}\partial_x p + \nu(\underbrace{\partial_{xx} u}_{\sim \frac{U_\infty}{L^2}} + \underbrace{\partial_{yy} u}_{\sim \frac{U_\infty}{\delta^2}})$$

Note that, in the viscous term

$$\partial_{xx} u \sim \left(\tfrac{\delta}{L}\right)^2 \partial_{yy} u \ll \partial_{yy} u$$

Inside the BL, the dominating viscous term must be of the same order as the convective terms. Hence

$$\nu\,\partial_{yy}u \sim \frac{U_\infty^2}{L} \;\Rightarrow\; \nu\frac{U_\infty}{\delta^2} \sim \frac{U_\infty^2}{L} \;\Rightarrow\; \frac{\delta^2}{L^2} \sim \frac{\nu}{U_\infty L}$$

It follows that the (average) **relative thickness of the BL is related to the Reynolds number** by the formula

$$\boxed{\frac{\delta}{L} \sim \frac{1}{\sqrt{\dfrac{U_\infty L}{\nu}}} = \frac{1}{\sqrt{\mathrm{Re}_L}}}$$

Example: Estimate average BL's thickness for $L = 0.5\,m\;,\,U_\infty = 50\,\frac{m}{s}\,,\,\nu = 10^{-5}\,\frac{m^2}{s}$.

We have: $\mathrm{Re} = \frac{50\cdot 0.5}{10^{-5}} = 2.5\cdot 10^6 \quad\Rightarrow\; \delta \approx \frac{1}{\sqrt{2.5\cdot 10^6}}L \approx 0.63\,mm$ (!)

We see that the thickness of the BL is really small !!!

Consider the equation of motion in the direction y (normal to the wall). It is clear that all terms are smaller than analogical ones in the equation for the direction x by the factor $\delta/L \ll 1$.

We conclude that the general order of magnitude of the equations of motion is

$$u\partial_x u + \upsilon\partial_y u = -\tfrac{1}{\rho}\partial_x p + \nu\partial_{yy} u \ \ , \ \ O(\tfrac{U_\infty^2}{L})$$

$$u\partial_x \upsilon + \upsilon\partial_y \upsilon \approx -\tfrac{1}{\rho}\partial_y p + \nu\partial_{yy} \upsilon \ \ , \ \ O(\tfrac{U_\infty^2}{L}\cdot\tfrac{\delta}{L})$$

Automatically, one can conclude that $\partial_y p \approx 0$, i.e., the **pressure across the BL is nearly constant and equal to the pressure at the external border of the BL**. The distribution of pressure along this border can be computed from the **Bernoulli Equation** (which is applicable as the external flow is virtually inviscid), namely

$$p_\infty + \tfrac{1}{2}\rho U_\infty^2 = p(x) + \tfrac{1}{2}\rho U^2(x)$$

After differentiation with respect to x, one obtains

$$-\tfrac{1}{\rho}p'(x) = U(x)U'(x)$$

After insertion to the equation of motion in x direction, we obtain the Prandtl Equation (for a laminar BL)

$$\boxed{u\partial_x u+\upsilon\partial_y u=U(x)U'(x)+\nu\partial_{yy}u}$$

In order to obtain closed system, we add the **continuity equation**

$$\partial_x u+\partial_y \upsilon=0$$

We also need to specify the boundary (or wall) and far-field conditions

$$u,\upsilon\big|_{wall}\equiv u,\upsilon(x,0)=0 \quad , \quad \lim_{y\to\infty} u(x,y)=U(x)$$

From the mathematical point of view **the Prandtl Equation is the partial differential equation (PDE) of the parabolic type** (like, for instance, the equation of heat conduction). The spatial **variable *x* plays here a role of a time-like variable**. Thus, in order to solve this equation we need to specify also the "initial" condition, i.e., the velocity profile at some location $x=x_0$

$$u(x_0,y)=u_0(y)$$

In general, the solution of the PE is obtained numerically and the result of such calculations is the velocity and pressure inside the BL for $x > x_0$. Ideally, we would choose x_0 at the front stagnation point, where the BL starts to develop. There is, however, some dose of trickiness in such choice, as the initial thickness of the BL is zero and $u_0(y)$ is not properly defined. The way to overcome this "singularity" is to use approximate analytical solution of the BL flow near the stagnation point.

Self-similar solutions of the Prandtl Equation

For certain cases of $U = U(x)$, one can find for the PE so-called **self-similar solutions**. We say that the solution of the PE is self-similar if it can be expressed by a **function of a single variable** applied to a "self-similar" coordinate $\eta = \dfrac{y}{\delta(x)}$, where the function $\delta = \delta(x)$ is to be determined in the solution procedure.

Since we concern only 2D case, let's use the streamfunction. Assume that it can be written as follows

$$\psi(x,y) = U(x)\delta(x) f[\tfrac{y}{\delta(x)}] = U(x)\delta(x) f[\eta(x,y)]$$

The corresponding velocity components are

$$u(x,y) = \partial_y \psi(x,y) = U(x)\delta(x) f'[\eta(x,y)]\partial_y \eta(x,y) =$$

$$= U(x)\delta(x) f'[\eta(x,y)]\frac{1}{\delta(x)} = U(x) f'[\eta(x,y)]$$

and

$$\upsilon(x,y) = -\partial_x \psi(x,y) = -U'(x)\delta(x)\, f[\eta(x,y)] - U(x)\delta'(x)\, f[\eta(x,y)] +$$

$$+U(x)\delta(x)\, f'[\eta(x,y)]\frac{y\delta'(x)}{\delta^2(x)}$$

Let’s calculate all necessary derivatives …

$$\partial_x u(x,y) = U'(x)\, f'[\eta(x,y)] - U(x)\, f''[\eta(x,y)]\frac{y\delta'(x)}{\delta^2(x)}$$

$$\partial_y u(x,y) = \frac{U(x)}{\delta(x)}\, f''[\eta(x,y)] \quad , \quad \partial_{yy} u(x,y) = \frac{U(x)}{\delta^2(x)}\, f'''[\eta(x,y)]$$

After insertion to the left side of the PE we get

$$\mathcal{L} \equiv u\,\partial_x u + \upsilon\,\partial_y u = U(x)U'(x)\, f'^2[\eta(x,y)] -$$

$$-U(x)U'(x)\, f[\eta(x,y)]\, f''[\eta(x,y)] - U^2(x)\frac{\delta'(x)}{\delta(x)}\, f[\eta(x,y)]\, f''[\eta(x,y)]$$

The right-hand side of this equation assumes the following form

$$\mathcal{R} = U(x)U'(x) - \frac{\nu U(x)}{\delta^2(x)} f'''[\eta(x,y)]$$

Finally, equating $\mathcal{L}$ to $\mathcal{R}$, we arrive at the following equation

$$U(x)U'(x) f'^2[\eta(x,y)] - U(x)U'(x) f[\eta(x,y)] f''[\eta(x,y)] -$$
$$-U^2(x)\frac{\delta'(x)}{\delta(x)} f[\eta(x,y)] f''[\eta(x,y)] = U(x)U'(x) - \frac{\nu U(x)}{\delta^2(x)} f'''[\eta(x,y)]$$

After division by $\dfrac{\nu U(x)}{\delta^2(x)}$ and simple algebra, the above equation reduces to

$$\frac{\delta U'}{\nu}(f'^2 - f f'' - 1) - \frac{U\delta\delta'}{\nu} f f'' = f'''$$

Note that the left side of this equation depends on both x and η. **A self-similar solution is possible only when we remove the dependence on x, which is possible only for some forms of the function $U = U(x)$.**

Falkner and Skan notices (in 1930), that self-similar solutions exist if

$$U(x) = U_\infty \left(\tfrac{x}{L}\right)^m$$

where $m \in R$ is a parameter.

Then $$U'(x) = U_\infty m \frac{x^{m-1}}{L^m}$$

Assume that the function δ is equal $\delta(x) = C\left(\tfrac{x}{L}\right)^\alpha L$ where the constants C and α have to be determined.

Then

$$\frac{\delta^2 U'}{\nu} = \frac{C^2 U_\infty L}{\nu} m \left(\frac{x}{L}\right)^{2\alpha+m-1} \quad , \quad \frac{U\delta\delta'}{\nu} = \frac{C^2 U_\infty L}{\nu} \alpha \left(\frac{x}{L}\right)^{2\alpha+m-1}$$

One can see that the x-dependence disappears if and only if

$$2\alpha + m - 1 = 0 \quad \Rightarrow \quad \alpha = \frac{1-m}{2}$$

Thus, we obtain the following expressions

$$\frac{\delta^2 U'}{\nu} = m\frac{C^2 U_\infty L}{\nu} \quad , \quad \frac{U\delta\delta'}{\nu} = \frac{C^2 U_\infty L}{\nu}\alpha = \frac{1-m}{2}\frac{C^2 U_\infty L}{\nu}$$

After insertion, we get **the 3rd-order nonlinear ordinary differential equation for the unknown function** f

$$\frac{C^2 U_\infty L}{\nu} m(f'^2 - f f'' - 1) - \frac{C^2 U_\infty L}{\nu}\frac{1-m}{2} f f'' = f'''$$

Equivalently

$$\frac{C^2 U_\infty L}{\nu} m(f'^2 - 1) - \frac{C^2 U_\infty L}{\nu}\frac{1+m}{2} f f'' = f'''$$

The constant C can be chosen accordingly to the “normalization condition” …

$$\frac{C^2 U_\infty L}{\nu}\frac{1+m}{2} = 1 \quad \Rightarrow \quad C = \sqrt{\frac{2}{m+1}\frac{\nu}{U_\infty L}}$$

We obtain also the formula for the function $\delta = \delta(x)$, namely

$$\delta(x) = C\left(\tfrac{x}{L}\right)^{\frac{1-m}{2}} L = \sqrt{C^2\left(\tfrac{x}{L}\right)^{1-m}}\, L = \sqrt{\tfrac{2}{m+1}\,\tfrac{x}{L}\,\tfrac{\nu}{U_\infty L}\left(\tfrac{L}{x}\right)^{m}}\, L =$$

$$= \sqrt{\tfrac{2}{m+1}\,\tfrac{x}{L}\,\tfrac{\nu}{U_\infty L}\left(\tfrac{L}{x}\right)^{m} L^2} = \sqrt{\frac{2}{m+1}\,\frac{\nu x}{U(x)}}$$

The second coefficient in the left side of the equation for the function f is now equal

$$\frac{C^2 U_\infty L}{\nu}\, m = \frac{2m}{1+m} \equiv \beta$$

which leads to the final form of the famous **Falkner-Skan Equation**

$$\boxed{f''' + f\,f'' + \beta(1 - f'^2) = 0}$$

where $$f = f(\eta)\ ,\ \ \eta = \frac{y}{\delta(x)} = \frac{y}{\sqrt{\frac{2}{m+1}\,\frac{\nu x}{U(x)}}}$$

The parameter β can be expressed in terms of the number m, namely

$$\beta = \frac{2m}{1+m} \quad \Rightarrow \quad m = \frac{\beta}{2-\beta} \Rightarrow \frac{2}{m+1} = 2 - \beta$$

Hence, the self-similar coordinate η can be expressed by the formula

$$\eta = \frac{y}{\sqrt{(2-\beta)\frac{\nu x}{U(x)}}}$$

The boundary and "far-field" conditions for the Falkner-Skan Equation are

$$\begin{cases} u\big|_{wall} = 0 \quad \Rightarrow \quad f'(0) = 0 \\ \upsilon\big|_{wall} = 0 \quad \Rightarrow \quad f(0) = 0 \\ u \xrightarrow[y\to\infty]{} U(x) \Rightarrow \lim\limits_{\eta\to\infty} f'(\eta) = 1 \end{cases}$$

Physical interpretation of the Falkner-Skan self-similar solutions

In order to understand the physical meaning of the FS solutions, consider the Laplace Equation for the streamfunction of an external potential flow. In polar coordinates, we have

$$\Delta\psi \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\psi\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\psi = 0$$

The particular solution to this equation is

$$\psi(r,\theta) = Kr^{m+1}\sin[(m+1)\theta].$$

Indeed, we have

$$\frac{\partial}{\partial r}\psi = K(m+1)r^{m}\sin[(m+1)\theta]$$

$$r\frac{\partial}{\partial r}\psi = K(m+1)r^{m+1}\sin[(m+1)\theta]$$

$$\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\psi\right) = K(m+1)^2 r^{m}\sin[(m+1)\theta]$$

$$\left.\begin{aligned}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\psi\right) &= K(m+1)^2 r^{m-1}\sin[(m+1)\theta]\\ \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\psi &= -K(m+1)^2 r^{m-1}\sin[(m+1)\theta]\end{aligned}\right\} \Rightarrow \Delta\psi = 0$$

The polar components of the corresponding velocity field are

$$\begin{cases} \upsilon_r = \frac{1}{r}\frac{\partial}{\partial\theta}\psi = K(m+1)r^m \cos[(m+1)\theta] \\ \upsilon_\theta = -\frac{\partial}{\partial r}\psi = -K(m+1)r^m \sin[(m+1)\theta] \end{cases}$$

Which radial lines $\theta = const$ **are the streamlines of this flow?** Clearly, along such line we must have $\upsilon_\theta(r,\theta) \equiv 0$. Hence, $\theta = \theta_*$ such that

$$\sin[(m+1)\theta_*] = 0 \qquad \Rightarrow \qquad \theta_{*,1} = 0 \quad , \quad \theta_{*,2} = \frac{\pi}{m+1}$$

Assume that $m \geq 0$. Then $\beta \in [0,2)$ and

$$\frac{1}{m+1} = 1 - \frac{1}{2}\beta \;\Rightarrow\; \theta_{*,2} = \pi - \frac{1}{2}\beta\pi$$

Radial component of the velocity along the line $\theta_{*,1} = 0$ is equal

$$\upsilon_r(r,\theta=0) = \underbrace{K(m+1)}_{\frac{U_\infty}{L^m}} \underbrace{r^m}_{x^m} = U_\infty (x/L)^m \quad , \quad m = \frac{\beta}{2-\beta}$$

Thus, we have just obtained the external flow corresponding to self-similar Falkner-Skan BL, i.e.,

$$U(x) \equiv \upsilon_r(x,0) = U_\infty (x/L)^m$$

Such flow corresponds to the flow in the vicinity of the flat plate oriented at the negative angle of attack $\alpha = -\beta\frac{\pi}{2}$ with respect to the free-stream direction. Equivalently, this is potential flow past a semi-infinite wedge (see the figure).

The case $\beta = 0$ corresponds to the BL along the flat plate oriented parallel to the free-stream direction (hence, the angle of incidence is zero). Then $m = 0$ and $U(x) \equiv U_\infty$, i.e., the **pressure along the BL is constant**.

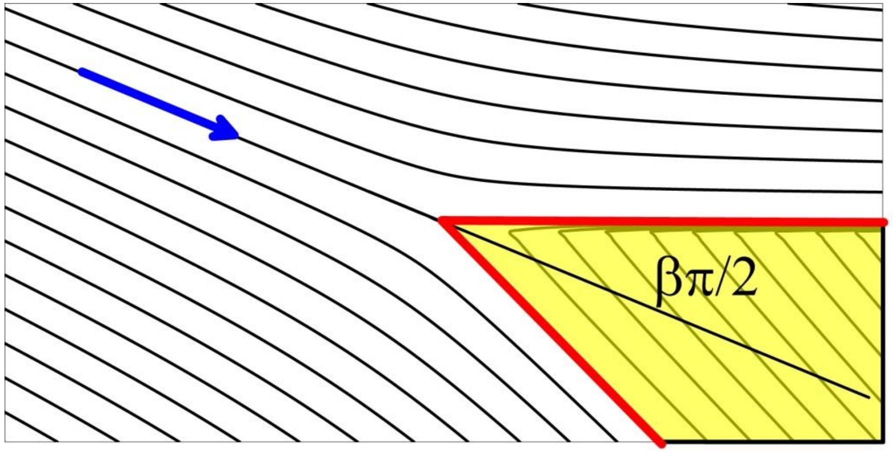
In such case, the FS equation reduces to the **Blasius equation**

$$f''' + f\,f'' = 0$$

The corresponding self-similar coordinate is $\eta = \dfrac{y}{\sqrt{2\frac{\nu x}{U_\infty}}}$.

EXERCISE: Analyze the case of slightly negative values of the parameter β.

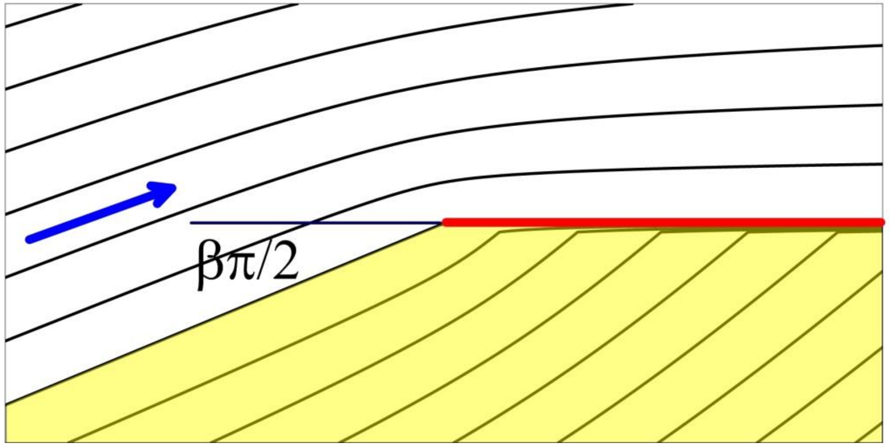
SUMMARY



(A)

The BL flow along the flat plate at a negative angle of attack (the wedge flow)

$$\begin{cases} m > 0\,,\;\; \beta \in (0,2)\,,\;\; U(x) = U_\infty \left(\frac{x}{L}\right)^m \\ U'(x) > 0 \quad i \quad p'(x) < 0 \end{cases}$$

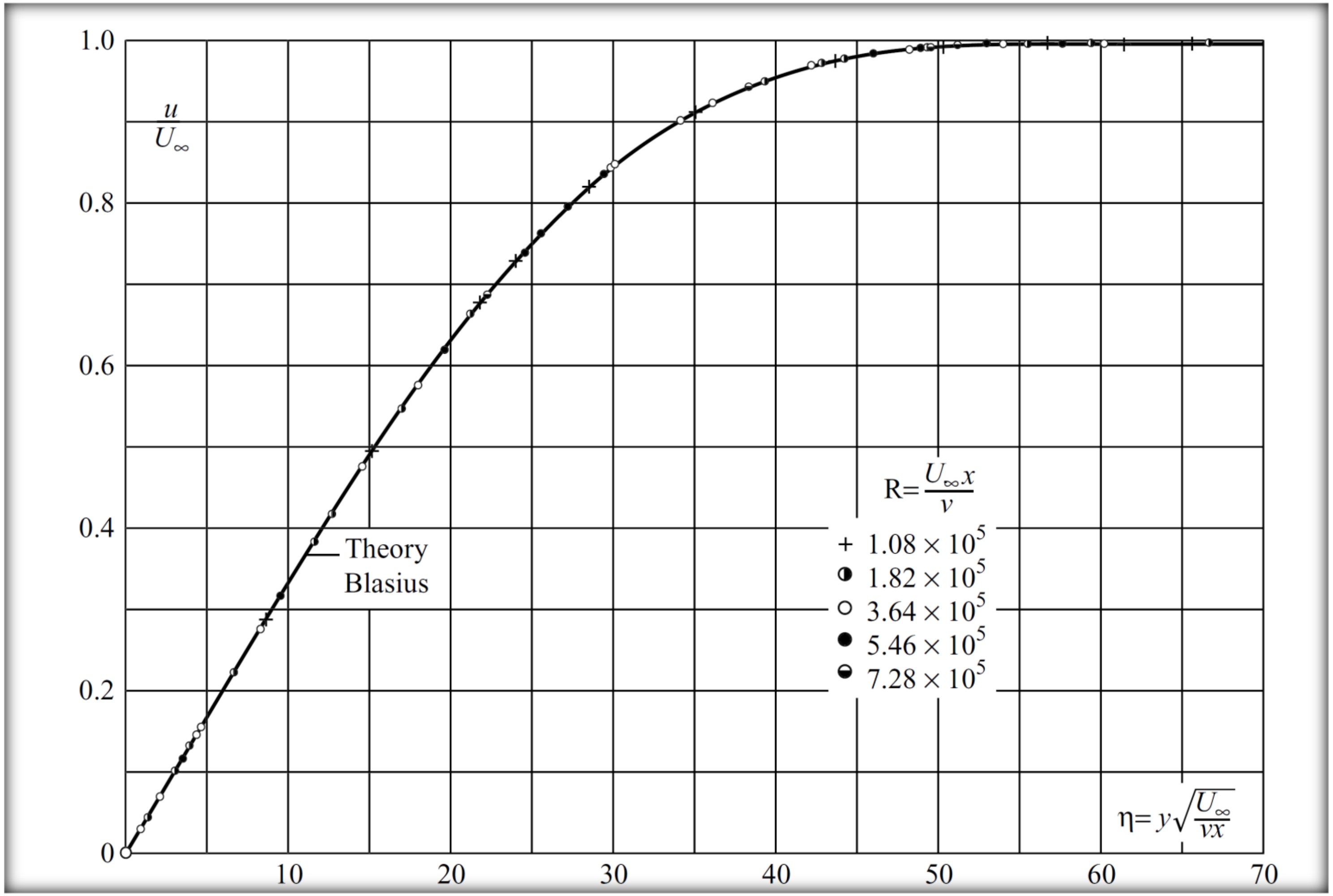


(B)

The BL flow along the flat plate at a positive angle of attack

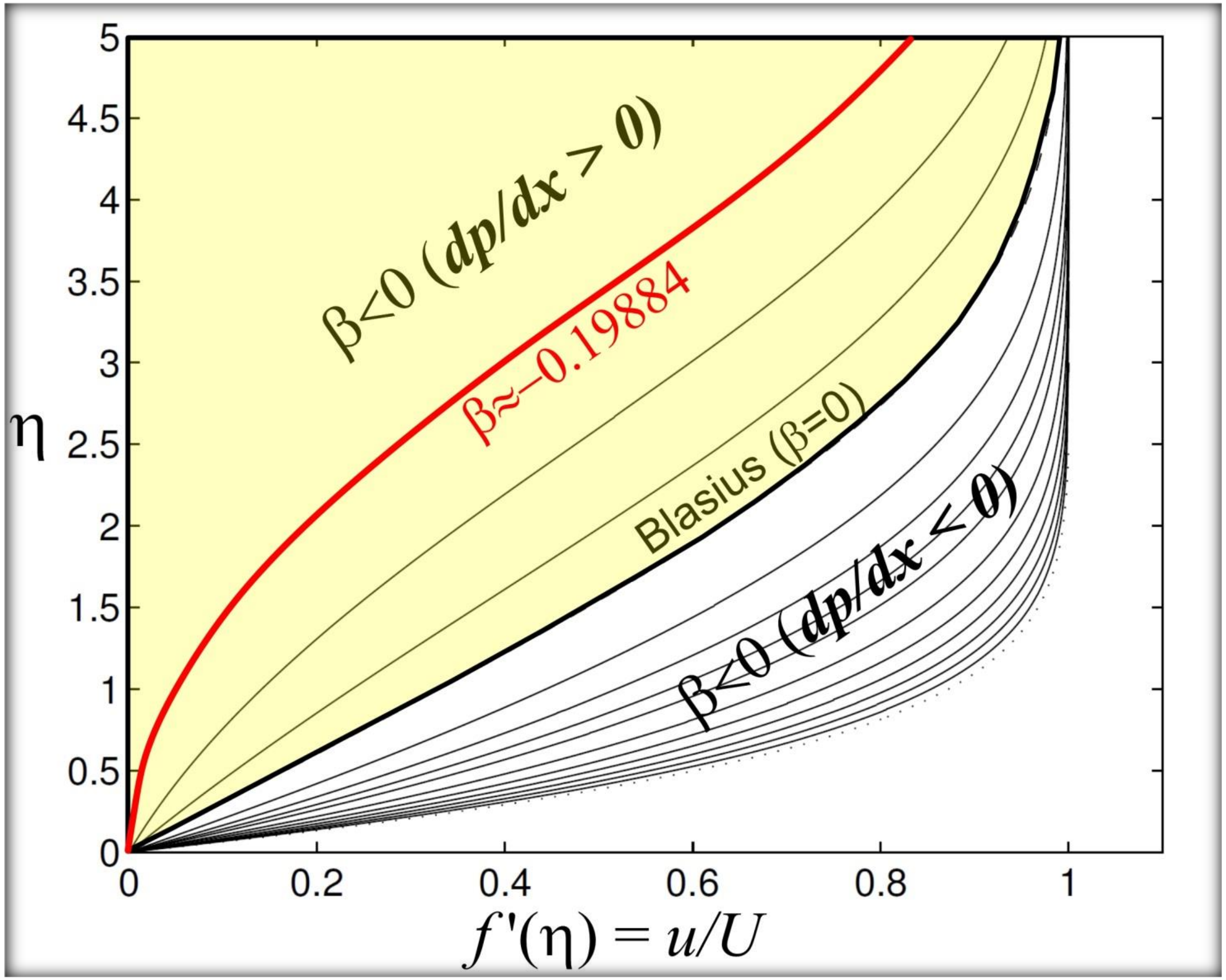
$$\begin{cases} -\frac{1}{2} \le m < 0\,,\;\; \beta \in [-2,0)\,,\;\; U(x) = U_\infty \left(\frac{x}{L}\right)^m \\ U'(x) < 0 \quad i \quad p'(x) > 0 \end{cases}$$

The velocity profiles of the sefl-similar Falknera-Skan BL



The velocity profile of the Blasius BL ($\beta = 0$) – theory vs experiment (after Schlichting, 2003)

The velocity profiles of the sefl-similar Falknera-Skan BL – cont.

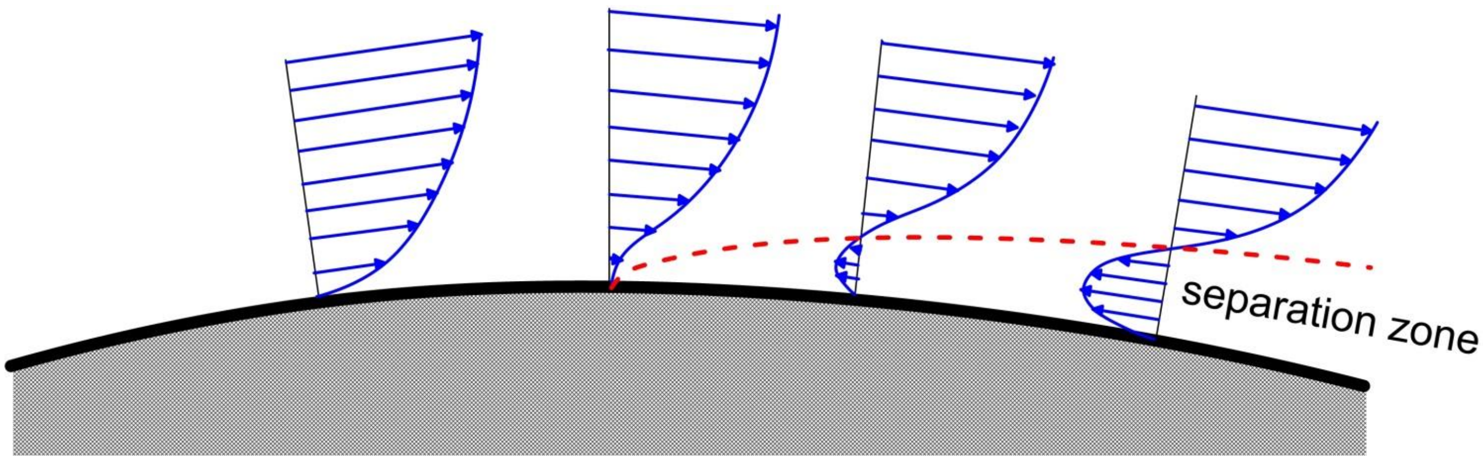


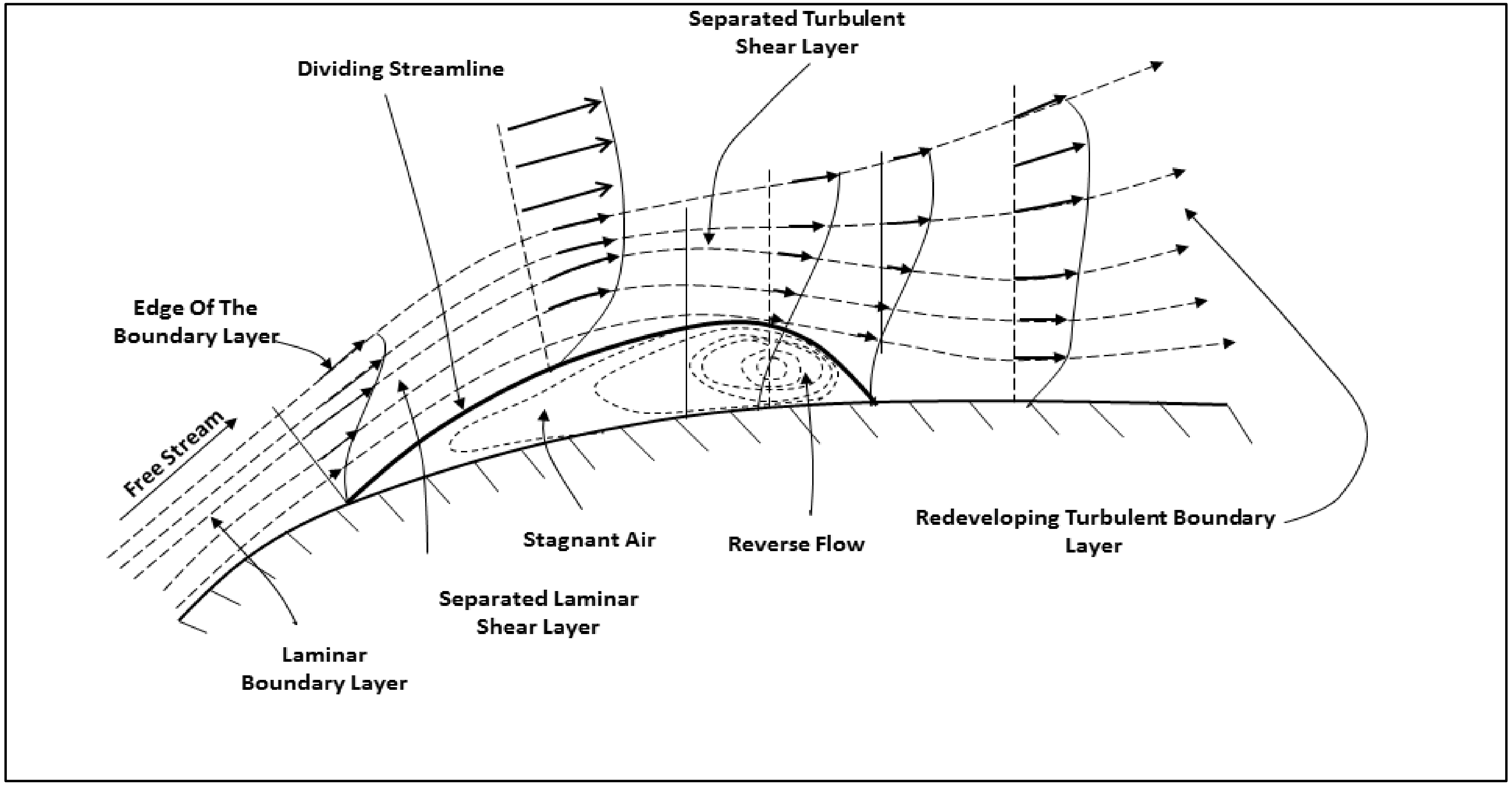
Critical angle of attack – appr. 18^0. For larger AA – BL separation along the whole plate!

Separation of the laminar BL

In some circumstances, the BL may separate from the surface of the body. “Separation” means that beyond a certain point along the body contour (in 3D – a line) the region of a flow reversal appears. This region may extend over the remaining downstream part of the body (in the case of an airfoil – to its trailing edge) or it can be limited in its extent, forming so-called “separation bubble”. Downstream form this bubble the reverse flow region disappears – the BL is re-attached to the body surface.

From the aerodynamic pint of view, massive separation of the boundary layer is disadvantageous and potentially dangerous as it is accompanied by sudden drop in the lift force as well as the growth of the aerodynamic drag (in one word – immense reduction of aerodynamic efficiency of a lifting surface)





Using the Prandtl's theory, one can show that the BL separation can occur only if the pressure rises along the wall, i.e., if $dp/dx > 0$. In aerodynamics, we call this situation an adverse pressure gradient (when the pressure drops along the BL – which effectively works against separation – we have a favorable pressure gradient)

We will show that $dp/dx > 0$ (or – equivalently - $U'(x) < 0$) is the necessary condition of the BL separation. As before, we assume that $y = 0$ at the wall. Since the velocity of a fluid at the wall is zero, then it follows from the Prandtl Equation written in the limit of $y \to 0$ that

$$0 = U(x)U'(x) + \nu \underbrace{\partial_{yy} u(x, y = 0)}_{\text{at the wall}}$$

Note that **in the vicinity of the separation point** $\partial_{yy} u$ **at the wall is positive**. Then, from the above equation follows that in this region $U'(x) < 0$, or equivalently $p'(x) > 0$.

The practical conclusion is that - in order to avoid laminar separation - one must be very careful while designing the "pressure recovery" part of the flow past an airfoil. This problem is particularly severe in the design of low-Reynolds number lifting surfaces (gliders, flying models, sails, small wind turbines). We will address this problem in the next lecture ..

Integral characteristics of the boundary layer. The von Karman Equation

In this section, we consider the integral approach to the boundary layer flows.

We begin with addressing the problem of formulating an appropriate measure of the BL thickness. We have already seen that the concept of BL thickness is a bit vague. In mathematical terms, the BL extends arbitrarily far away from the wall, but this fact is of little physical relevance. It reasonable to define the BL thickness as a distance from the wall where the velocity is nearly equal to a supposed velocity of an external (potential) flow. The commonly accepted definition is the 99% thickness δ_{99}. In our coordinate system, it is defined by the following equlity

$$u(x, y = \delta_{99}) := 0.99 \cdot U(x)$$

It is instructive to find the value of δ_{99} for the Blasius boundary layer

$$\delta_{99}(x) \approx 4.91\sqrt{\frac{\nu x}{U}} \quad \Rightarrow \quad \frac{\delta_{99}(x)}{x} \approx 4.91\sqrt{\frac{\nu}{Ux}} = \frac{4.91}{\sqrt{\mathrm{Re}_x}}$$

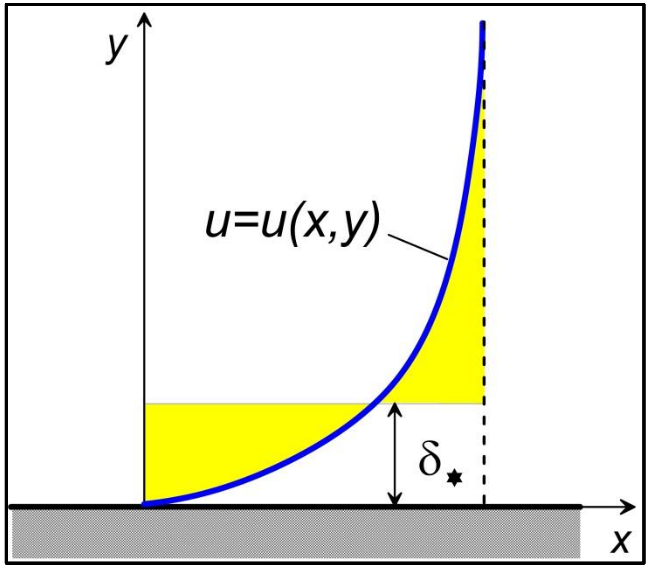
Example: Calculate 99% thickness of the Blasius BL for the following data: kinematic viscosity $\nu = 1.5 \cdot 10^{-5} \frac{m^2}{s}$ (air at the temperature 20^0 C), the velocity $U = 50 \frac{m}{s}$.

We have

$$\delta_{99}(x) \approx 4.91\sqrt{\frac{\nu}{U}}\sqrt{x} \approx 0.0027\sqrt{x}$$

Hence, at the distance of 1 m from the beginning of the BL, its 99% thickness reaches the value of approx. 2.7 mm.

Displacement thickness



Another measure of the BL thickness can be defined as follows Let's fix the coordinate x, choose sufficiently large number Δ and calculate the quantity

$$\Delta Q(x) = \int_0^{\Delta} [U(x) - u(x,y)]dy$$

Formally, it is a difference between volumetric flow rates of two flows - an ideal fluid flow with the uniform velocity equal to $U(x)$ and the actual BL flow with the velocity $u(x,y)$ - and computed for the transverse section extending from the wall to the line $y = \Delta$.

The **displacement thickness** is defined as (interpretation – see Figure)

$$\delta_*(x) := \frac{\Delta Q(x)}{U(x)} = \int_0^{\Delta} \left[1 - \frac{u(x,y)}{U(x)}\right] dy \to \boxed{\delta_* = \int_0^{\infty} \left[1 - \frac{u}{U}\right] dy}$$

The quantity $\delta_*(x)$ is a measure of the vertical wall displacement that would make the flow rate in an ideal and BL flows equal. **In other words, the external flow “feels” the shape of the hypothetical body thickened by the amount of contour-distributed displacement thickness rather that the shape of an actual body. Thus, the presence of the BL effectively modifies the external flows and changes the pressure distribution in a way that results in an additional component to the aerodynamic drag**.

We will discuss this problem in the next lecture ….

Note also that the following equality holds

$$\int_0^{\delta_*} u(x,y)dy = \int_{\delta_*}^{\Delta} [U(x) - u(x,y)]dy \Rightarrow \int_{\delta_*}^{\Delta} U(x)dy = \int_0^{\Delta} u(x,y)dy$$

Momentum thickness

Yet another, very important measure is the **momentum thickness**. Consider again the layer of fluid located between the wall and the line $y = \Delta > \delta_*$ and calculate the stream of the x-component of the linear momentum. As before, we consider two cases of flows:

1) ideal fluid flow with the uniform velocity $U(x)$, moved away from the real wall by a distance equal to the displacement thickness δ_*

$$\dot{P}_{x,id} = \rho\int_{\delta_*}^{\Delta} U^2(x)dy = \rho U\int_{\delta_*}^{\Delta} U(x)dy = \rho U\int_0^{\Delta} u(x,y)dy = \rho\int_0^{\Delta} u(x,y)U(x)dy$$

2) actual boundary layer flow

$$\dot{P}_{x,BL} = \rho\int_0^{\Delta} u^2(x,y)dy$$

The difference between these momentum fluxes is

$$\Delta\dot{P}_x = \rho\int_0^{\Delta} u(x,y)U(x)dy - \rho\int_0^{\Delta} u^2(x,y)dy = \rho\int_0^{\Delta} u(x,y)[U(x)-u(x,y)]dy$$

In the limit of $\Delta \to \infty$ ….

$$\Delta \dot{P}_x = \rho U^2 \underbrace{\int_0^\infty \frac{u}{U}\left[1 - \frac{u}{U}\right] dy}_{\theta} = \rho U^2 \theta$$

The quantity θ which appeared above is called the **momentum thickness.** Thus

$$\boxed{\theta = \frac{\Delta \dot{P}_x}{\rho U^2} = \int_0^\infty \frac{u}{U}\left[1 - \frac{u}{U}\right] dy}$$

Note that $\theta < \delta_*$ (explain). The BL velocity profile can be characterized also by its **shape factor** defines as follows

$$\boxed{H := \frac{\delta_*}{\theta} > 1}$$

Example 1:

Calculate displacement and momentum thickness for the local velocity profile given by the formula

$$u(y)=U_0(1-e^{-\alpha y})$$

Solution:

$$\delta_* = \lim_{\delta\to\infty}\int_0^{\delta}(1-1+e^{-\alpha y})dy = -\tfrac{1}{\alpha}\lim_{\delta\to\infty}[e^{-\alpha\delta}-e^{0}] = \tfrac{1}{\alpha}$$

$$\theta = \lim_{\delta\to\infty}\int_0^{\delta}(1-e^{-\alpha y})e^{-\alpha y}dy = \lim_{\delta\to\infty}\int_0^{\delta}e^{-\alpha y}dy - \lim_{\delta\to\infty}\int_0^{\delta}e^{-2\alpha y}dy = \tfrac{1}{\alpha}-\tfrac{1}{2\alpha} = \tfrac{1}{2\alpha}$$

The shape factor is $H=\dfrac{\delta_*}{\theta}=2$

Exercise: Repeat the above calculations for $u(y)=U_0[1-(1+y)^{-\alpha}]\ ,\ \alpha>1$

Example 2: Determine the displacement thickness, momentum thickness, shape factor and friction coefficient for the Blasius BL

From the Blasius solution we get

$$\eta(x,y)=\frac{y}{\sqrt{2\frac{\nu x}{U_\infty}}} \quad , \quad \frac{u}{U}(x,y)=f'[\eta(x,y)]$$

Hence

$$\delta_*(x)=\int_0^\infty \{1-f'[\eta(x,y)]\}dy=\sqrt{\frac{2\nu x}{U}}\int_0^\infty [1-f'(\eta)]d\eta\approx 1.721\sqrt{\frac{\nu x}{U}}$$

meaning that

$$\frac{\delta_*(x)}{x}\approx 1.721\sqrt{\frac{\nu}{Ux}}=\frac{1.721}{\sqrt{\mathrm{Re}_x}}$$

Next …

$$\theta(x)=\int_0^\infty f'[\eta(x,y)]\{1-f'[\eta(x,y)]\}dy=\sqrt{\frac{2\nu x}{U}}\int_0^\infty f'(\eta)[1-f'(\eta)]d\eta=$$

$$=\sqrt{\frac{2\nu x}{U}}f''(0)\approx 0.664\sqrt{\frac{\nu x}{U}}$$

so

$$\frac{\theta(x)}{x}\approx 0.664\sqrt{\frac{\nu}{Ux}}=\frac{0.664}{\sqrt{\mathrm{Re}_x}} \quad , \quad H\approx 2.592 \text{ (constant – self-similarity!)}$$

Other parameters are

- wall shear stress $\tau_w = \mu \frac{\partial u}{\partial y}\Big|_{y=0} = \frac{\mu U f''(0)}{\sqrt{2\nu x/U}}$

- local friction coefficient $C_f = \frac{2\tau_w}{\rho U^2} \approx \frac{\sqrt{2}\, f''(0)}{\sqrt{\mathrm{Re}_x}} = \frac{0,664}{\sqrt{\mathrm{Re}_x}} = \frac{\theta}{x}$

- (global) friction coefficient for the flat plate having the length L

$$C_D(L) = \frac{1}{L}\int_0^L C_f(x)dx = \frac{0.664}{L}\sqrt{\frac{\nu}{U}}\int_0^L x^{-1/2}dx = \frac{0.664}{L}\sqrt{\frac{\nu}{U}}\,2\sqrt{L} = 2C_f(L) = \frac{1.328}{\sqrt{\mathrm{Re}_x}}$$

Derivation of the von Karman Equation

We will derive the von Karman Equation which expresses the principle of the momentum conservation in terms of the integral measures of the BL thickness.

The point of departure is the Prandtl Equation

$$u\,\partial_x u + \upsilon\,\partial_y u = U(x)U'(x) + \tfrac{\mu}{\rho}\,\partial_{yy} u$$

Let's integrate this equation with respect to the y coordinate in the interval $[0, \Delta]$. As, before $y = 0$ corresponds to the wall. Thus, we get

$$\underbrace{\int_0^{\Delta} \left(u\,\partial_x u + \upsilon\,\partial_y u - UU' \right) dy}_{L_\Delta} = \underbrace{\tfrac{\mu}{\rho} \int_0^{\Delta} \partial_{yy} u \, dy}_{R_\Delta}$$

The right side of the above equation can be evaluated as follows

$$R_\Delta = \tfrac{\mu}{\rho} \int_0^{\Delta} \partial_{yy} u \, dy = \tfrac{1}{\rho} \left(\mu\,\partial_y u \Big|_{y=\Delta} - \mu\,\partial_y u \Big|_{y=0} \right)$$

Taking the limit $\Delta \to \infty$, one obtains

$$R = \lim_{\Delta\to\infty} R_\Delta = -\tfrac{1}{\rho}\mu\partial_y u\Big|_{sciana} = -\tfrac{1}{\rho}\tau_w$$

where τ_w denotes the wall shear stress.

Using the continuity equation, we can express the wall-normal velocity component as follows

$$\upsilon\big|_{y=0} = 0 \;\Rightarrow\; \upsilon = \int_0^y \partial_y \upsilon\, dy = -\int_0^y \partial_x u\, dy$$

Hence

$$\int_0^\Delta \upsilon\partial_y u\, dy = \int_0^\Delta \left(-\int_0^y \partial_x u\, dy'\right)\partial_y u\, dy \underset{\text{int.}\,by\ parts...}{=} \left(-\int_0^y \partial_x u\, dy'\right)u\Big|_0^\Delta + \int_0^\Delta u\partial_x u\, dy =$$

$$= -u(x,\Delta)\int_0^\Delta \partial_x u\, dy + \int_0^\Delta u\partial_x u\, dy = -\int_0^\Delta [u(x,\Delta)\partial_x u - u(x,y)\partial_x u]dy$$

After insertion of the obtained expression to the left side L_Δ …

$$L_\Delta = \int_0^\Delta (u\,\partial_x u + \upsilon\,\partial_x u - UU')dy =$$

$$= \int_0^\Delta (u\,\partial_x u - UU')dy - \int_0^\Delta [u(\Delta)\partial_x u - u\,\partial_x u]dy =$$

$$= \int_0^\Delta [u\,\partial_x u - UU' - u(\Delta)\partial_x u + u\,\partial_x u]dy =$$

$$= \int_0^\Delta [u\,\partial_x u - UU' - u(\Delta)\partial_x u + u\,\partial_x u + uU' - uU']dy =$$

$$= -\int_0^\Delta \{u\partial_x(U_0 - u) + [u(\Delta) - u]\partial_x u\}dy - \int_0^\Delta (U - u)U'dy$$

Since $\lim_{\Delta\to\infty} u(\Delta) = U$, thus

$$L = \lim_{\Delta\to\infty} L_\Delta = -\int_0^\infty \{u\partial_x(U-u) + [U-u]\partial_x u\}dy - \int_0^\infty (U-u)U'dy =$$

$$= -\tfrac{d}{dx}\int_0^\infty u(U-u)dy - U'\int_0^\infty (U-u)dy =$$

$$= -\tfrac{d}{dx}\left[U^2\int_0^\infty \frac{u}{U}(1-\frac{u}{U})dy\right] - U'U\int_0^\infty (1-\frac{u}{U})dy$$

Equating L and R we arrive at the **von Karman Equation**

$$\boxed{\tfrac{d}{dx}[U^2(x)\theta(x)] + U(x)U'(x)\delta_*(x) = \tfrac{1}{\rho}\tau_w}$$

Equivalent forms of this important equations are …

$$\boxed{\tfrac{d}{dx}\theta + \frac{U'}{U}(\delta_* + 2\theta) = \frac{\tau_w}{\rho U^2}} \quad , \quad \boxed{\tfrac{d}{dx}\theta + (2+H)\frac{U'}{U}\theta = \frac{C_f}{2}}$$

Thwaites method (1949)

Define the parameter $$\lambda = \frac{\theta^2 U'}{\nu}$$

Next, let's multiply the Karman Equation by the quantity $\mathrm{Re}_\theta = U\theta/\nu$. We obtain

$$\tfrac{1}{\nu} U\theta\theta' + (2+H)\tfrac{1}{\nu} U'\theta^2 = \frac{\tau_W \theta}{\rho\nu U} \equiv \frac{\tau_W \theta}{\mu U}$$

The left side of the above equation can be written in the following form

$$\tfrac{1}{\nu} U\theta\theta' + (2+H)\tfrac{1}{\nu} U'\theta^2 = \tfrac{1}{2} U \tfrac{d}{dx}(\tfrac{1}{\nu}\theta^2) + (2+H)\lambda =$$
$$= \tfrac{1}{2} U \tfrac{d}{dx}(\tfrac{1}{U'}\lambda) + (2+H)\lambda$$

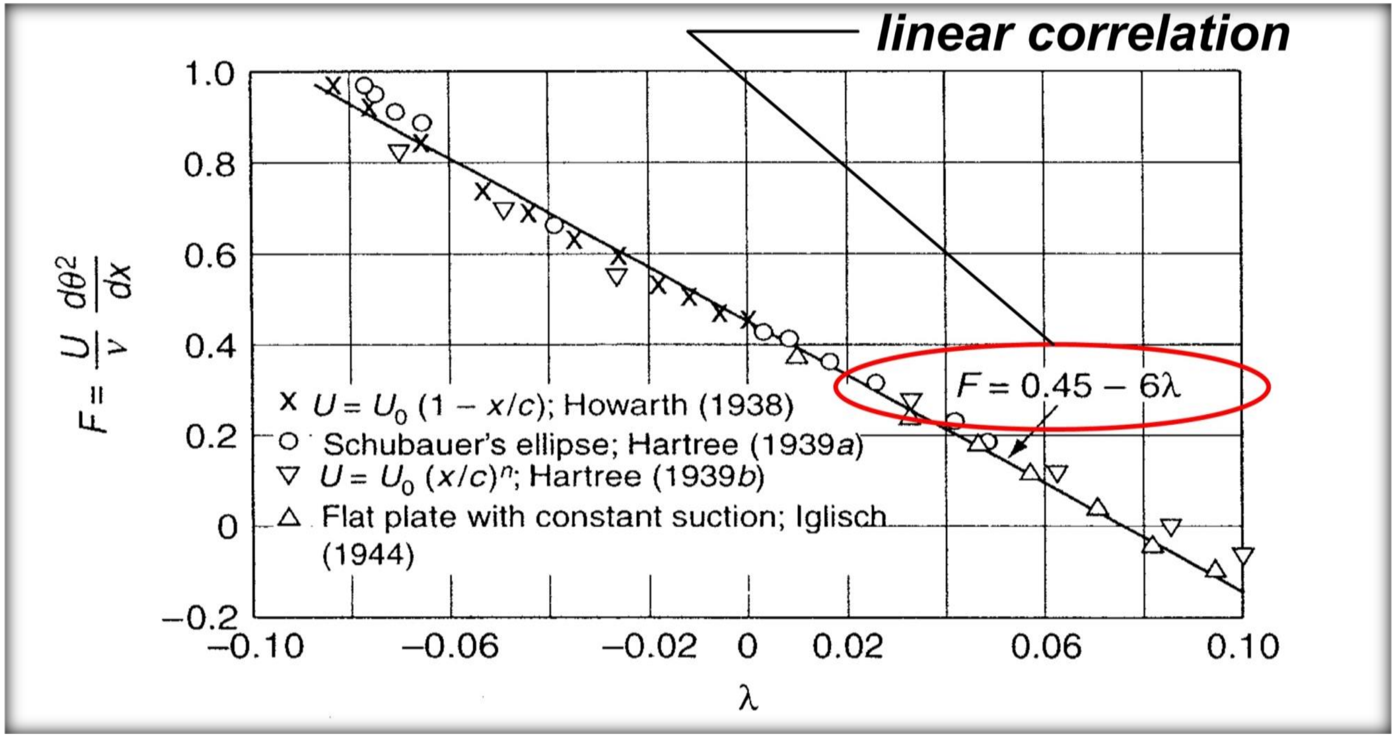
Assume that the right side of the Karman Equation can be expressed as the function of λ

$$S(\lambda) = \tau_w \theta / \mu U$$

This way, we have obtained the differential equation for the function $\lambda = \lambda(x)$

$$U\frac{d}{dx}\left(\frac{1}{U'}\lambda\right) = 2[S(\lambda) - (2+H)\lambda] \equiv F(\lambda)$$

Thwaites collected all available analytical and experimental results and constructed his famous plot (1949)



Thwaites' approximate correlation formula: $F(\lambda) = 0.45 - 6\lambda$

It turns out that the differential equation for $\lambda = \lambda(x)$ with the linear function F can be solved analytically in the closed form (**exercise for the Student**). The general solution reads

$$\theta^2(x) = 0.45\nu U^{-6}(\int_{x_0}^{x} U^5 dx + C)$$

The integration constant can be evaluated as follows.

$$\theta^2(x_0) = 0.45\nu U(x_0)^{-6} C \quad \Rightarrow \quad C = \frac{\theta^2(x_0)U^6(x_0)}{0.45\nu}$$

Hence

$$\theta^2(x) = U^{-6}[0.45\nu \int_{x_0}^{x} U^5 dx' + U^6 \theta^2(x_0)]$$

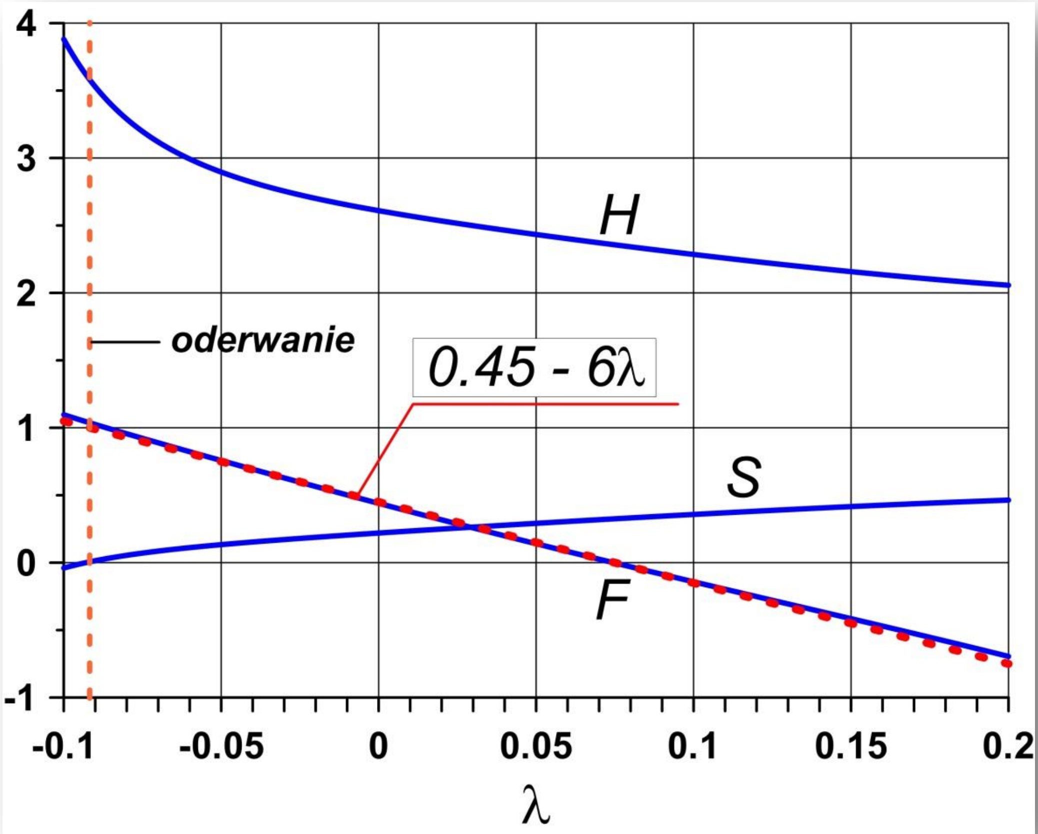
Usually, the coordinate x_0 corresponds to the beginning of the BL, where either the velocity U or the momentum thickness θ is equal zero (explain why!). Thus, the final form of the **Thwaites' Formula** is

$$\boxed{\theta^2(x) = \frac{0.45\nu}{U^6} \int_{x_0}^{x} U^5 dx'}$$

This formula is used to evaluate the distribution of the momentum thickness along the BL.

After θ is known, one can determine $\lambda(x) = \frac{1}{\nu} U'(x)\theta^2(x)$ and $\mathrm{Re}_\theta(x) = \dfrac{U(x)\theta(x)}{\nu}$.

Next, the friction coefficient can be evaluated from the formula $C_f(x) = \frac{2}{\mathrm{Re}_\theta(x)} S[\lambda(x)]$.



Thwaites provided the functions $S = S(\lambda)$ and $H = H(\lambda)$ in tabulated form.

In practice, one can use convenient analytical approximation defines as follows:

$$S(\lambda) = 0.22 + 1.52\lambda - 5\lambda^3 - \frac{0.072\lambda^2}{(\lambda + 0.18)^2}$$

$$H(\lambda) = 2.61 - 4.1\lambda + 14\lambda^3 + \frac{0.56\lambda^2}{(\lambda + 0.18)^2}$$