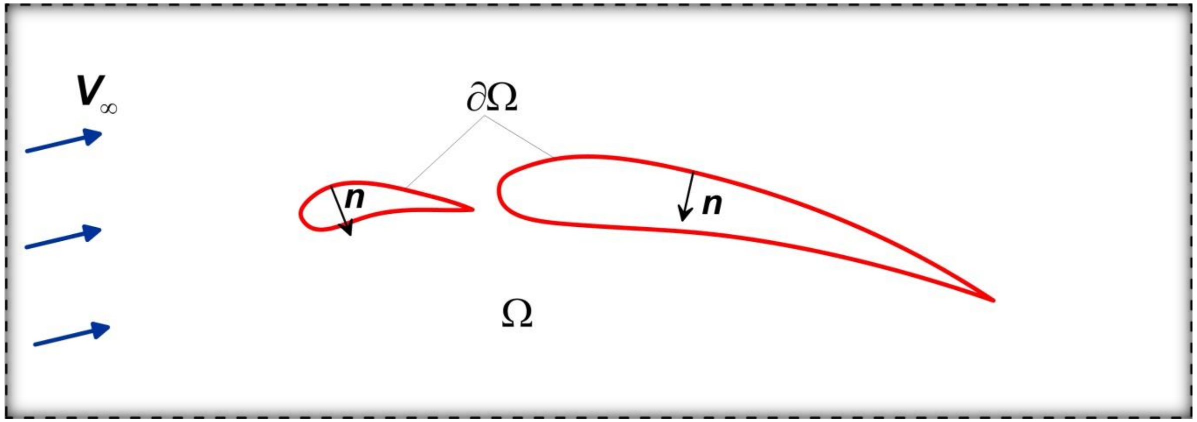
LECTURE 2

INCOMPRESSIBLE POTENTIAL FLOWS
PART 2

General formulation of a potential flow problem



In the domain Ω, find velocity potential φ such that:

- $\nabla^2\varphi = 0$ in Ω
- The following boundary condition is satisfied at $\partial\Omega$ (impermeability of the solid bodies)

$$\upsilon_n = \boldsymbol{\upsilon}\cdot\boldsymbol{n} = \nabla\varphi\cdot\boldsymbol{n} \equiv \frac{\partial\varphi}{\partial n} = 0$$

- Far-field (asymptotic) condition is satisfied

$$\lim_{\|\boldsymbol{x}\|_2\to\infty} \boldsymbol{\upsilon} = \boldsymbol{V}_\infty \quad \Rightarrow \quad \lim_{\|\boldsymbol{x}\|_2\to\infty} \nabla\varphi = \boldsymbol{V}_\infty$$

We will focus of 2D case.

Mathematical preparation

Consider the contour integral along the circle with the center at the origin and the radius R_∞

$$J_\alpha := \oint_{K_\infty} \frac{f(\theta)}{r^\alpha} ds$$

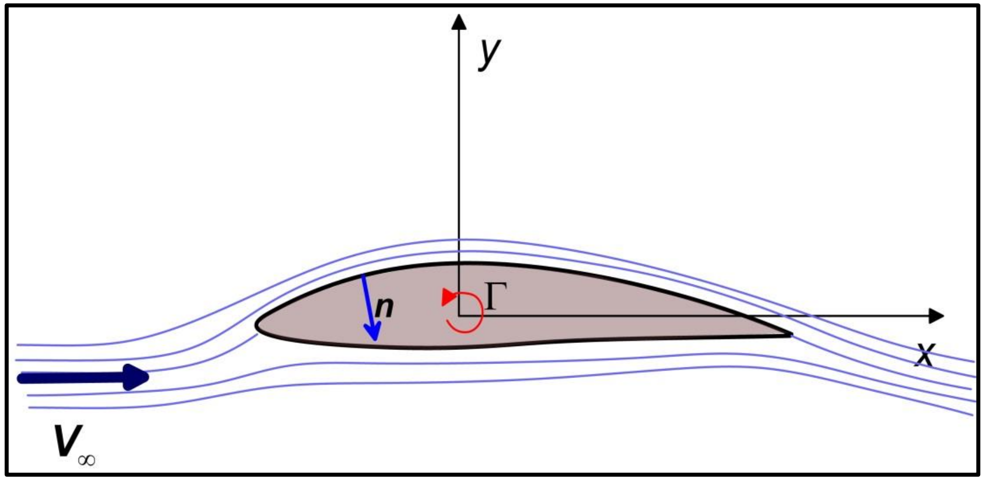
Assume that $R_\infty \to \infty$. We need to find the value(s) of the exponent α such that the limit of the above integral is finite and nonzero.

Clearly
$$\oint_{K_\infty} \frac{f(\theta)}{r^\alpha} ds = \frac{1}{R_\infty^{\alpha-1}} \int_0^{2\pi} f(\theta) d\theta$$

The answer is simple: two conditions must hold simultaneously

$$\alpha = 1 \quad , \quad \int_0^{2\pi} f(\theta) d\theta \neq 0.$$

Otherwise, either $\alpha > 1$ hence $\lim\limits_{R_\infty \to \infty} \frac{1}{R_\infty^{\alpha-1}} = 0$ and the integral J_α vanishes, or $\alpha < 1$ and then $\lim\limits_{R_\infty \to \infty} \frac{1}{R_\infty^{\alpha-1}} = \infty$. In the latter case, the finite limit of the integral J_α exists only when $\int_0^{2\pi} f(\theta) d\theta \equiv 0$, which means that the integral J_α vanishes identically for arbitrarily large values of the radius R_∞.



The velocity potential can be constructed by superposition of four components …

$$\varphi(x,y) = U_\infty x + V_\infty y + \frac{\Gamma}{2\pi}\theta(x,y) + \hat{\varphi}(x,y)$$

We assume that the velocity field corresponding to the component $\hat{\varphi}(x,y)$ diminishes at large distances faster than r^{-1} (and hence it does not contribute to the total circulation in the flow)

$$\hat{\boldsymbol{v}} = \nabla \hat{\varphi} \sim O(\tfrac{1}{r^{1+\alpha}}) \quad , \quad \alpha > 1$$

Clearly, the function $\hat{\varphi}$ must be harmonic in the flow domain Ω , i.e.,

$$\nabla^2\hat{\varphi} \equiv \partial_{xx}\hat{\varphi} + \partial_{yy}\hat{\varphi} = 0$$

The following boundary and far-field conditions are imposed

$$\upsilon_n\big|_{\partial\Omega} = \tfrac{\partial}{\partial n}\varphi\big|_{\partial\Omega} = 0 \quad , \quad \lim_{r\to\infty}\nabla\varphi = U_\infty \boldsymbol{e}_x + V_\infty \boldsymbol{e}_y$$

An additional degree of freedom is the circulation of the contour-bounded vortex Γ. The key problem is how to find the value of this quantity (and – in the first place – why we need it!)

Typical aerodynamic airfoil has nearly sharp rear tip (we say – trailing edge). In theoretical analysis we can assume ideally sharp tip, i.e., the radius of curvature of the airfoil contour becomes infinitely large at the training edge. Theoretically speaking, the angle of the trailing edge may be larger or equal zero. In the latter case, the trailing point is a **cusp**.

If the contour-bounded circulation is “improper” then the fluid is forced to flow around the sharp edge and the velocity at the very tip becomes infinite, which is nonsense. Hence, **the criterion of choice of the circulation is to obtain a physically meaningful flow at and near the sharp trailing edge**.

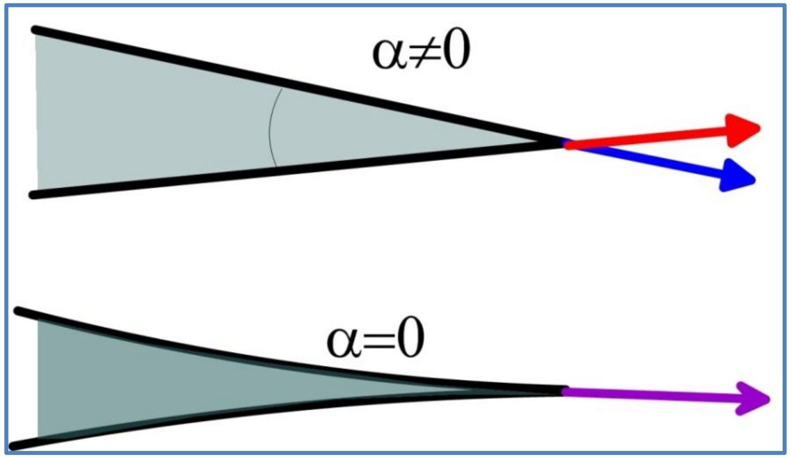
Kutta-Joukovski condition

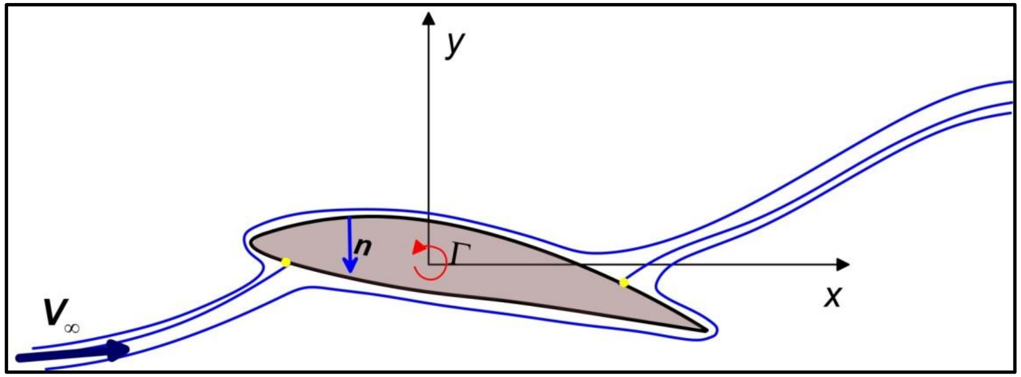
The circulation of the bounded vortex should be such that:

- If the trailing edge angle is larger than zero – the trailing edge tip is the rear stagnation point of the flow past an airfoil. This means that the velocity at this point is equal zero.
- If the trailing edge angle is equal zero (cusp) – the limit values of the tangent velocity (obtained while the trailing edge tip is approached along either the bottom of the upper part of the airfoil) must be equal.

Further comment:

If the trailing edge angle is nonzero, then violation of the K-J condition would lead to ambiguity of the velocity vector. In the case of a cusp, it is sufficient that the upper and bottom limits of tangent velocity are equal – as a rule the velocity at a cusped trailing edge is not equal zero.





Violation of the K-J condition

Let us introduce a curvilinear coordinate s along the contour and such that $s = 0$ and $s = L$ correspond to the trailing edge point.

Mathematical form of the Kutta-Joukovski condition is:

- For a trailing edge with nonzero angle

$$\nabla \varphi \big|_{s=0} = \mathbf{0}$$

- For the cusped trailing edge ($\boldsymbol{\tau}$ - unary vector tangent to the airfoil contour)

$$\lim_{s \to 0^+} \nabla \varphi \cdot \boldsymbol{\tau} = \lim_{s \to L^-} \nabla \varphi \cdot \boldsymbol{\tau}$$

Determination of the potential flow satisfying the Kutta-Joukovski condition

We write $\hat{\varphi}$ as the sum of two components

$$\hat{\varphi}(x,y) = \Gamma\hat{\varphi}_1(x,y) + \hat{\varphi}_2(x,y)$$

The scalar field $\hat{\varphi}_1$ is the solution to the following boundary-value problem

$$\begin{cases} \nabla^2\hat{\varphi}_1 = 0 \quad in \;\; \Omega \\ \frac{\partial}{\partial n}\hat{\varphi}_1\Big|_{\partial\Omega} = -(u_1 n_x + \upsilon_1 n_y)\Big|_{\partial\Omega} \end{cases}$$

where $\boldsymbol{\upsilon}_1 = u_1\boldsymbol{e}_x + \upsilon_1\boldsymbol{e}_y$ is the velocity field induced by the bounded vortex with unitary circulation ($\Gamma = 1$).

Note that $\hat{\varphi}_1$ depends only on the airfoil shape and position of the vortex center inside the airfoil.

The scalar field $\hat{\varphi}_2$ is the solution to the following boundary-value problem

$$\begin{cases} \nabla^2 \hat{\varphi}_2 = 0 \quad in \ \ \Omega \\ \frac{\partial}{\partial n} \hat{\varphi}_2 \Big|_{\partial\Omega} = -\boldsymbol{V}_\infty \cdot \boldsymbol{n} \Big|_{\partial\Omega} = -(U_\infty n_x + V_\infty n_y) \Big|_{\partial\Omega} \end{cases}$$

The function $\hat{\varphi}_2$ depends on the geometry of the airfoil and on the orientation of the free-stream velocity with respect to the airfoil (on the angle of attack). The value of the free-stream velocity works here as a scaling factor for the field $\hat{\varphi}_2$.

Total velocity field is given by the formula

$$\boldsymbol{v} = \boldsymbol{V}_\infty + \Gamma(\boldsymbol{v}_1 + \nabla \hat{\varphi}_1) + \nabla \hat{\varphi}_2$$

Now, we impose the K-J condition (assume that the angle of the trailing edge is larger than zero). Due to impermeability of the body we have

$$\lim_{s \to 0^+} \boldsymbol{v} \cdot \boldsymbol{n} = \lim_{s \to L^-} \boldsymbol{v} \cdot \boldsymbol{n}$$

Thus, the K-J condition is equivalent to

$$\lim_{s\to 0^+}(\boldsymbol{V}_\infty + \nabla\hat{\varphi}_2)\cdot\boldsymbol{\tau} + \Gamma \lim_{s\to L^-}(\boldsymbol{v}_1 + \nabla\hat{\varphi}_1)\cdot\boldsymbol{\tau} = 0$$

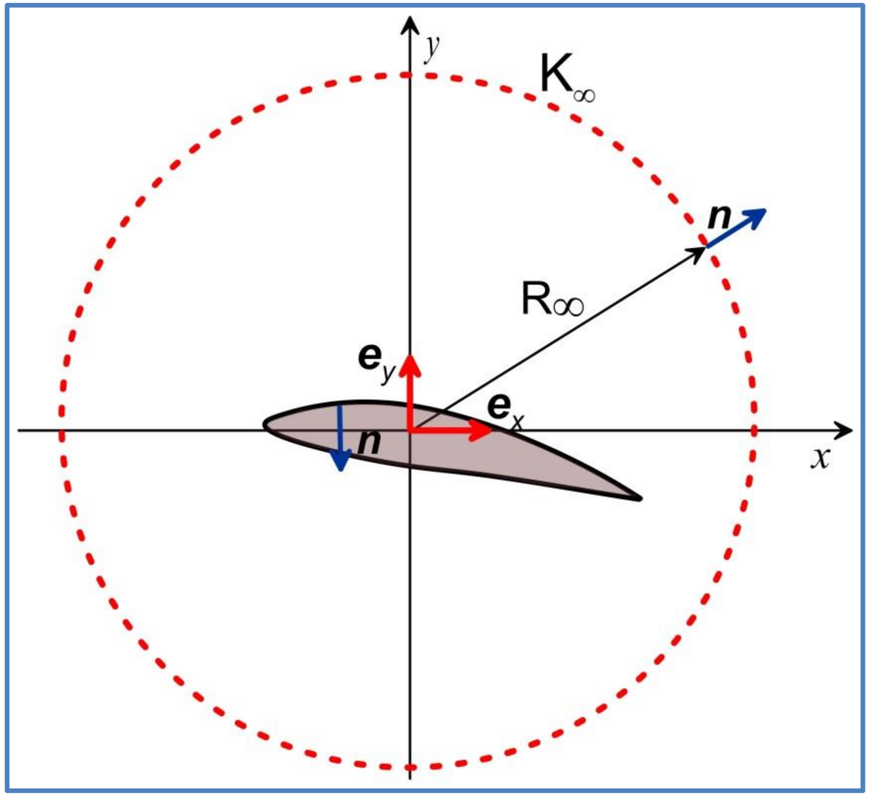
Finally, we obtain the formula for the circulation of the bounded vortex

$$\Gamma = -\left.\frac{(\boldsymbol{V}_\infty + \nabla\hat{\varphi}_2)\cdot\boldsymbol{\tau}}{(\boldsymbol{v}_1 + \nabla\hat{\varphi}_1)\cdot\boldsymbol{\tau}}\right|_{s=0}$$

Comment:

One usually solves the auxiliary BV problems formulated above by means of the numerical methods. These problems are in fact the particular cases of the more general problem where a harmonic function with prescribed distribution of the wall-normal derivative is sought in the exterior domain. An efficient approach to such problem is to solve numerically the **Boundary Integral Equation**. The solution of this equation yields the boundary value of the harmonic function. Afterwards, this function can be "reconstructed" in any point within the domain by computing some contour integrals.

Aerodynamic force



We will use the method of integral balance of linear momentum to derive a formula for the aerodynamic force in a 2D potential flow.

Without scarifying generality, we will assume that the free-stream velocity is oriented along the *0x* axix, i.e.,

$$\boldsymbol{V}_\infty = U_\infty \boldsymbol{e}_x \quad , \quad V_\infty = 0$$

We introduce the control volume located between the circular contour K_∞ centered at the origin and the radius R_∞.

Accordingly to the formula derived in the course of Fluid Mechanics I, the aerodynamic force can be expressed as

$$\boldsymbol{F} = F_x \boldsymbol{e}_x + F_y \boldsymbol{e}_y = -\int_{K_\infty} \rho \upsilon_n \boldsymbol{\upsilon} ds - \int_{K_\infty} (p - p_\infty) \boldsymbol{n} ds = -\boldsymbol{J}_V - \boldsymbol{J}_P$$

Note that the Cartesian component F_x corresponds to the aerodynamic drag, while the component F_y is the lift force.

We will calculate $\boldsymbol{J}_V$ and $\boldsymbol{J}_P$. Note that the velocity field can be expressed by the following formula (where the circulation Γ has been chosen accordingly to the K-J condition)

$$\boldsymbol{\upsilon} = \boldsymbol{V}_\infty + \Gamma \boldsymbol{\upsilon}_1 + \nabla \hat{\varphi} \approx (U_\infty + \Gamma u_1)\boldsymbol{e}_x + \Gamma \upsilon_1 \boldsymbol{e}_y + \boldsymbol{O}(\tfrac{1}{R_\infty^{1+\alpha}})$$

The last term contained all terms vanishing with the distance faster than $1/R_\infty$.

In the above formula, the Cartesian components of the velocity induced by the unary vortex appear. At the external circle K_∞ these components are equal

$$u_1\big|_{K_\infty} = -\frac{1}{2\pi R_\infty}\sin\theta \quad , \quad \upsilon_1\big|_{K_\infty} = \frac{1}{2\pi R_\infty}\cos\theta$$

Hence, the total velocity at the circle K_∞ is equal

$$\boldsymbol{\upsilon}\big|_{K_\infty} = \left[U_\infty - \frac{\Gamma}{2\pi R_\infty}\sin\theta\right]\boldsymbol{e}_x + \frac{\Gamma}{2\pi R_\infty}\cos\theta\,\boldsymbol{e}_y + \boldsymbol{O}(\tfrac{1}{R_\infty^{1+\alpha}})$$

Note also that the external normal vector at K_∞ is: $\boldsymbol{n}\big|_{K_\infty} = \cos\theta\,\boldsymbol{e}_x + \sin\theta\,\boldsymbol{e}_y$

Let's calculate the normal velocity at the contour of K_∞ …

$$\upsilon_n\Big|_{K_\infty} = \boldsymbol{\upsilon}\cdot\boldsymbol{n}\Big|_{K_\infty} = U_\infty\cos\theta - \frac{\Gamma}{2\pi R_\infty}\sin\theta\cos\theta + \frac{\Gamma}{2\pi R_\infty}\cos\theta\sin\theta + O(\tfrac{1}{R_\infty^{1+\alpha}}) =$$

$$= U_\infty\cos\theta + O(\tfrac{1}{R_\infty^{1+\alpha}})$$

We also need …

$$\upsilon_n\boldsymbol{\upsilon}\Big|_{K_\infty} = \left[U_\infty^2\cos\theta - \frac{\Gamma U_\infty}{2\pi R_\infty}\sin\theta\cos\theta\right]\boldsymbol{e}_x + \frac{\Gamma U_\infty}{2\pi R_\infty}\cos^2\theta\,\boldsymbol{e}_y + \boldsymbol{O}(\tfrac{1}{R_\infty^{1+\alpha}})$$

We compute $\boldsymbol{J}_V$ …

$$\boldsymbol{J}_V = \rho\int_{K_\infty}\upsilon_n\boldsymbol{\upsilon}\,ds = \left[\rho R_\infty\int_0^{2\pi}(U_\infty^2\cos\theta - \tfrac{\Gamma U_\infty}{2\pi R_\infty}\sin\theta\cos\theta)d\theta\right]\boldsymbol{e}_x +$$

$$+\left[\rho R_\infty\int_0^{2\pi}\tfrac{\Gamma U_\infty}{2\pi R_\infty}\cos^2\theta\,d\theta\right]\boldsymbol{e}_y + \boldsymbol{O}(R_\infty^{-\alpha}) = \tfrac{1}{2}\rho U_\infty\Gamma\boldsymbol{e}_y + \boldsymbol{O}(R_\infty^{-\alpha}) \xrightarrow[R_\infty\to\infty]{} \tfrac{1}{2}\rho U_\infty\Gamma\boldsymbol{e}_y$$

The pressure integral $\boldsymbol{J}_P$ can be calculated with the use of the Bernoulli Equation …

$$\boldsymbol{J}_P = \int_{K_\infty} (p - p_\infty)\boldsymbol{n}\,ds \underset{Bernoulli\ Eq.}{=} \tfrac{1}{2}\rho \int_{K_\infty} (U_\infty^2 - V^2)\boldsymbol{n}\,ds$$

We need to find the square of the velocity magnitude …

$$V^2 = \left[U_\infty - \frac{\Gamma}{2\pi R_\infty}\sin\theta + O\left(\tfrac{1}{R_\infty^{1+\alpha}}\right)\right]^2 + \left[\frac{\Gamma}{2\pi R_\infty}\cos\theta + O\left(\tfrac{1}{R_\infty^{1+\alpha}}\right)\right]^2 =$$

$$= U_\infty^2 - \frac{U_\infty \Gamma}{\pi R_\infty}\sin\theta + O\left(\tfrac{1}{R_\infty^{1+\alpha}}\right)$$

We compute $\boldsymbol{J}_P$ …

$$\boldsymbol{J}_P = \tfrac{1}{2}\rho \int_{K_\infty} (U_\infty^2 - V^2)\boldsymbol{n}\,ds = \tfrac{1}{2}\rho\left[R_\infty \int_0^{2\pi} \frac{U_\infty \Gamma}{\pi R_\infty}\sin\theta\cos\theta\,d\theta\right]\boldsymbol{e}_x +$$

$$+ \tfrac{1}{2}\rho\left[R_\infty \int_0^{2\pi} \frac{U_\infty \Gamma}{\pi R_\infty}\sin^2\theta\,d\theta\right]\boldsymbol{e}_y + \boldsymbol{O}(R_\infty^{-\alpha}) \xrightarrow[R_\infty \to \infty]{} \tfrac{1}{2}\rho U_\infty \Gamma \boldsymbol{e}_y$$

Finally, we obtain the formula for the aerodynamic force (known as the Kutta-Joukovski formula)

$$\boldsymbol{F} = -\boldsymbol{J}_V - \boldsymbol{J}_P = -\rho U_\infty \Gamma \boldsymbol{e}_y$$

Note that:

- $F_x \equiv 0$ - lack of the drag (d'Alembert paradox) !!!
- $F_y = -\rho U_\infty \Gamma$ - lift force is proportional to the circulation of the bounded vortex

Application of complex functions in classical aerodynamics

Assume that the potential flow is given with

$$\varphi = \varphi(x, y) \quad , \quad \psi = \psi(x, y)$$

Let us formally introduce the following change of the coordinates

$$\begin{cases} z = x + iy \\ \overline{z} = x - iy \end{cases} \quad \rightleftarrows \quad \begin{cases} x = \frac{1}{2}(z + \overline{z}) \\ y = \frac{1}{2i}(z - \overline{z}) \end{cases} \quad , \quad i = \sqrt{-1}$$

Next, define a function of two complex variables

$$W(z, \overline{z}) = \varphi[\tfrac{1}{2}(z + \overline{z}), \tfrac{1}{2i}(z - \overline{z})] + i\,\psi[\tfrac{1}{2}(z + \overline{z}), \tfrac{1}{2i}(z - \overline{z})]$$

Let's calculate the partial derivative

$$\partial_{\overline{z}} W(z, \overline{z}) = \tfrac{1}{2}\partial_x \varphi(..) - \tfrac{1}{2i}\partial_y \varphi(..) + \tfrac{i}{2}\partial_x \psi(..) + i(-\tfrac{1}{2i})\partial_y \psi(..) =$$
$$= \tfrac{1}{2}[\partial_x \varphi(..) - \partial_y \psi(..)] + \tfrac{1}{2} i [\partial_y \varphi(..) + \partial_x \psi(..)]$$

We know that the following equalities hold

$$\partial_x \varphi = \partial_y \psi = u \quad , \quad \partial_y \varphi = -\partial_x \psi = \upsilon$$

which implies that $\partial_{\bar{z}} W(z, \bar{z}) \equiv 0$

Thus, the function W is in fact a function of just one complex variable, i.e., $W = W(z)$. Let's calculate its first (ordinary) derivative …

$$W'(z) = \tfrac{1}{2}\partial_x \varphi(..) + \tfrac{1}{2i}\partial_y \varphi(..) + \tfrac{i}{2}\partial_x \psi(..) + i\tfrac{1}{2i}\partial_y \psi(..) =$$

$$= \tfrac{1}{2}[\underbrace{\partial_x \varphi(..) + \partial_y \psi(..)}_{=2\partial_x \varphi = 2u}] - \tfrac{1}{2} i [\underbrace{\partial_y \varphi(..) - \partial_x \psi(..)}_{=2\partial_y \varphi = 2\upsilon}] = u - i\upsilon$$

The quantity

$$V(z) \equiv W'(z) = (u - i\upsilon)(x, y) \quad , \quad z = x + i\, y$$

is called the **complex velocity**. The function W is called the **complex (velocity) potential**.

Examples:

1. Uniform stream $W(z) = V_\infty z \quad , \; V_\infty = u_\infty - i\upsilon_\infty$

2. Source/sink at the point $z_0 = x_0 + iy_0$

$$W(z) = \frac{Q}{2\pi} Ln(z - z_0) \quad , \qquad V(z) = \frac{Q}{2\pi} \frac{1}{z - z_0}$$

3. Potential vortex with the center at the point $z_0 = x_0 + iy_0$

$$W(z) = \frac{\Gamma}{2\pi i} Ln(z - z_0) \quad , \qquad V(z) = \frac{\Gamma}{2\pi i} \frac{1}{z - z_0}$$

4. Doublet at the point $z_0 = x_0 + iy_0$

$$W(z) = \frac{D}{z - z_0} \quad , \qquad V(z) = -\frac{D}{(z - z_0)^2}$$

5. Symmetric flow past a circular contour $|z| = a$

$$W(z) = V_\infty z + \frac{\bar{V}_\infty a^2}{z} + \frac{\Gamma}{2\pi i} Ln(z)$$

$$V(z) = V_\infty - \frac{\bar{V}_\infty a^2}{z^2} + \frac{\Gamma}{2\pi i}\frac{1}{z}$$

Note: Natural logarithm of the complex argument is defined as $(z = |z| e^{i \arg z} = re^{i\theta})$

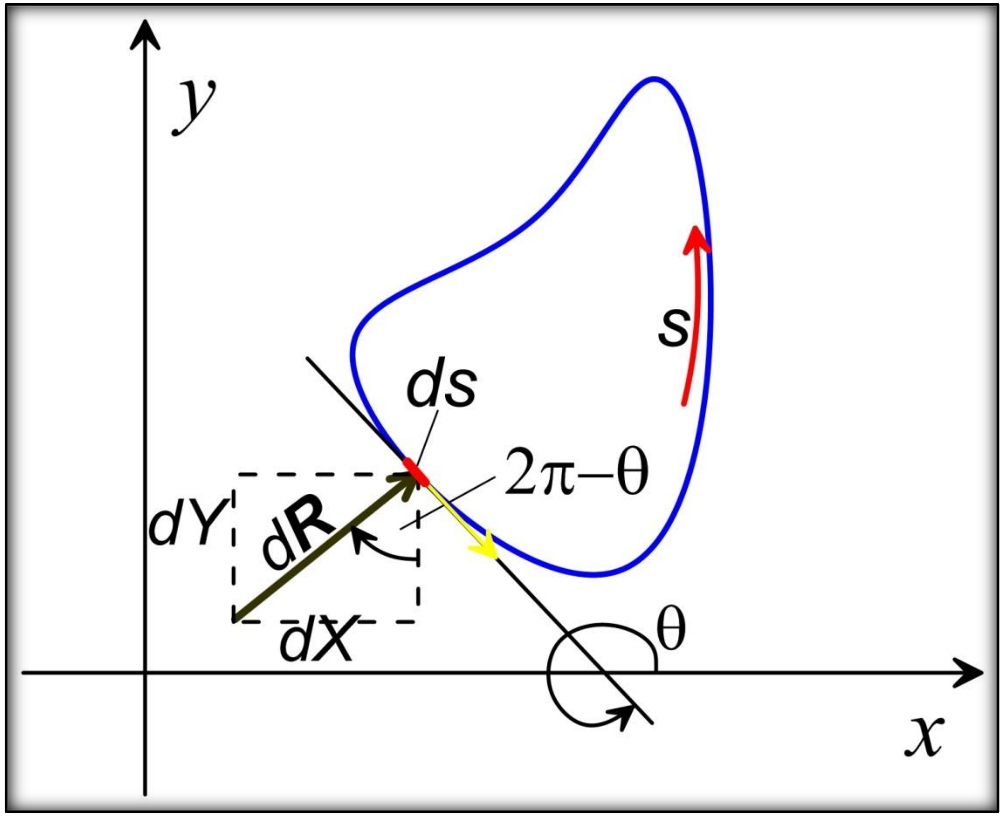
$$Ln(z) = \ln|z| + i \arg z = \ln r + i\theta$$

Note that the imaginary part of the complex logarithm is the multivalued function!

Exercise:

1. Using complex calculus show that the contour $z\bar{z} = a^2$ is one of the streamlines.
2. Write the complex form of the Milne-Thomson theorem (see Lecture 1)

Aerodynamic force and moment. The Blasius formuale



$$dX = dR\sin(2\pi-\theta) = -p(s)\sin\theta ds$$
$$dY = dR\cos(2\pi-\theta) = p(s)\cos\theta ds$$

$$dz = dx + i\,dy = e^{i\theta}ds = (\cos\theta + i\sin\theta)\,ds$$
$$\Downarrow$$
$$dx = \cos\theta\, ds\;,\; dy = \sin\theta\, ds$$

We define

$$d\bar{R} = dX - i\,dY \qquad , \qquad d\boldsymbol{M}_0 = \boldsymbol{r}\times d\boldsymbol{R} = \begin{vmatrix} \boldsymbol{e}_x & \boldsymbol{e}_y & \boldsymbol{e}_z \\ x & y & 0 \\ dX & dY & 0 \end{vmatrix} = (x dY - y dX)\boldsymbol{e}_z$$

$$d\bar{R} = -p\sin\theta ds - i\,p\cos\theta ds = -i\,p(\cos\theta - i\sin\theta)ds = -i\,pe^{-i\theta}ds = -i\,p(z)d\,\bar{z}$$

$$dM_0 = xp\underbrace{\cos\theta ds}_{dx} + yp\underbrace{\sin\theta ds}_{dy} = p(xdx + ydy) = p\,\mathfrak{Re}(zd\,\bar{z})$$

From the Bernoulli Equation we have $\quad p_\infty + \frac{1}{2}\rho(u_\infty^2 + \upsilon_\infty^2) = p(z) + \frac{1}{2}\rho|V(z)|^2$

Hence $\quad p(z) = B - \frac{1}{2}\rho|V(z)|^2 = B - \frac{1}{2}\rho|W'(z)|^2$

Note that **at the body contour only tangent velocity exists**, so it is identical to the total velocity magnitude. Let's calculate the following complex value …

$$W'(z)e^{i\theta} = (u - i\upsilon)(\cos\theta + i\sin\theta) = u\cos\theta + \upsilon\sin\theta + i(\underbrace{u\sin\theta - \upsilon\cos\theta}_{=\vec{V}\cdot\vec{n}=0}) = u\cos\theta + \upsilon\sin\theta$$

In the above, we have used the fact that the external normal vector is equal

$$\boldsymbol{n} = [-\sin\theta, \cos\theta]$$

We conclude that the value $W'(z)e^{i\theta}$ computed at the body contour is a real value and

$$\left|W'(z)e^{i\theta}\right| = \left|W'(z)\right|$$

Hence, the pressure at the contour can be expressed as follows

$$p(z) = B - \tfrac{1}{2}\rho\,[W'(z)]^2\,e^{2i\theta}$$

Aerodynamic "conjugate force" and moment are expressed by the following contour integrals

$$\bar{R} = -\oint i\,p(z)\,d\bar{z} \quad , \quad M_0 = -\mathfrak{Re}\oint p(z)z\,d\bar{z}$$

We insert pressure from the Bernoulli equation. Then, the value of the force is

$$\bar{R} = -i\oint\{B - \tfrac{1}{2}\rho[W'(z)]^2 e^{2i\theta}\}d\bar{z} = -iB\underbrace{\oint d\bar{z}}_{0} + \tfrac{1}{2}i\rho\oint[W'(z)]^2\underbrace{e^{2i\theta}e^{-i\theta}ds}_{e^{i\theta}ds=dz} = \frac{i\rho}{2}\oint[W'(z)]^2 dz$$

This way, we have obtained the **1st Blasius Formula**

$$\boxed{X - iY = i\frac{\rho}{2}\oint[W'(z)]^2 dz}$$

Next, we calculate the aerodynamic moment …

$$M_0 = \mathfrak{Re}\oint\{B - \tfrac{1}{2}\rho[W'(z)]^2 e^{2i\theta}\} z d\bar{z} = B\underbrace{\mathfrak{Re}\oint z d\bar{z}}_{0} - \tfrac{\rho}{2}\mathfrak{Re}\oint [W'(z)]^2 z \underbrace{e^{2i\theta} e^{-i\theta} ds}_{e^{i\theta}ds = dz} =$$

$$= -\tfrac{\rho}{2}\mathfrak{Re}\oint [W'(z)]^2 z dz$$

We have obtained the **2nd Blasius Formula**

$$\boxed{M_0 = -\tfrac{\rho}{2}\mathfrak{Re}\oint [W'(z)]^2 z dz}$$

An interesting (and practical) property of complex contour integration is that changing the integration contour does not affect the value of the integral as long as the set of integrand's singularities enclosed by the contour is not changed. This property allows for easier evaluation of complex contour integrals by choosing more convenient integration contours.

If all singularities in the flow are "hidden" inside the body interior then the force and moment can be computed by choosing other (perhaps more convenient) contour. Such contour could be the circle K_∞ with the large radius R_∞, surrounding the body.

If all singularities (point-localized or spatially distributed) are confined inside the body then – from the large distance – they nearly "look like" concentrated at the origin. The far-field complex velocity can be written in the following form

$$W'(z) = V_\infty + \frac{Q_{total}}{2\pi}\frac{1}{z} + \frac{\Gamma_{total}}{2\pi i}\frac{1}{z} + o(z^{-1})$$

where Q_{total} and Γ_{total} denote – respectively – total flow rate of sources/sinks and total charge of the circulation in the flow. The last symbol stands for all terms which diminish with the distance faster than $1/|z|$.

Assume that $Q_{total} = 0$ and calculate the square of the complex velocity $W'(z)$ …

$$[W'(z)]^2 = V_\infty^2 + \frac{V_\infty \Gamma_{total}}{\pi i}\frac{1}{z} + o(z^{-1})$$

Form the 1st Blasius formula we get the expression for the force

$$X - iY = i\frac{\rho}{2}\oint [W'(z)]^2 dz = i\frac{\rho}{2}\frac{V_\infty \Gamma}{\pi i}\underbrace{\oint \frac{dz}{z}}_{=2\pi i} = i\rho\Gamma V_\infty$$

Since $V_\infty = u_\infty - i\upsilon_\infty$ the Cartesian components of the force vector are

$$X = \rho\Gamma\upsilon_\infty \quad , \quad Y = -\rho\Gamma u_\infty$$

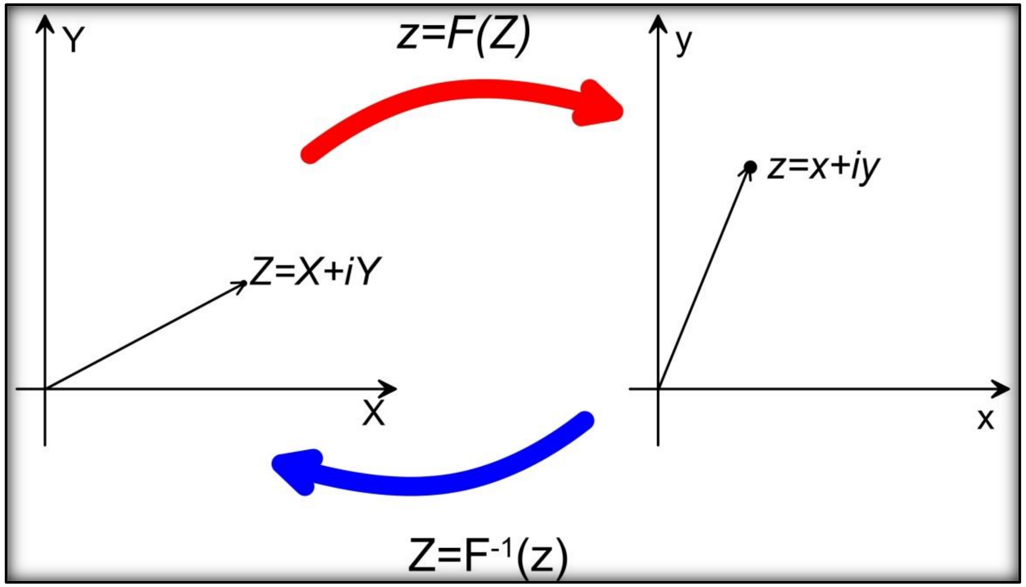
Note that this vector is oriented perpendicularly with respect to the free-stream velocity direction. Indeed, we have

$$\vec{R}\cdot\vec{V}_\infty = Xu_\infty + Y\upsilon_\infty = \rho\Gamma(\upsilon_\infty u_\infty - u_\infty \upsilon_\infty) = 0$$

Hence, there is no aerodynamic drag (d'Alembert paradox). The absolute value of the lift force is equal to the length of the vector $\vec{R}$, i.e.,

$$L = \rho|\Gamma|\sqrt{u_\infty^2 + \upsilon_\infty^2}$$

Application of conformal mappings



Conformal mapping $$z = F(Z)$$

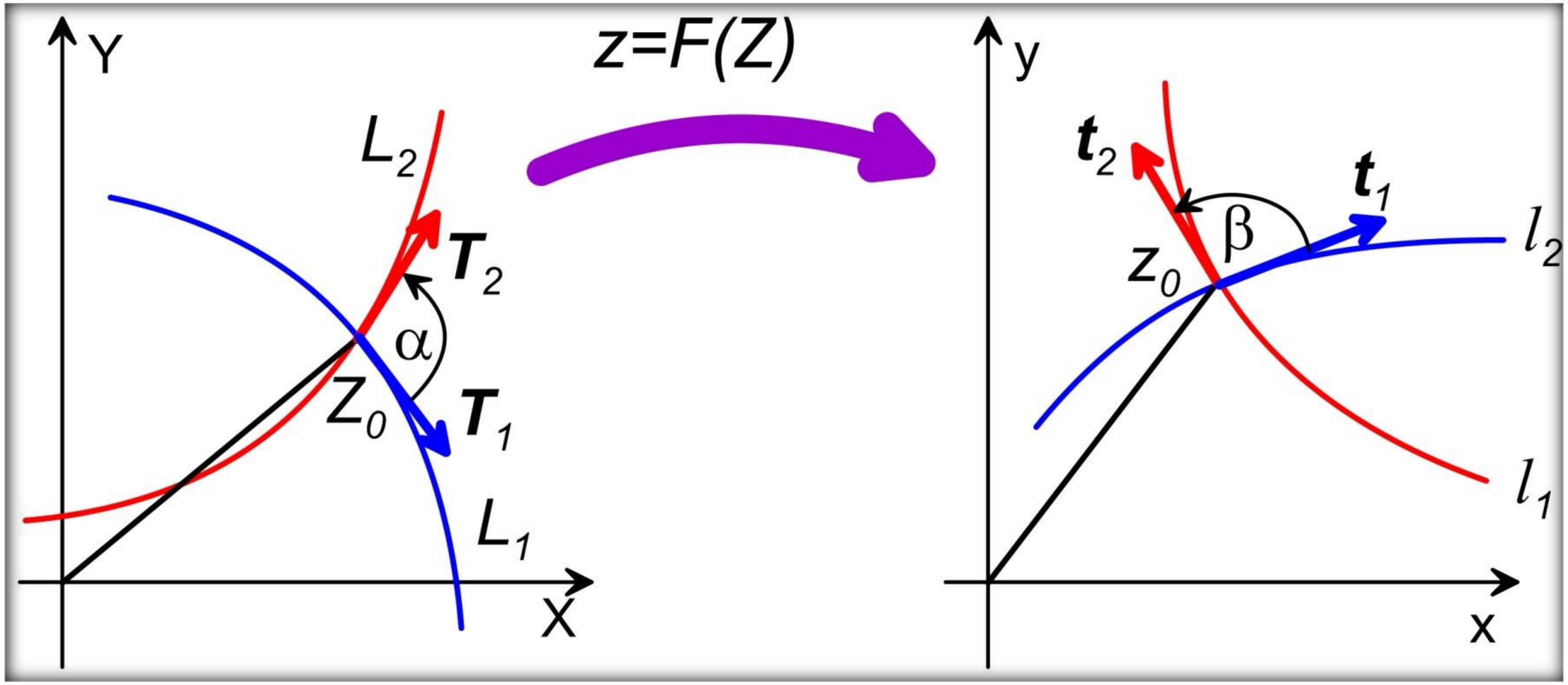
In a real notation:

$$x = f(X,Y)\ ,\ y = g(X,Y)$$

$$f = \mathfrak{Re}F\ ,\ g = \mathfrak{Im}F$$

Functions f and g satisfy the Cauchy-Riemann conditions:

$$\partial_X f = \partial_Y g \quad , \quad \partial_Y f = -\partial_X g$$



What is an actual meaning of "conformal"? Consider the transformation of two smooth lines (see Figure). We will show that the angle between these lines at their intersection point is preserved if the mapping is conformal (i.e., $\alpha = \beta$) and its derivative $0 < |F'(Z_0)| < \infty$ (meaning that the point Z_0 is a regular point of this mapping).

First, we need to know **how a tangent vector to a given line changes during a differentiable mapping of the plane**.

In a general case, tangent vectors are transformed accordingly to the linear transformation defined by the local Jacobi matrix of the mapping. In our case, this matrix can be written as

$$J_F = \begin{bmatrix} \partial_X f & \partial_Y f \\ \partial_X g & \partial_Y g \end{bmatrix} \underset{C-R}{=} \begin{bmatrix} \partial_X f & \partial_Y f \\ -\partial_Y f & \partial_X f \end{bmatrix} \underset{C-R}{=} \begin{bmatrix} \partial_Y g & -\partial_X g \\ \partial_X g & \partial_Y g \end{bmatrix}$$

Hence, the transformation formulae for the tangent vector are

$$t_x = \partial_X f \, T_X + \partial_Y f \, T_Y \quad , \quad t_y = -\partial_Y f \, T_X + \partial_X f \, T_Y$$

Let's define the complex tangent vectors to the original and transformed lines

$$T = T_X + iT_y \quad , \quad t = t_x + it_y$$

Since the transformation F is conformal, then its derivative can be written as

$$F'(Z) = \partial_X f(X,Y) - i\partial_Y f(X,Y)$$

Note that the transformation formulae given above can be obtained by the following complex multiplication

$$t(z) = F'(Z)T(Z) \quad , \quad z = F(Z)$$

In the case depicted in the Figure, we can write

$$t_1(z_0) = F'(Z_0)T_1(Z_0) \quad , \quad t_2(z_0) = F'(Z_0)T_2(Z_0)$$

Next, let us note that the inner (scalar) product of two vectors can be obtained by the following multiplication of their complex representations

$$t_1\bar{t}_2 = [(t_1)_x + i(t_1)_y][(t_2)_x - i(t_1)_x] = (t_1)_x(t_2)_x + (t_1)_y(t_2)_y - i[(t_1)_x(t_1)_x - (t_1)_y(t_2)_x] =$$
$$= \boldsymbol{t}_1 \cdot \boldsymbol{t}_2 - i\,(\boldsymbol{t}_1 \times \boldsymbol{t}_2)_z$$

meaning that $\boldsymbol{t}_1 \cdot \boldsymbol{t}_2 = \mathfrak{Re}(t_1\bar{t}_2)$.

Now we are ready to show that the angle between the tangent vectors remains unchanged. To this aim, we calculate the product

$$t_1(z_0)\bar{t}_2(z_0) = F'(Z_0)T_1(Z_0)\bar{F}'(Z_0)\bar{T}_2(Z_0) = \left|F'(Z_0)\right|^2 T_1(Z_0)\bar{T}_2(Z_0)$$

The vector's lengths are transformed as follows:

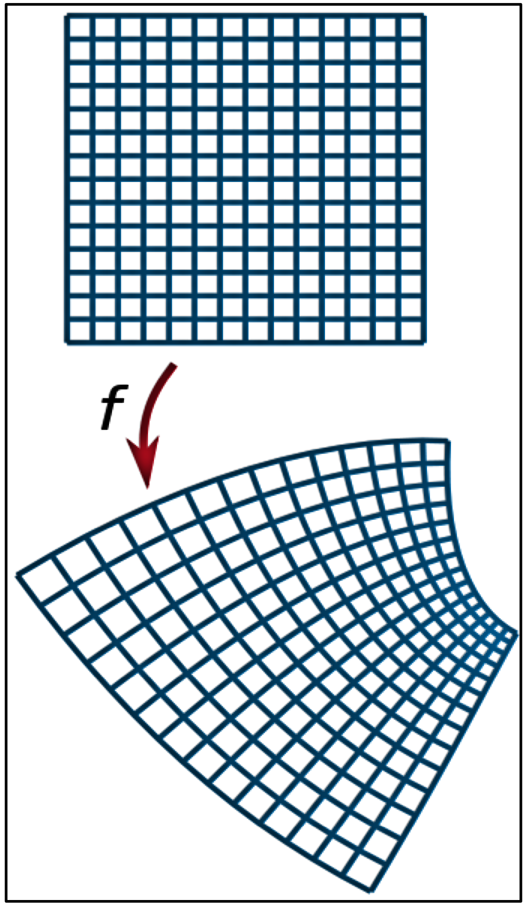
$$\left\|\boldsymbol{t}_1\right\|_2 = \sqrt{\boldsymbol{t}_1 \cdot \boldsymbol{t}_1} = \sqrt{t_1\bar{t}_1}(z_0) = \sqrt{\left|F'(Z_0)\right|^2 T_1\bar{T}_1} = \left|F'(Z_0)\right|\sqrt{T_1\bar{T}_1} = \left|F'(Z_0)\right|\left\|\boldsymbol{T}_1\right\|_2$$

By analogy $\quad \left\|\boldsymbol{t}_2\right\|_2 = \left|F'(Z_0)\right|\left\|\boldsymbol{T}_2\right\|_2$

From the obtained formulae follows that

$$\frac{t_1(z_0)\overline{t}_2(z_0)}{\|\boldsymbol{t}_1\|_2\,\|\boldsymbol{t}_2\|_2} = \frac{|F'(Z_0)|^2\,T_1(Z_0)\overline{T}_2(Z_0)}{|F'(Z_0)|\|\boldsymbol{T}_1\|_2\,|F'(Z_0)|\|\boldsymbol{T}_2\|_2} = \frac{T_1(Z_0)\overline{T}_2(Z_0)}{\|\boldsymbol{T}_1\|_2\,\|\boldsymbol{T}_2\|_2}$$

Which implies identity of the sine and cosine function of the angles α and β. Hence, these angles are the same, which ends the proof.



If an orthogonal grid is defined in the plane (X,Y) then its image in the conformal mapping is also a (curvilinear) orthogonal grid (at all regular point of the mapping). Since orthogonal grids are of particular value, the method of conformal mappings is often used in the grid generation for the use in the Computational Fluid Dynamics (CFD).

A key property of the conformal mappings is that they preserve also the circulation of the transformed potential flow.

Let $W = W(Z)$ be the complex potential of a certain flow in the plane (X, Y). The conformal mapping $z = F(Z)$ transforms this flow into another potential flow in the plane (x, y), with the complex potential $w = w(z)$ such that:

$$W(Z) = w[F(Z)] \quad \text{and} \quad w(z) = W[F^{-1}(z)]$$

It follows that the complex velocity transforms accordingly to the formulae

$$W'(Z) = w[F(Z)]F'(Z) \quad \text{and} \quad w'(z) = W'[F^{-1}(z)](F^{-1})'(z) = \frac{W'[F^{-1}(z)]}{F'[F^{-1}(z)]}$$

Circulation along the line L in the plane (X, Y) can be expressed as

$$\Gamma = \int_L U_X dX + U_Y dY = \Re\mathfrak{e}\int_L (U_X - iU_Y)(dX + idY) = \Re\mathfrak{e}\int_L W'(Z)dZ$$

Note that $$\int_L W'(Z)dZ = \int_L w'[F(Z)]F'(Z)dZ = \int_l w'(z)dz$$

Where the symbol l denotes the image of L in the mapping F. The obtained equality means that the **circulation is preserved in this mapping**.

Example – Joukovski’s mapping

Consider the mapping defined by the formula

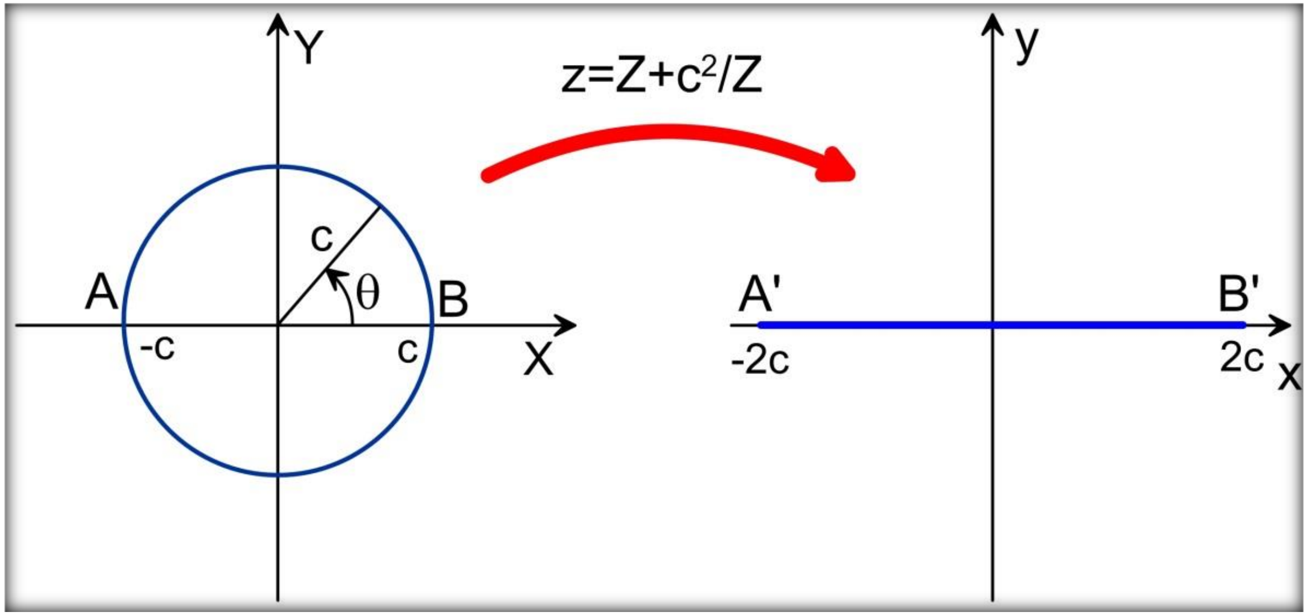
$$z = F(z) = Z + \frac{c^2}{Z} \quad , \quad c \in R \;\; , \;\; c > 0$$

The derivative of this mapping is

$$F'(z) = 1 - \frac{c^2}{Z^2}$$

Singular points (where the derivative vanishes): $(0,0), (\pm c, 0)$.

CASE 1: Joukovski's mapping of the circle $|Z| = c$.



The image of the circle is the horizontal segment („flat plate") described parametrically as

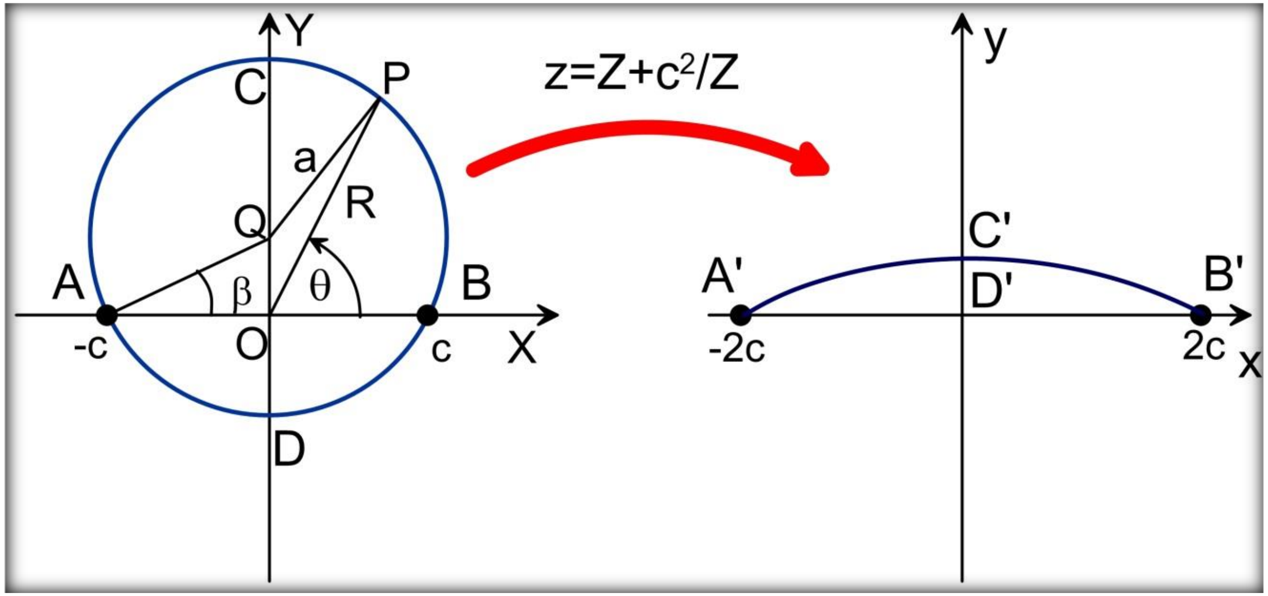
$$z(\theta) = ce^{i\theta} + ce^{-i\theta} = 2c\cos\theta$$

Note that the singular point are located at the circle.

<u>Exercise 1:</u> In the plane *(X,Y)*, construct the potential flow in such a way that stagnation points assume angular locations $\theta = 0$ and $\theta = \pi + 2\,\iota$ (an appropriate choice of free-stream velocity $V_{\infty} = u_{\infty} - i\upsilon_{\infty}$ and the circulation Γ is required). Then, calculate velocity and pressure distributions on the plate. Calculate also the lift force coefficient

$$C_L = \frac{L}{\frac{1}{2}\rho\left|V_{\infty}\right|^2 4c} = \frac{\Gamma}{2c\left|V_{\infty}\right|}$$

CASE 2: Mapping of the circle $|Z - a\sin\beta| = a \quad , \quad a = \dfrac{c}{\cos\beta}$.



This circle is transformed in the arc of the circle described mathematically by the following formula

$$x^2 + (y + 2c\, ctg\, 2\beta)^2 = (2c / \sin 2\beta)^2$$

Detailed derivation – see the handbook „Fluid Mechanics" 5th Ed., by P.K. Kundu, I.M. Cohen and D.R. Dowling.

CASE 3: Mapping of the circle $|Z+\varepsilon c| = c(1+\varepsilon) \equiv a$

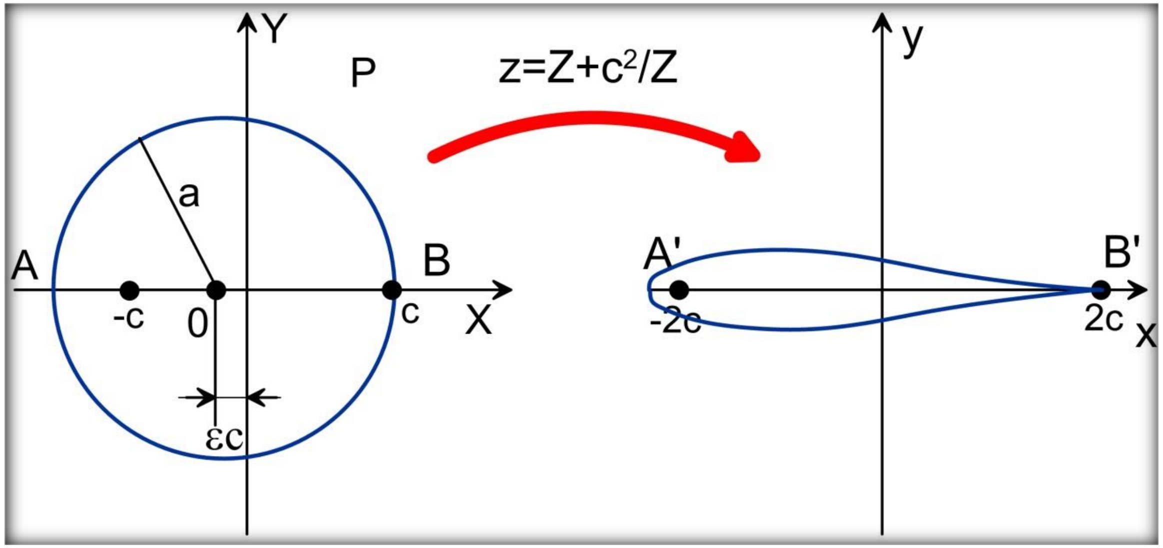
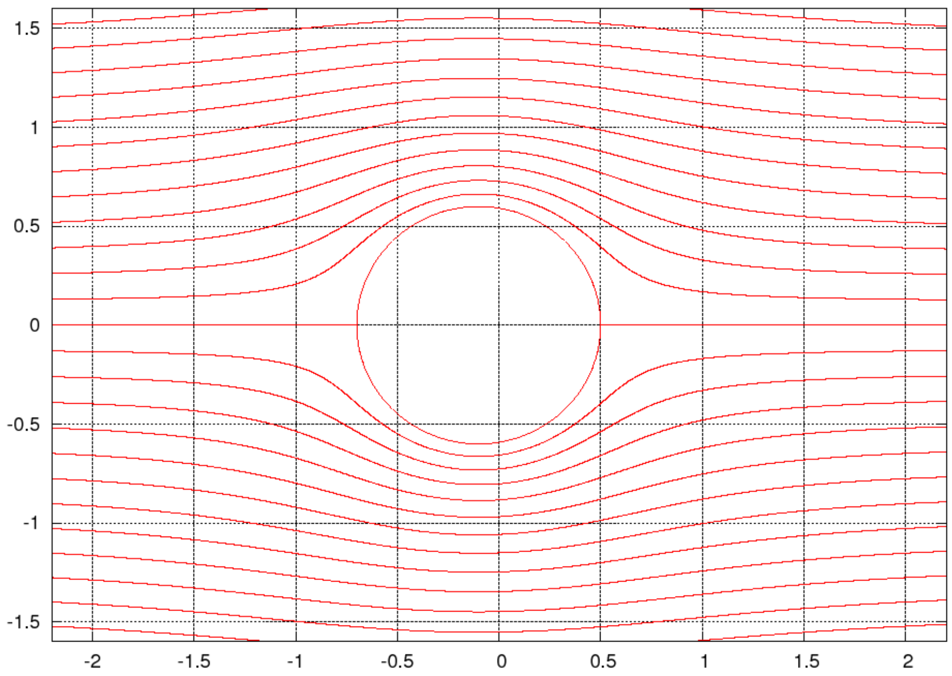


Image – symmetrical Joukovski airfoil (having for small ε, maximal thickness equal $\approx 1.3\,\varepsilon \cdot 4c$ located near the point of ¼ of the chord, i.e., $x_{\max} \approx -c$).

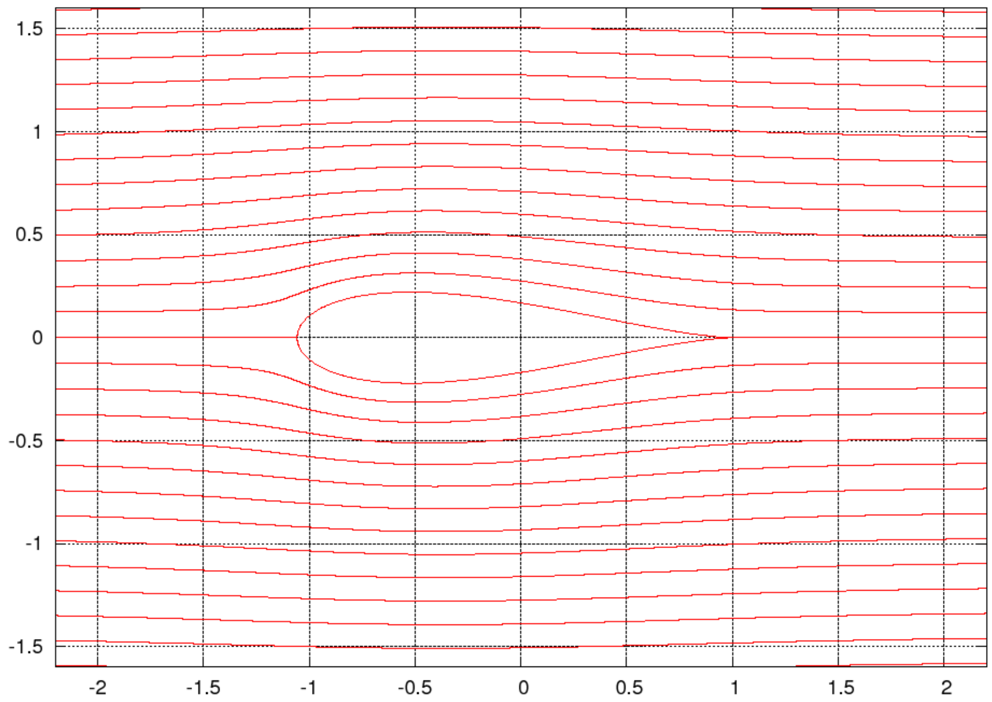
Kutta-Joukovski condition - point B must be a stagnation point of a potential flow past a circular contour in the plane *(X,Y)*. Then, the velocity at the trailing edge point B' is unique and finite (the angle of the trailing edge for this airfoil is equal zero)

Some pictures …

Symmetric flow past a symmetric Joukovski airfoil

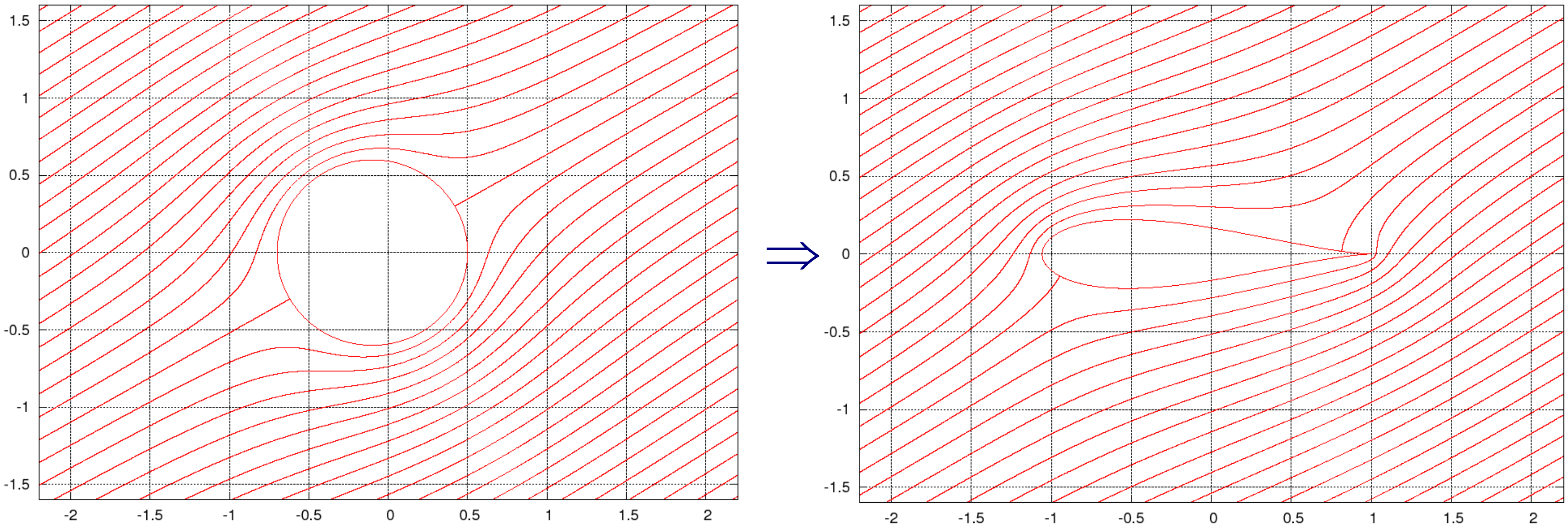


$\Rightarrow$



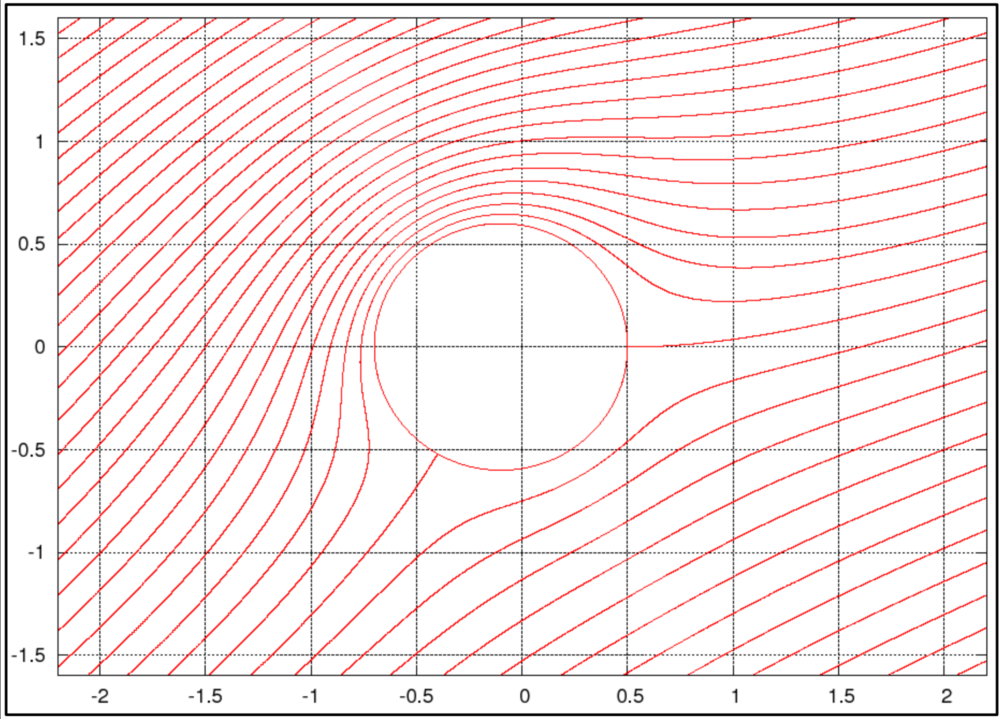
Flow past a symmetric the Kutta-Youkovski airfoil

<u>Flow at nonezero angle of attack of a symmetric Joukovski airfoil</u>

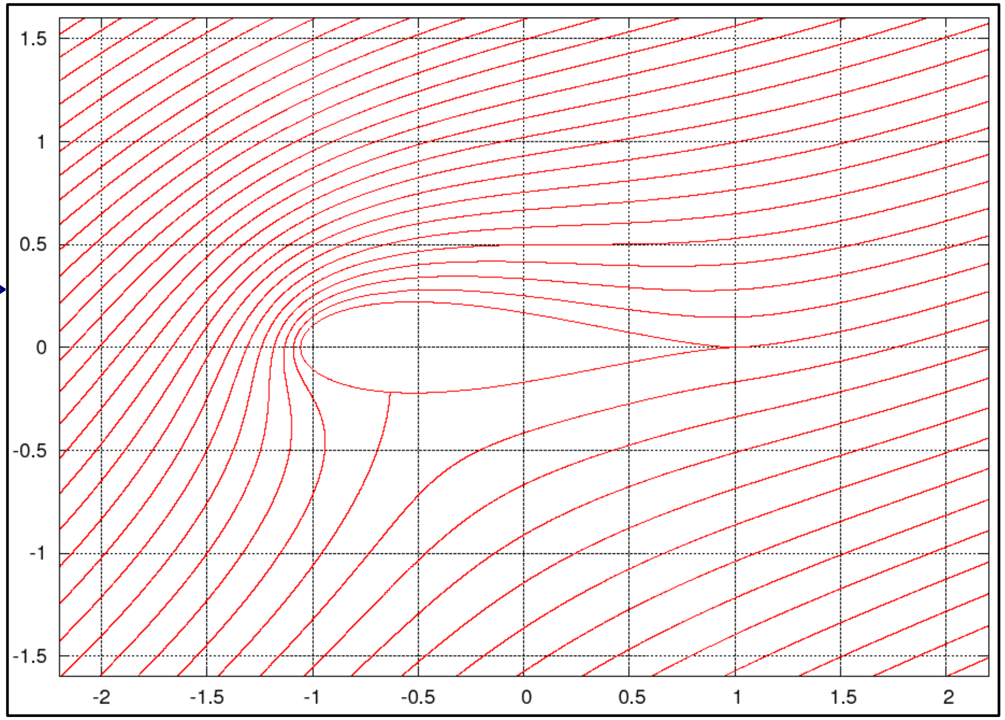


Flow with zero circulation and lift force – the Kutta-Youkovski condition is violated

Flow at nonezero angle of attack of a symmetric Joukovski airfoil (cont.)

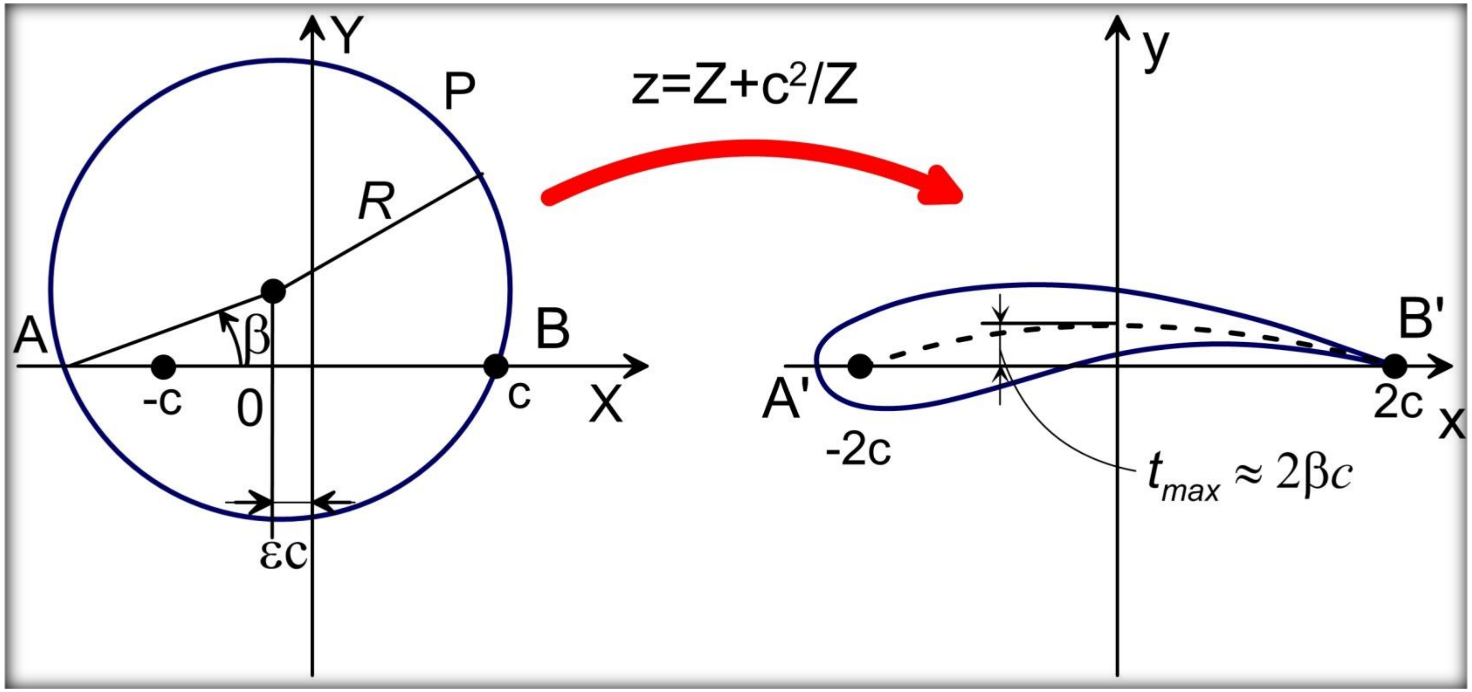


$\Rightarrow$



The circulation Γ has been chosen so that the Kutta-Joukovsky condition is satisfied. The list force equal to $L = \rho \left| \Gamma \right| V_\infty$ is generated in this case.

CASE 4: Mapping of the circle $\left|Z-(-\varepsilon c+ia\,\mathrm{tg}\beta)\right|=(a/\cos\beta)^2$, $a=c(1+\varepsilon)$

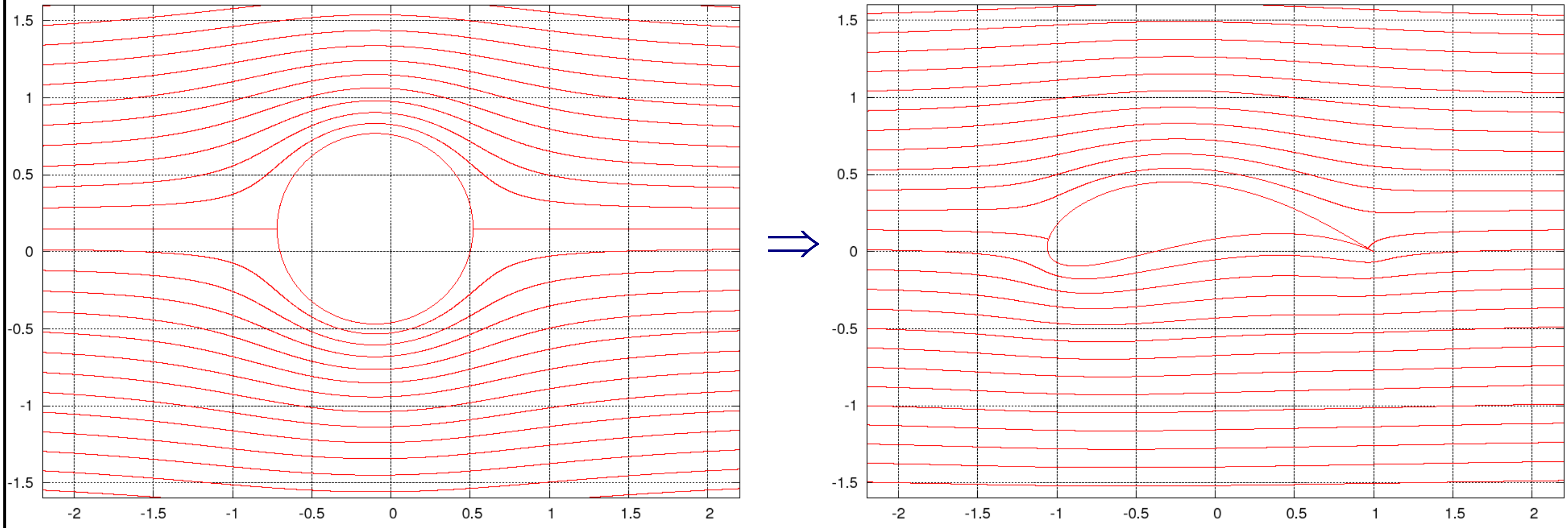


For $\varepsilon \ll 1$ and small angles β we have:

- Chord of the airfoil $\approx 4c$,
- Maximal deflection (camber of the airfoil) $\approx 2c\beta$,
- Maximal thickness $t_{\max}/4c \approx 1.3\varepsilon$ (located near ¼ of the chord from the airfoil's nose, i.e., at $x \approx -c$)

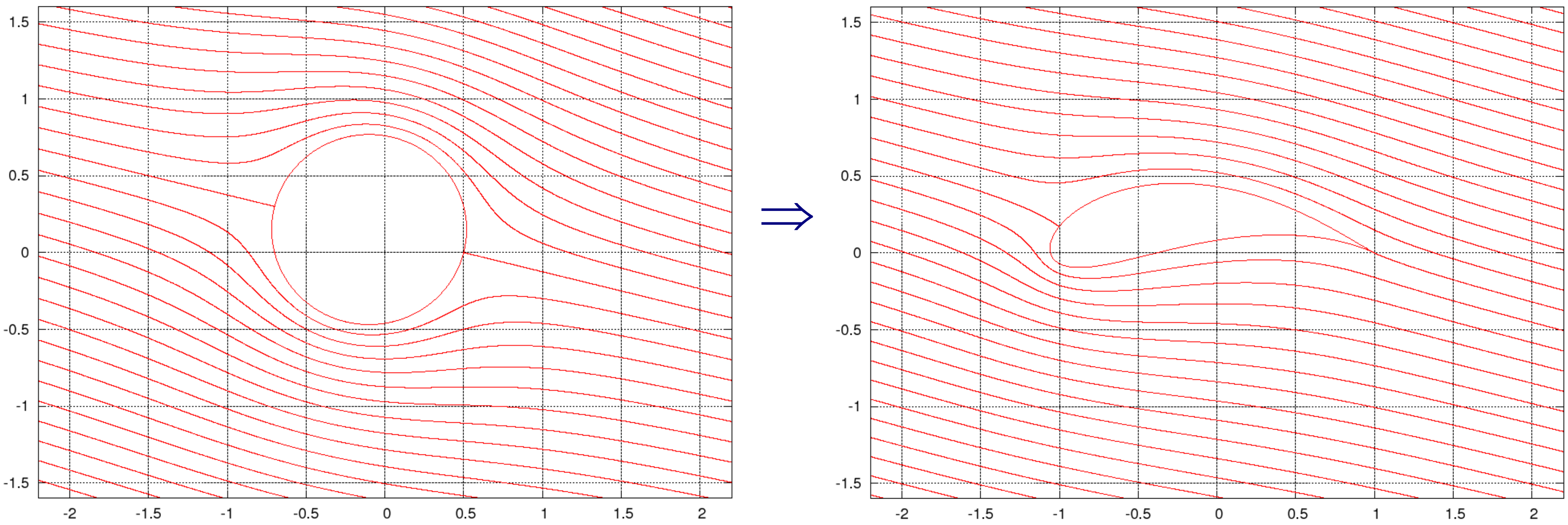
Again, some pictures …

Flow past a non-symmetric Joukovsky airfoils at zero angle of attack



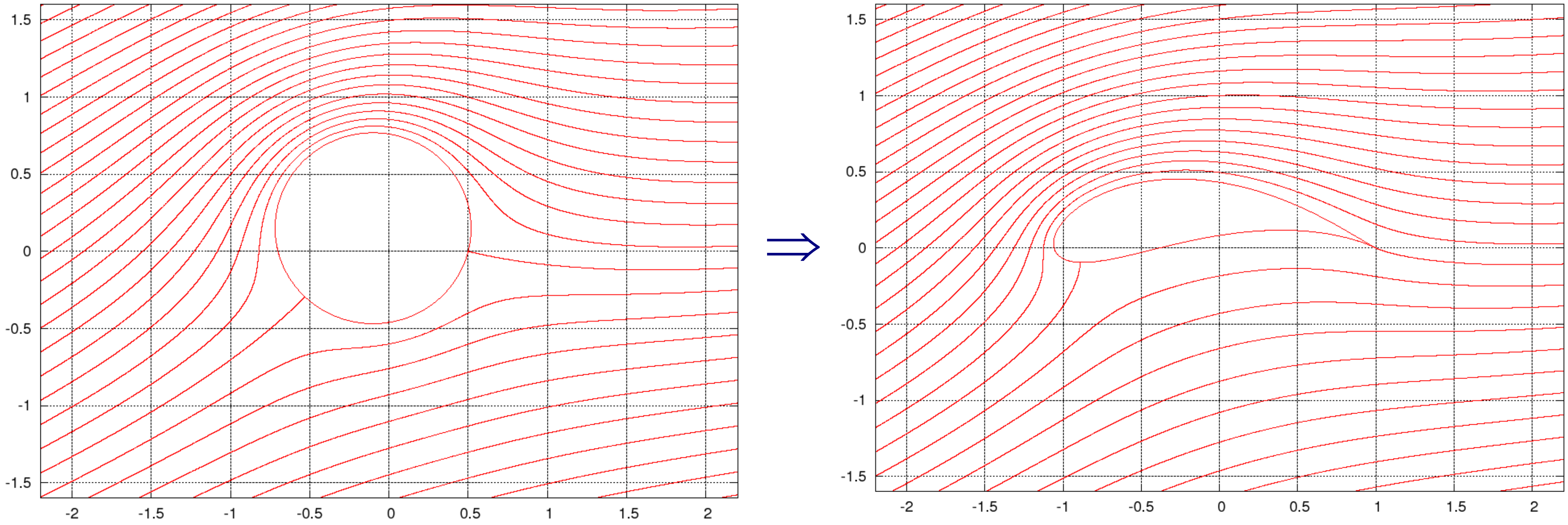
Flow with zero circulation and lift – The Kutta-Joukovski condition is violated. Hence, the non-symmetric Joukovski airfoil must generate lift at the zero angle of attack!

Flow past a non-symmetric Joukovski airfoil at nonzero angle of attack



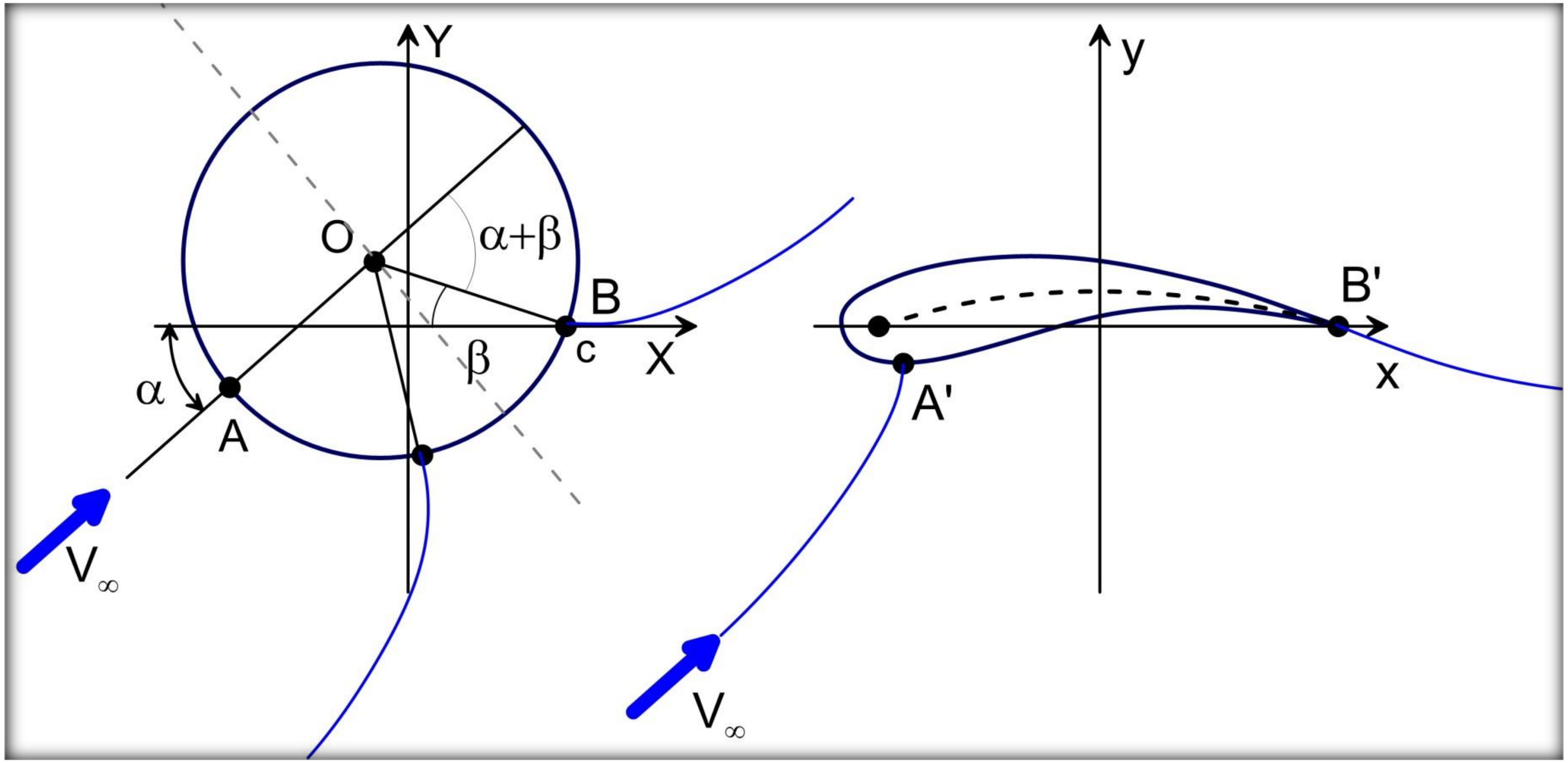
Flow past a non-symmetric airfoil at the angle of attack equal $\alpha \approx\sim -13.7^{0}$. The Kutta-Joukovski condition is satisfied. The circulation of the flow is zero, hence the airfoil does not generate the lift force.

Flow past a non-symmetric Joukovski airfoil at nonzero angle of attack (cont.)



Correctly constructed flow past a non-symmetric airfoil with the angle of attack $\alpha = 30^0$.

Lift force on the Joukovski airfoil



The Kutta-Joukovski condition requires that the point B is the stagnation point of the flow in the *(X,Y)* plane. The circulation in this flow must be equal

$$\Gamma = 4\pi V_{\infty} R \sin(\alpha+\beta) \qquad (R \text{ - radius of the circle})$$

The lift force coefficient is equal ($chord \approx 4R$)

$$C_L = \frac{L}{\frac{1}{2}\rho V_\infty^2 \cdot chord} = \frac{2\Gamma}{V_\infty \cdot chord} \approx 2\pi \sin(\alpha+\beta) \approx 2\pi(\alpha+\beta)$$

Conclusions:

- For small camber values (the relative camber is equal $\frac{1}{2}\beta$) the derivative $\dfrac{dC_L}{d\alpha} = 2\pi$,
- The deflection of the airfoil lead to the vertical shift of the characteristic $C_L = C_L(\alpha)$,
- The zero lift conditions correspond to the negative angle of attack $\alpha = -\beta$

Exercise:

1. Perform the analysis of the flow past a symmetric Joukovski airfoil and show that for small values of the parameter ε one obtains

$$\frac{1}{2\pi}\frac{dC_L}{d\alpha} \approx 1+\varepsilon$$

What can you say about influence of the airfoil thickness on the slope of the line $C_L = C_L(\alpha)$?

2. Perform more detailed analysis of influence of the airfoil camber on the slope $\frac{dC_L}{d\alpha}$ (for simplicity, assume that the thickness of the airfoil is zero, i.e., $\varepsilon = 0$) and show that for slightly deflected airfoils the following approximate relation holds

$$\frac{dC_L}{d\alpha} \approx 2\pi(1 + 2\bar{f}^2)$$

In the above, the symbol $\bar{f}$ denotes the camber ratio (the ratio between the maximal deflection of the mean camber line and the chord of the airfoil).