PROGRAM ROZWOJOWY
POLITECHNIKI WARSZAWSKIEJ

LECTURE 7
STRESS IN FLUIDS. CONSTITUTIVE RELATION AND NEWTONIAN FLUID.

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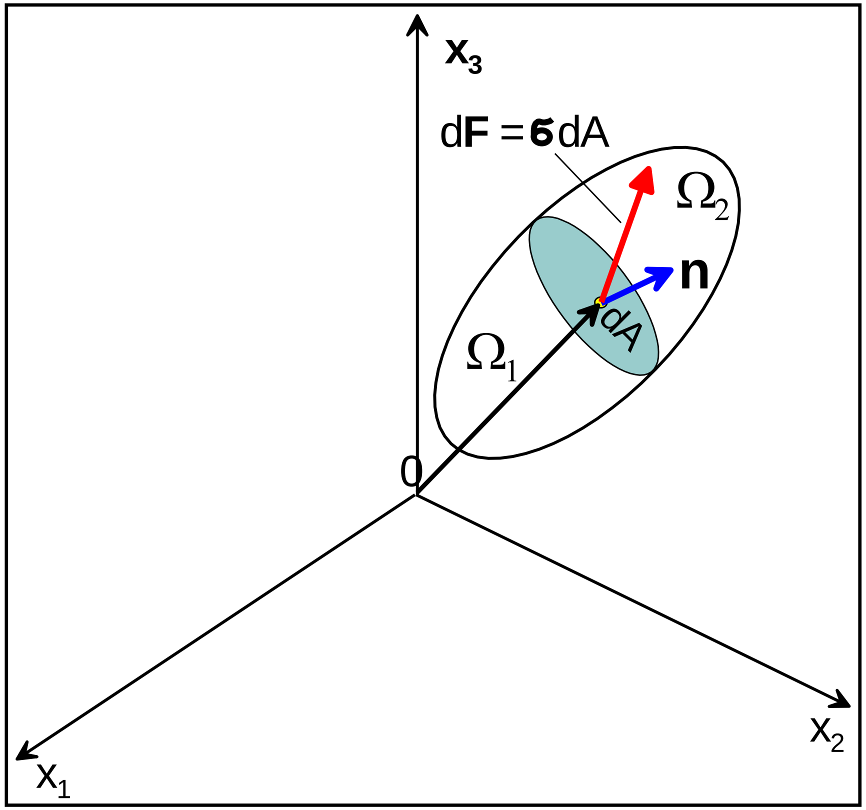
MATHEMATICAL MODEL FOR INTERNAL FORCES IN FLUIDS. STRESS TENSOR.

According to **Cauchy hypothesis**, the **surface** (or interface) **reaction force** acting between two adjacent portions of a fluid can be characterized by its surface vector density called the **stress**.

Thus, for an infinitesimal piece dA of the interface $\partial\Omega_1 \cap \partial\Omega_2$, we have (see figure)

$$d\boldsymbol{F} = \boldsymbol{\sigma}\, dA \quad \text{and} \quad \boldsymbol{F}_{\Omega_2 \to \Omega_1} = \int_{\partial\Omega_1 \cap \partial\Omega_2} \boldsymbol{\sigma}\, dA$$

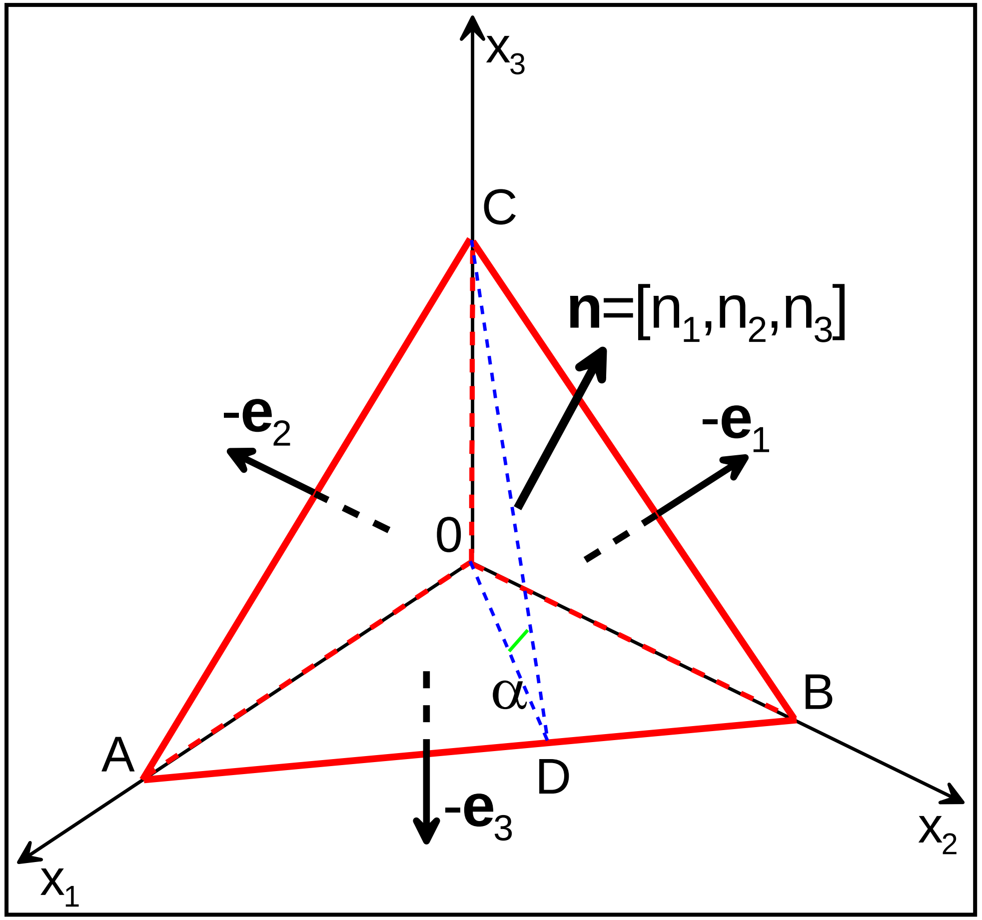


The stress vector $\boldsymbol{\sigma}$ **is not a vector field**: it depends not only on the point $\boldsymbol{x}$ but also on the **orientation** of the surface element dA or – equivalently – on the vector $\boldsymbol{n}$ normal (perpendicular) to dA at the point $\boldsymbol{x}$.

From the 3rd **principle of Newton's dynamics** (action-reaction principle) we have

$$\boldsymbol{\sigma}(\boldsymbol{x},\boldsymbol{n}) = -\boldsymbol{\sigma}(\boldsymbol{x},-\boldsymbol{n})$$

We will show that the value of stress vector $\boldsymbol{\sigma}$ can be expressed by means of a tensor field. To this aim, consider a portion of fluid in the form of small **tetrahedron** as depicted in the figure below.



The **front face** ΔABC belongs to the plane which is describes by the following formula

$$(\boldsymbol{n},\boldsymbol{x}) \equiv n_j x_j = h \quad , \quad h \text{ – small}$$

number.

The areas of the faces of the tetrahedron are S, S_1, S_2 and S_3 for ΔABC, ΔOBC, ΔAOC and ΔABO, respectively.

Obviously, $S \sim O(h^2)$.

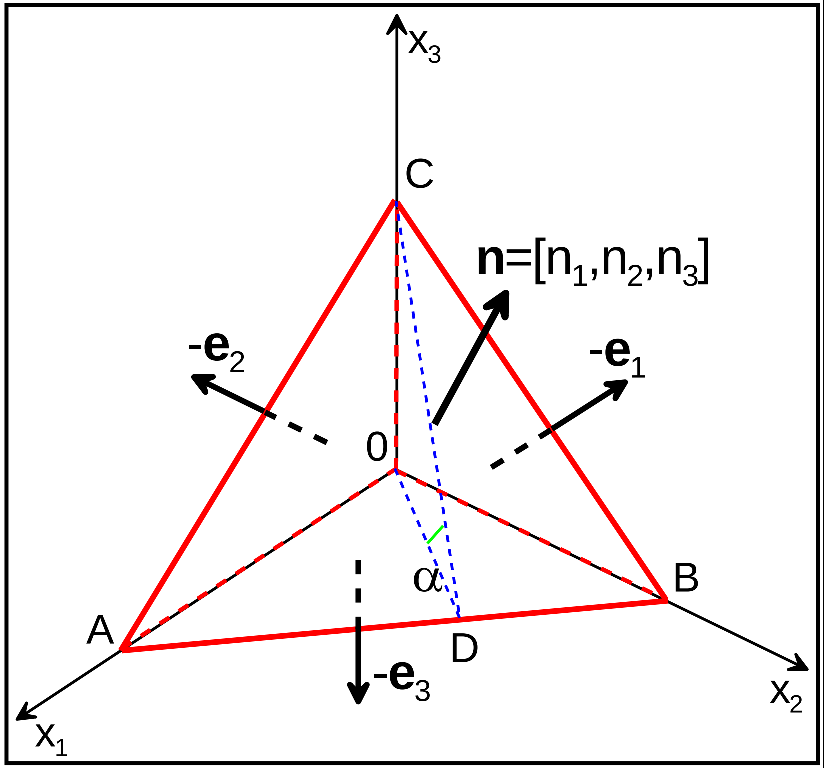
Moreover, the **following relations hold** for j = 1,2,3:

$$S_j = S\cos[\sphericalangle(\boldsymbol{n},\boldsymbol{e}_j)] = S\cdot(\boldsymbol{n},\boldsymbol{e}_j) = S\, n_j$$

The volume of the tetrahedron is $V_\Omega \sim O(h^3)$.

The **momentum principle** for the fluid contained inside the tetrahedron volume reads

$$\underbrace{\frac{d}{dt}\int_{\Omega}\rho \boldsymbol{v}\, d\boldsymbol{x}}_{\text{time derivative of the momentum}} = \underbrace{\boldsymbol{F}_{vol}}_{\text{total volume force}} + \underbrace{\boldsymbol{F}_{surf}}_{\text{total surface force}}$$



We need to calculate the **total surface force** $\boldsymbol{F}_{surf}$.

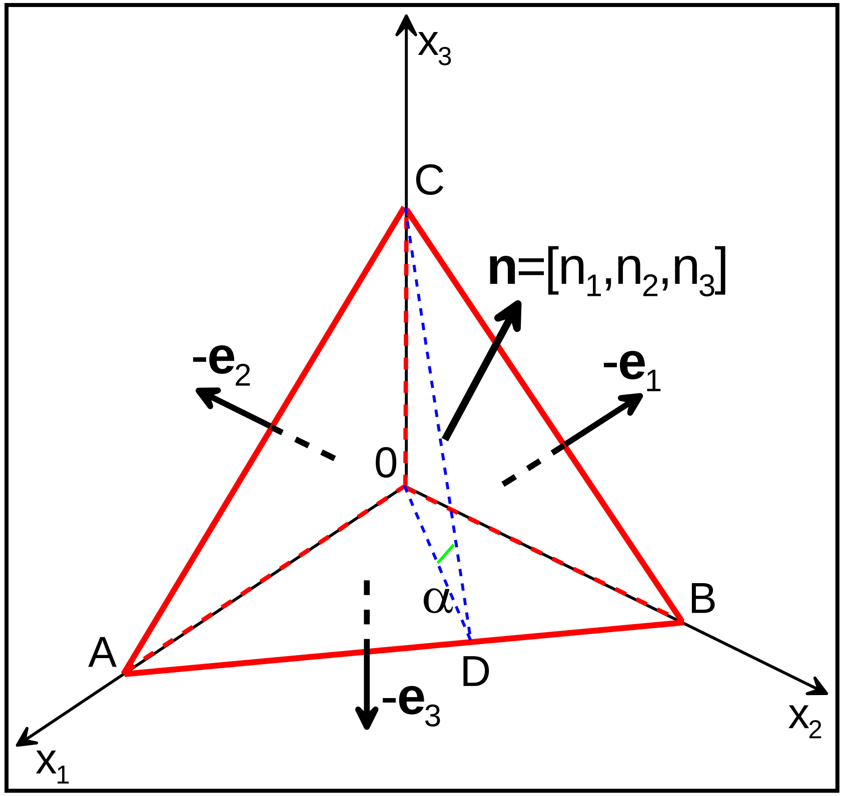
We have:

on ΔABC:
$$\boldsymbol{\sigma}(\boldsymbol{x},\boldsymbol{n}) = \boldsymbol{\sigma}(\boldsymbol{0},\boldsymbol{n}) + O(h)$$
$$\boldsymbol{F}_{surf}^{\Delta ABC} = S\,\boldsymbol{\sigma}(\boldsymbol{0},\boldsymbol{n}) + O(h^3)$$

on ΔOBC:
$$\boldsymbol{\sigma}(\boldsymbol{x},-\boldsymbol{e}_1) = -\boldsymbol{\sigma}(\boldsymbol{x},\boldsymbol{e}_1) = -\boldsymbol{\sigma}(\boldsymbol{0},\boldsymbol{e}_1) + O(h)$$
$$\boldsymbol{F}_{surf}^{\Delta OBC} = -S_1\,\boldsymbol{\sigma}(\boldsymbol{0},\boldsymbol{e}_1) + O(h^3) = -S\,n_1\,\boldsymbol{\sigma}(\boldsymbol{0},\boldsymbol{e}_1) + O(h^3)$$

on ΔAOC:
$$\boldsymbol{\sigma}(\boldsymbol{x},-\boldsymbol{e}_2) = -\boldsymbol{\sigma}(\boldsymbol{x},\boldsymbol{e}_2) = -\boldsymbol{\sigma}(\boldsymbol{0},\boldsymbol{e}_2) + O(h)$$
$$\boldsymbol{F}_{surf}^{\Delta AOC} = -S_2\,\boldsymbol{\sigma}(\boldsymbol{0},\boldsymbol{e}_2) + O(h^3) = -S\,n_2\,\boldsymbol{\sigma}(\boldsymbol{0},\boldsymbol{e}_2) + O(h^3)$$

on ΔAOB:
$$\boldsymbol{\sigma}(\boldsymbol{x},-\boldsymbol{e}_3) = -\boldsymbol{\sigma}(\boldsymbol{x},\boldsymbol{e}_3) = -\boldsymbol{\sigma}(\boldsymbol{0},\boldsymbol{e}_3) + O(h)$$
$$\boldsymbol{F}_{surf}^{\Delta AOB} = -S_3\,\boldsymbol{\sigma}(\boldsymbol{0},\boldsymbol{e}_3) + O(h^3) = -S\,n_3\,\boldsymbol{\sigma}(\boldsymbol{0},\boldsymbol{e}_3) + O(h^3)$$



When the above formulas are inserted to the equation of motion we get

$$\underbrace{\frac{d}{dt}\int_{\Omega}\rho\boldsymbol{\upsilon}\,d\boldsymbol{x}}_{O(h^3)} = \underbrace{\boldsymbol{F}_{vol}}_{O(h^3)} + \underbrace{S[\boldsymbol{\sigma}(\boldsymbol{0},\boldsymbol{n}) - n_j\,\boldsymbol{\sigma}(\boldsymbol{0},\boldsymbol{e}_j)]}_{O(h^2)} + O(h^3)$$

When $h \to 0$ the above equation reduces to

$$\boldsymbol{\sigma}(\boldsymbol{0},\boldsymbol{n}) - n_j\,\boldsymbol{\sigma}(\boldsymbol{0},\boldsymbol{e}_j) = 0$$

In general case, the vertex O is not the origin of the coordinate system and the field of stress is time dependent.

Hence, we can write
$$\boldsymbol{\sigma}(t,\boldsymbol{x},\boldsymbol{n}) = n_j\,\boldsymbol{\sigma}(t,\boldsymbol{x},\boldsymbol{e}_j)$$

In the planes oriented perpendicularly to the vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ or $\boldsymbol{e}_3$, the stress vector can be written as

$$\boldsymbol{\sigma}(t,\boldsymbol{x},\boldsymbol{e}_j) = \sigma_{ij}(t,\boldsymbol{x})\,\boldsymbol{e}_i$$

Thus, the **general formula for the stress vector** takes the form

$$\boldsymbol{\sigma}(t,\boldsymbol{x},\boldsymbol{n}) = n_j\,\boldsymbol{\sigma}(t,\boldsymbol{x},\boldsymbol{e}_j) = \sigma_{ij}(t,\boldsymbol{x})\,n_j\,\boldsymbol{e}_i \equiv \boldsymbol{\Xi}(t,\boldsymbol{x})\,\boldsymbol{n}$$

We have introduced the matrix $\boldsymbol{\mathit{\Xi}}$ which represents the **stress tensor**. The **stress tensor** depends on time and space coordinates, i.e., we actually have the **tensor field**.

Note that the **stress tensor** $\mathit{\Xi}$ can be viewed as the linear mapping (parameterized by t and $\boldsymbol{x}$) between vectors in 3-dimensional Euclidean space

$$\mathit{\Xi} : E^3 \ni \boldsymbol{w} = w_j \boldsymbol{e}_j \;\mapsto\; \sigma_{ij} w_j \boldsymbol{e}_i \in E^3$$

In particular

$$\boldsymbol{\mathit{\Xi}}(\boldsymbol{n}) \equiv \boldsymbol{\mathit{\Xi}}\boldsymbol{n} = \sigma_{ij} n_j \boldsymbol{e}_i = \boldsymbol{\sigma}$$

i.e., **the action of** $\mathit{\Xi}$ **on the normal vector** $\boldsymbol{n}$ **at some point of the fluid surface yields the stress vector** $\boldsymbol{\sigma}$ **at this point**.

It is often necessary to calculate the **normal and tangent stress components at the point of some surface**.

Normal component is equal

$$\boldsymbol{\sigma}_n = (\boldsymbol{n} \cdot \boldsymbol{\mathit{\Xi}}\,\boldsymbol{n})\boldsymbol{n} \equiv \underbrace{(\boldsymbol{n}, \boldsymbol{\mathit{\Xi}}\,\boldsymbol{n})}_{\text{inner (scalar) product}}\;\boldsymbol{n}$$

Tangent component can be expressed as

$$\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma} - \sigma_n \boldsymbol{n} = \sigma_{ij} n_j \boldsymbol{e}_i - (\sigma_{km} n_k n_m) n_i \boldsymbol{e}_i = \underbrace{[\sigma_{ij} n_j - (\sigma_{km} n_k n_m) n_i]}_{(\boldsymbol{\sigma}_\tau)_i} \boldsymbol{e}_i$$

or, equivalently as

$$\boldsymbol{\sigma}_\tau = \boldsymbol{n} \times (\boldsymbol{\sigma} \times \boldsymbol{n})$$

Indeed, using the identity

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a}, \boldsymbol{c}) \boldsymbol{b} - (\boldsymbol{a}, \boldsymbol{b}) \boldsymbol{c}$$

for $\boldsymbol{a} = \boldsymbol{n}$, $\boldsymbol{b} = \boldsymbol{\sigma}$, $\boldsymbol{c} = \boldsymbol{n}$ we obtain

$$\boldsymbol{\sigma}_\tau = \boldsymbol{n} \times (\boldsymbol{\sigma} \times \boldsymbol{n}) = \underbrace{(\boldsymbol{n}, \boldsymbol{n})}_{1} \boldsymbol{\sigma} - \underbrace{(\boldsymbol{n}, \boldsymbol{\sigma})}_{\sigma_n} \boldsymbol{n} = \boldsymbol{\sigma} - \boldsymbol{\sigma}_n$$

CONSTITUTIVE RELATION

The **constitutive relation** for the (**simple**) fluids is the relation **between stress tensor** $\boldsymbol{\Xi}$ **and the deformation rate tensor** $\boldsymbol{D}$. This relation should be postulated in a form which is frame-invariant and such that the **stress tensor is symmetric**.

Let's remind two facts:

- The **velocity gradient** $\nabla\boldsymbol{v}$ can be decomposed into two parts: the symmetric part $\boldsymbol{D}$ called the **deformation rate tensor** and the skew-symmetric part $\boldsymbol{R}$ called the (rigid) **rotation tensor**.

$$\nabla\boldsymbol{v} = \boldsymbol{D} + \boldsymbol{R}$$

- Tensor $\boldsymbol{D}$ can be expressed as the sum of the spherical part $\boldsymbol{D}_{SPH}$ and the deviatoric part $\boldsymbol{D}_{DEV}$

$$\boldsymbol{D} = \boldsymbol{D}_{SPH} + \boldsymbol{D}_{DEV}$$

where
$$\boldsymbol{D}_{SPH} = \frac{1}{3} tr\,\boldsymbol{D}\cdot\boldsymbol{I} = \frac{1}{3}(\nabla\cdot\boldsymbol{v})\,\boldsymbol{I}$$

and
$$\boldsymbol{D}_{DEV} = \boldsymbol{D} - \frac{1}{3} div\,\boldsymbol{v}\cdot\boldsymbol{I} \;\Rightarrow\; (\boldsymbol{D}_{DEV})_{ij} = \frac{1}{2}\left(\frac{\partial \upsilon_i}{\partial x_j} + \frac{\partial \upsilon_j}{\partial x_i}\right) - \frac{1}{3}\frac{\partial \upsilon_k}{\partial x_k}\delta_{ij}$$

The **general constitutive relation** for a (simple) fluid can be written in the form of the matrix "polynomial"

$$\boldsymbol{\Xi} = \mathcal{P}(\boldsymbol{D}) = \boldsymbol{\Xi}_0 + c_0\boldsymbol{I} + c_1\boldsymbol{D} + c_2\boldsymbol{D}^2 + c_3\boldsymbol{D}^3 + \ldots$$

where the **coefficients are the function of 3 invariants of the tensor** $\boldsymbol{D}$, i.e.

$$c_k = c_k[\,I_1(\boldsymbol{D}), I_2(\boldsymbol{D}), I_3(\boldsymbol{D})\,].$$

Consider the **characteristic polynomial of the tensor** $\boldsymbol{D}$

$$p_{\boldsymbol{D}}(\lambda) = \det[\boldsymbol{D} - \lambda\boldsymbol{I}] = -\lambda^3 + I_1\lambda^2 - I_2\lambda + I_3.$$

The **Cayley-Hamilton Theorem** states that the matrix (or tensor) satisfies its own characteristic polynomial meaning that

$$p_{\boldsymbol{D}}(\boldsymbol{D}) = -\boldsymbol{D}^3 + I_1\boldsymbol{D}^2 - I_2\boldsymbol{D} + I_3 = \boldsymbol{0}$$

Thus, the 3rd power of $\boldsymbol{D}$ (and automatically **all higher powers**) **can be expressed as a linear combinations of** $\boldsymbol{I}$**,** $\boldsymbol{D}$ **and** $\boldsymbol{D}^2$.

Hence, the **most general polynomial constitutive relation** is given by the 2nd order formula

$$\boldsymbol{\Xi} = \mathcal{P}(\boldsymbol{D}) = \boldsymbol{\Xi}_0 + c_0\boldsymbol{I} + c_1\boldsymbol{D} + c_2\boldsymbol{D}^2$$

NEWTONIAN FLUIDS

The behavior of many fluids (water, air, others) can be described quite accurately by the linear constitutive relation. Such fluids are called Newtonian fluids.

For **Newtonian fluids** we assume that:

- c_0 is a linear function of the invariant I_1,
- c_1 is a constant,
- $c_2 = 0$.

If there is **no motion** we have the **Pascal Law**: pressure in any direction is the same. It means that the matrix $\boldsymbol{\Xi}_0$ should correspond to a spherical tensor and

$$\boldsymbol{\Xi}_0\boldsymbol{n} = -p\boldsymbol{n} \;\Rightarrow\; \boldsymbol{\Xi}_0 = -p\,\boldsymbol{I}$$

The **constitutive relation for the Newtonian fluids** can be written as follows

$$\boldsymbol{\Xi} = \underbrace{-p\boldsymbol{I}}_{\boldsymbol{\Xi}_0} + \zeta\underbrace{(\nabla\cdot\boldsymbol{v})}_{I_1(\boldsymbol{D})}\boldsymbol{I} + 2\mu\,\boldsymbol{D}_{DEV} = \underbrace{-p\boldsymbol{I}}_{\boldsymbol{\Xi}_0} + \underbrace{(\zeta - \tfrac{2}{3}\mu)(\nabla\cdot\boldsymbol{v})}_{c_0}\,\boldsymbol{I} + \underbrace{2\mu}_{c_1}\,\boldsymbol{D}$$

where

- μ - **(shear) viscosity** (the physical unit in SI is kg/m·s)
- ζ - **bulk viscosity** (the same unit as μ) ; usually $\zeta << \mu$ and can be assumed zero.

The constitutive relation can be written in the **index notation**

$$\sigma_{ij} = \left[-p + (\zeta - \tfrac{2}{3}\mu) \frac{\partial \upsilon_k}{\partial x_k} \right] \delta_{ij} + \mu \left[\frac{\partial \upsilon_i}{\partial x_j} + \frac{\partial \upsilon_j}{\partial x_i} \right]$$

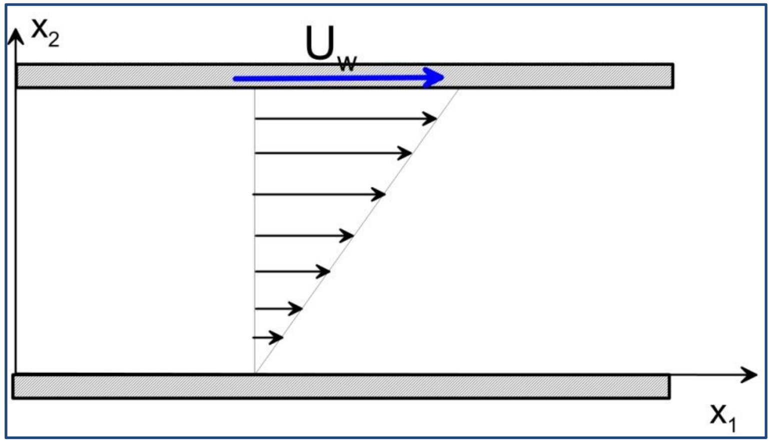
For an **incompressible fluid** we have $\nabla \cdot \boldsymbol{\upsilon} \equiv div\, \boldsymbol{\upsilon} \equiv \frac{\partial \upsilon_j}{\partial x_j} = 0$ and the constitutive relation reduces to the simpler form

$$\boldsymbol{\Xi} = -p\boldsymbol{I} + 2\mu\boldsymbol{D}$$

or, in the index notation

$$\sigma_{ij} = -p\delta_{ij} + \mu \left[\frac{\partial}{\partial x_j} \upsilon_i + \frac{\partial}{\partial x_i} \upsilon_j \right]$$

Example: Calculate the tangent stress in the wall shear layer.



The velocity field is defined as follows:

$$\upsilon_1(x_1,x_2)=U_{wall}x_2/H \quad , \quad \upsilon_2(x_1,x_2)\equiv 0$$

and the pressure is constant. At the bottom wall, the normal vector which points outwards is $\boldsymbol{n}=[0,-1]$.

Then

$$\boldsymbol{\sigma}=\boldsymbol{\Xi n}=-p\underbrace{\boldsymbol{n}}_{=[0,-1]}+2\mu\boldsymbol{Dn}=\begin{bmatrix}0\\p\end{bmatrix}+2\mu\begin{bmatrix}\frac{\partial\upsilon_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial\upsilon_1}{\partial x_2}+\frac{\partial\upsilon_2}{\partial x_1}\right)\\ \frac{1}{2}\left(\frac{\partial\upsilon_1}{\partial x_2}+\frac{\partial\upsilon_2}{\partial x_1}\right) & \frac{\partial\upsilon_2}{\partial x_2}\end{bmatrix}\begin{bmatrix}0\\-1\end{bmatrix}=$$

$$=\begin{bmatrix}0\\p\end{bmatrix}+2\mu\begin{bmatrix}0 & \frac{1}{2}\frac{\partial\upsilon_1}{\partial x_2}\\ \frac{1}{2}\frac{\partial\upsilon_1}{\partial x_2} & 0\end{bmatrix}\begin{bmatrix}0\\-1\end{bmatrix}=\begin{bmatrix}-\mu\frac{\partial\upsilon_1}{\partial x_2}\\p\end{bmatrix}=\begin{bmatrix}-\mu U_w/H\\p\end{bmatrix}$$

According to the action-reaction principle, the tangent stress at the bottom wall is

$$\tau_{wall}=\mu\frac{\partial}{\partial x_2}\upsilon_1\Big|_{wall}=\frac{\mu U_w}{H}$$