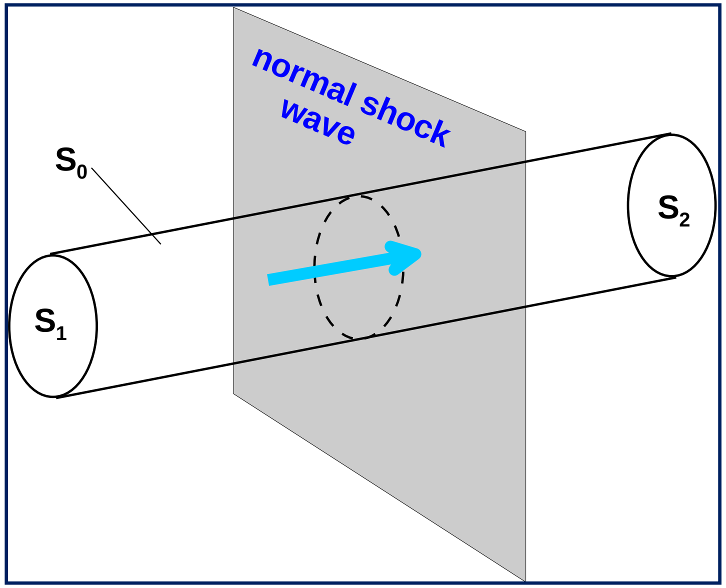
LECTURE 14

NORMAL SHOCK WAVE

NORMAL SHOCK WAVE. HUGONIOT ADIABATE.



Gas dynamics admitts existence of strongly discontinuous flow. The normal shock wave is the simplest example of such flow.

$$\partial\Omega = S_0 \cup S_1 \cup S_2$$

Conservation laws for the NSW

(1) Mass

$$\int_{\partial\Omega} \rho\upsilon_n\, ds = 0 \quad \underset{\text{1D case}}{\Rightarrow} \quad \rho_1 u_1 = \rho_2 u_2$$

(2) Linear momentum

$$\int_{\partial\Omega} (\rho\upsilon_n \boldsymbol{\upsilon} + p\boldsymbol{n})\, ds = \boldsymbol{0} \quad \underset{\text{1D case}}{\Rightarrow} \quad \rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2$$

(3) Energy

$$\frac{\kappa p_1}{(\kappa - 1)\rho_1} + \tfrac{1}{2} u_1^2 = \frac{\kappa p_2}{(\kappa - 1)\rho_2} + \tfrac{1}{2} u_2^2$$

Divide (2) by (1) …

$$u_1 + \frac{p_1}{\rho_1 u_1} = u_2 + \frac{p_2}{\rho_2 u_2} \Rightarrow u_1 - u_2 = \frac{p_2}{\rho_2 u_2} - \frac{p_1}{\rho_1 u_1} \quad (4)$$

New form of (3) is

$$(u_1 - u_2)(u_1 + u_2) = \frac{2\kappa}{\kappa - 1}\left(\frac{p_2}{\rho_2} - \frac{p_1}{\rho_1}\right) \quad (5)$$

Using (4) the Eq. (5) can be rewritten as

$$\left(\frac{p_2}{\rho_2 u_2} - \frac{p_1}{\rho_1 u_1}\right)(u_1 + u_2) = \frac{2\kappa}{\kappa - 1}\left(\frac{p_2}{\rho_2} - \frac{p_1}{\rho_1}\right) \quad (6)$$

The LHS of (6) can be transformed using the mass conservation equation (1) …

$$LHS_{(6)} = \underbrace{\frac{p_2}{\rho_2 u_2}}_{=\rho_1 \cancel{u_1}} \cancel{u_1} + \frac{p_2}{\rho_2 \cancel{u_2}} \cancel{u_2} - \frac{p_1}{\rho_1 \cancel{u_1}} \cancel{u_1} - \underbrace{\frac{p_1}{\rho_1 u_1}}_{=\rho_2 \cancel{u_2}} \cancel{u_2} = \frac{p_2}{\rho_1} + \frac{p_2}{\rho_2} - \frac{p_1}{\rho_1} - \frac{p_1}{\rho_2}$$

Thus, we get from (6)

$$\frac{p_2}{\rho_1}+\frac{p_2}{\rho_2}-\frac{p_1}{\rho_1}-\frac{p_1}{\rho_2}=\frac{2\kappa}{\kappa-1}\left(\frac{p_2}{\rho_2}-\frac{p_1}{\rho_1}\right) \qquad (7)$$

We multiply (7) by ρ_1/p_1 and get the formula which involves only the ratios ρ_1/ρ_2 and p_2/p_1 ….

$$\boxed{\frac{p_2}{p_1}=\frac{\frac{\kappa+1}{\kappa-1}-\frac{\rho_1}{\rho_2}}{\frac{\kappa+1}{\kappa-1}\frac{\rho_1}{\rho_2}-1}=\frac{\frac{\kappa+1}{\kappa-1}\frac{\rho_2}{\rho_1}-1}{\frac{\kappa+1}{\kappa-1}-\frac{\rho_2}{\rho_1}}} \qquad (8)$$

We have obtained the formula describing the thermodynamic process which affects the gas passing through the NSW. Note that **it is different that the isentropic process**! We call the above formula the **Hugoniot Adiabat (HA)**.

Let us analyze some properties of the HA. Note that for $\frac{\rho_2}{\rho_1}=\frac{\kappa+1}{\kappa-1}>1$ we have the vertical asymptote. If $\kappa=1.4$ then corresponding ratio is $\frac{\rho_2}{\rho_1}=6$.

Thus, **at the shock wave the density of gas always increases but never more than** $\frac{\kappa+1}{\kappa-1}$ **times.**

For brevity we introduce $y = p_2/p_1$, $x = \rho_2/\rho_1$ and $\alpha = \frac{\kappa+1}{\kappa-1}$. The formula (8) can be now written as

$$y(x) = \frac{\alpha x - 1}{\alpha - x}$$

Then, we have

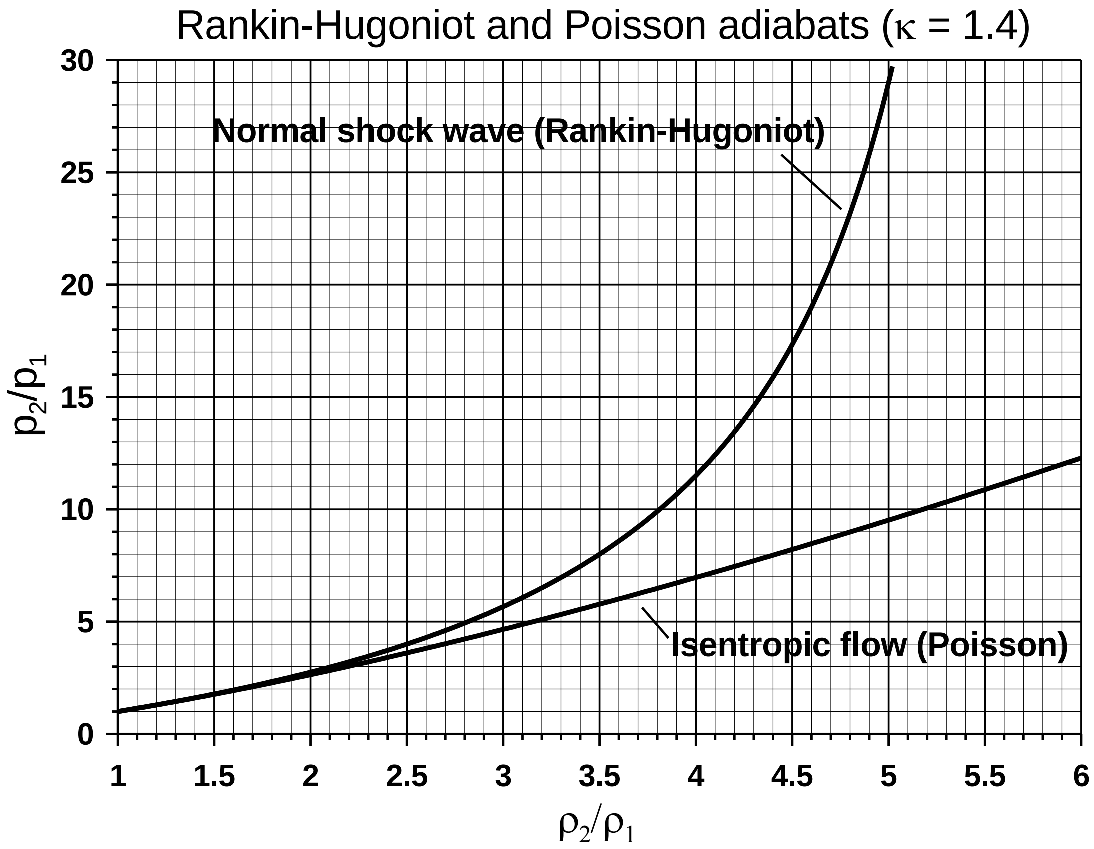
$$y'(x) = \frac{(\alpha+1)(\alpha-1)}{(\alpha-x)^2} \;\Rightarrow\; y'(1) = \frac{\alpha+1}{\alpha-1} = \kappa$$

Note that in case of the isentropic process, we have $y(x) = x^{\kappa}$, so $y'(x) = \kappa x^{\kappa-1}$ and $y'(1) = \kappa$. Moreover

$$y''_{Hugoniot}(x) = \frac{2(\alpha+1)(\alpha-1)}{(\alpha-x)^3} \;\Rightarrow\; y''_{Hugoniot}(1) = \kappa(\kappa-1)$$

$$y''_{isentropic}(x) = \kappa(\kappa-1)\,x^{\kappa-2} \;\Rightarrow\; y''_{isentropic}(1) = \kappa(\kappa-1)$$

We see that the isentropic and normal shock wave adiabats fit very well to each other in the vicinity $x = 1$ (they have the same values of $y(1)$, $y'(1)$ and $y''(1)$). We say that these lines are **strictly tangent** at $x = 1$.



Physically it means that **weak shock waves are nearly isentropic** and

$$\left.\frac{p_2}{p_1}\right|_{Hugoniot} - \left.\frac{p_2}{p_1}\right|_{isentropic} = C\left(\frac{\rho_2}{\rho_1} - 1\right)^3 + h.o.t$$

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ENTROPY AND GASODYNAMIC RELATION ON THE NORMAL SHOCK.

From the 1st **Principle of Thermodynamics** we have

$$ds=\frac{dq}{T}=\frac{1}{T}(di-\frac{dp}{\rho})=c_p\frac{dT}{T}-\frac{dp}{T\rho}=c_p\frac{dT}{T}-R\frac{dp}{p}$$

From the Clapeyron equation ….

$$p=\rho RT \;\Rightarrow\; \ln p=\ln\rho+\ln T+const$$

Thus

$$\frac{dp}{p}=\frac{d\rho}{\rho}+\frac{dT}{T} \;\Rightarrow\; \frac{dT}{T}=\frac{dp}{p}-\frac{d\rho}{\rho}$$

and the differential of (mass specific) entropy can be written as

$$ds=c_p\left(\frac{dp}{p}-\frac{d\rho}{\rho}\right)-R\frac{dp}{p}=c_v\frac{dp}{p}-c_p\frac{d\rho}{\rho}$$

After integration we get $s=c_v\ln p-\underbrace{c_p}_{=\kappa c_v}\ln\rho+const=c_v\ln(p/\rho^{\kappa})+const$

Thus, the change of entropy between two thermodynamic states can be expressed as follows

$$\frac{\Delta s}{c_v} \equiv \frac{s_2 - s_1}{c_v} = \ln\left(\frac{p_2}{\rho_2^{\kappa}}\right) - \ln\left(\frac{p_1}{\rho_1^{\kappa}}\right) = \ln\left(\frac{p_2}{p_1}\right) - \ln\left(\frac{\rho_2^{\kappa}}{\rho_1^{\kappa}}\right)$$

Note that for the Hugoniot adiabate we have

$$\frac{\rho_2}{\rho_1} < 1 \;\Rightarrow\; \left.\frac{p_2}{p_1}\right|_{Hugoniot} < \left(\frac{\rho_2}{\rho_1}\right)^{\kappa} \;\Rightarrow\; \Delta s < 0$$

i.e., the shock wave cannot expand the gas (it would lead to entropy decrease which contradicts the **2nd Principle of Thermodynamics**). Thus the (nontrivial) **shock wave must be a compression wave**! Indeed, in such case

$$\frac{p_2}{p_1} > \left(\frac{\rho_2}{\rho_1}\right)^{\kappa} \;\Rightarrow\; \Delta s > 0$$

We further observe that $\left.\frac{\rho_2}{\rho_1}\right|_{\substack{shock\\wave}} > 1 \;\; and \;\; \rho_1 u_1 = \rho_2 u_2 \;\; \Rightarrow \;\; u_2 = \left(\frac{\rho_2}{\rho_1}\right)^{-1} u_1 < u_1$

Using the energy equation (integral) we also conclude that

$$T_2 - T_1 = (u_1^2 - u_2^2)/2c_p > 0 \;\; \Rightarrow \;\; T_2 > T_1$$

i.e., **after crossing the shock wave the gas warms up**.

We would like to know what kind of flow exists at different sides of the NSW.

Writing the energy equation in the following form

$$\tfrac{1}{2}u^2 + \frac{\kappa p}{(\kappa-1)\rho} = \left(\tfrac{1}{2}u^2 + \frac{\kappa p}{(\kappa-1)\rho}\right)_{M=1} \equiv \frac{\kappa+1}{2(\kappa-1)}a_*^2$$

we obtain the following

$$\tfrac{1}{2}u_1 + \frac{\kappa p_1}{(\kappa-1)u_1\rho_1} = \frac{\kappa+1}{2(\kappa-1)}\frac{a_*^2}{u_1}$$

$$\tfrac{1}{2}u_2 + \frac{\kappa p_2}{(\kappa-1)u_2\rho_2} = \frac{\kappa+1}{2(\kappa-1)}\frac{a_*^2}{u_2}$$

Subtracting the above equations we get

$$u_1 - u_2 + \frac{2\kappa}{\kappa - 1}\underbrace{\left(\frac{p_1}{\rho_1 u_1} - \frac{p_2}{\rho_2 u_2}\right)}_{= u_2 - u_1 (Eq.4)} = \frac{\kappa + 1}{\kappa - 1}\left(\frac{a_*^2}{u_1} - \frac{a_*^2}{u_2}\right)$$

After some algebra we derive the **Prandtl's Relation**

$$u_2 - u_1 = \frac{u_2 - u_1}{u_1 u_2} a_*^2 \quad and \quad u_2 \neq u_1 \quad \Rightarrow \quad u_1 u_2 = a_*^2$$

The immediate conclusion from PR is that $u_1 > a_* \quad and \quad u_2 < a_*$

What about the Mach number?

The energy equation $\frac{1}{2} u^2 + \frac{a^2}{\kappa - 1} = \frac{\kappa + 1}{2(\kappa - 1)} a_*^2$

can be divided by the square of velocity to obtain

$$1 + \frac{2}{\kappa - 1}\frac{1}{M^2} = \frac{\kappa + 1}{\kappa - 1}\frac{a_*^2}{u^2}$$

After simple manipulations we have $$\left(\frac{u}{a_*}\right)^2 = \frac{\kappa+1}{\kappa-1+\frac{2}{M^2}}$$

From the above the following we infer

$$u_1 > a_* \;\Rightarrow\; M_1 > 1 \quad \text{and} \quad u_2 < a_* \;\Rightarrow\; M_2 < 1$$

Thus, **the flow in front of the NSW is always supersonic, while the flow behind it – always subsonic**.

The quantity $\lambda = \frac{u}{a_*}$ is called the velocity coefficient. In contrast to the Mach number, the **velocity coefficient assumes values in the bounded interval**, namely

$$\lim_{M_1\to\infty} \lambda_1 = \sqrt{\frac{\kappa+1}{\kappa-1}} \quad , \quad \lim_{M_1\to\infty} \lambda_2 = \sqrt{\frac{\kappa-1}{\kappa+1}}$$

The Mach number of the flow behind the wave M_2 can be expressed as the function of the Mach number in front of the wave M_1. To obtain this formula we use the **Prandtl's Relation** …

$$\frac{u_1}{a_*}\frac{u_2}{a_*}=1 \quad \Rightarrow \quad \frac{\kappa+1}{\sqrt{\kappa-1+\frac{2}{M_1^2}}}\frac{\kappa+1}{\sqrt{\kappa-1+\frac{2}{M_2^2}}}=1$$

After some algebra we get

$$\boxed{M_2=\sqrt{\frac{2+(\kappa-1)M_1^2}{2\kappa M_1^2-\kappa+1}}<1}$$

Similarly, we can expressed the ratios of density, pressure and temperature values. The density ratio can be evaluated as follows

$$\frac{\rho_2}{\rho_1}(M_1)=\frac{u_1}{u_2}(M_1)=\frac{M_1}{M_2(M_1)}\frac{a_1}{a_2}(M_1)=\frac{M_1}{M_2(M_1)}\left(\frac{a}{a_0}\right)_{is}(M_1)\left(\frac{a}{a_0}\right)_{is}^{-1}[M_2(M_1)]>1$$

To evaluate the pressure ratio (as the function of M_1) we rewrite the momentum equation in the following way

$$p + \rho u^2 = p\left(1 + \frac{u^2}{p/\rho}\right) = p\left(1 + \frac{\kappa u^2}{a^2}\right) = p(1 + \kappa M^2) = const$$

Since the above expression has the same value at both sides of the shock wave, we get

$$\frac{p_2}{p_1}(M_1) = \frac{1 + \kappa M_1^2}{1 + \kappa M_2^2(M_1)} > 1$$

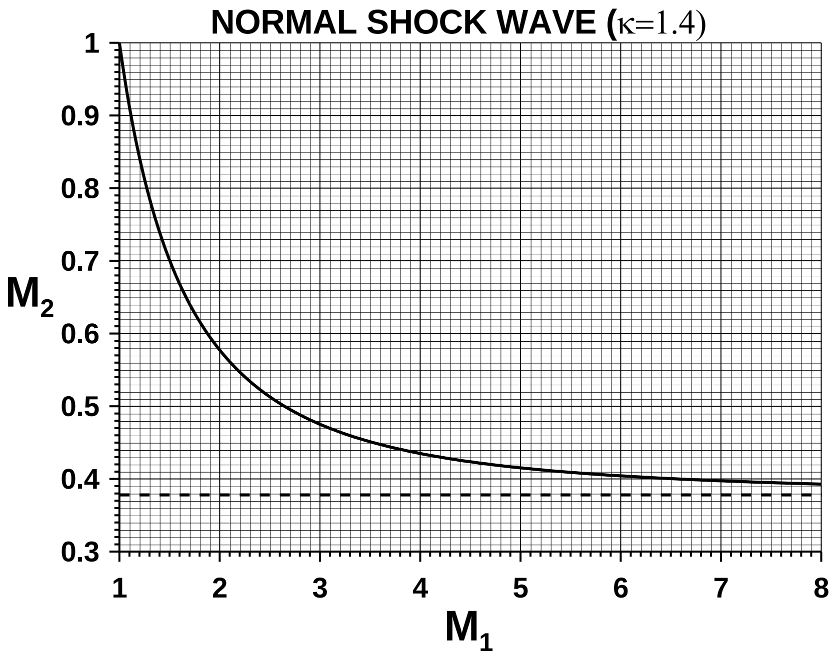
The flow through the shock is adiabatic (total energy is conserved) and the total (or stagnation) temperature T_0 remains the same.

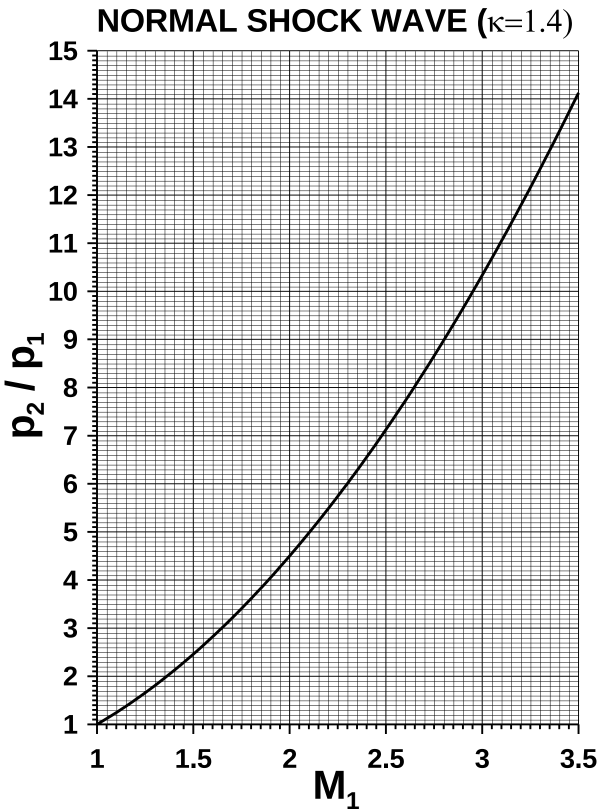
Thus, we can write

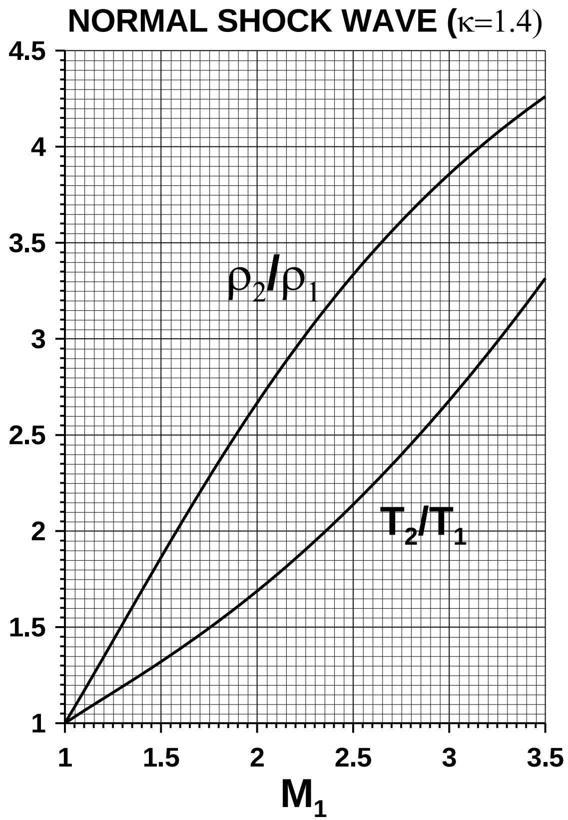
$$\frac{T_2}{T_1}(M_1) = \frac{\left(T/T_0\right)(M_2)}{\left(T/T_0\right)[M_2(M_1)]} > 1$$

where the ratio T/T_0 can be calculated from the formula derived in the Lecture 13.

$$\frac{T}{T_0}(M) = \left(1 + \frac{\kappa - 1}{2} M^2\right)^{-1}$$







In the end, we will explore what happens to the stagnation pressure. Conceptually, we consider the process described as follows

$$p_{01} \underset{\text{acceleration (isentropic)}}{\longrightarrow} p_1 \underset{\text{shock wave}}{\longrightarrow} p_2 \underset{\text{deceleration (isentropic)}}{\longrightarrow} p_{02} < p_{01}$$

We claim that **the total (stagnation) pressure diminishes on the shock wave**.

The justification of this fact goes as follows.

We know that the entropy of the gas increases on the shock wave. The formula derived earlier can be written for stagnation parameters, namely

$$0 < \frac{s_2 - s_1}{c_v} = \ln\left(p_{02}/p_{01}\right) - \kappa \ln\left(\rho_{02}/\rho_{01}\right)$$

Since the total temperature at both sides is the same then

$$T_{01} = T_{02} \equiv T_0 \quad \underset{\text{Clapeyron Equation}}{\Rightarrow} \quad \frac{\rho_{02}}{\rho_{01}} = \frac{p_{02}}{p_{01}}$$

Thus

$$0 < \frac{\Delta s}{c_v} = (1-\kappa)\ln\left(p_{02}/p_{01}\right) \quad \Rightarrow \quad p_{02}/p_{01} < 1.$$