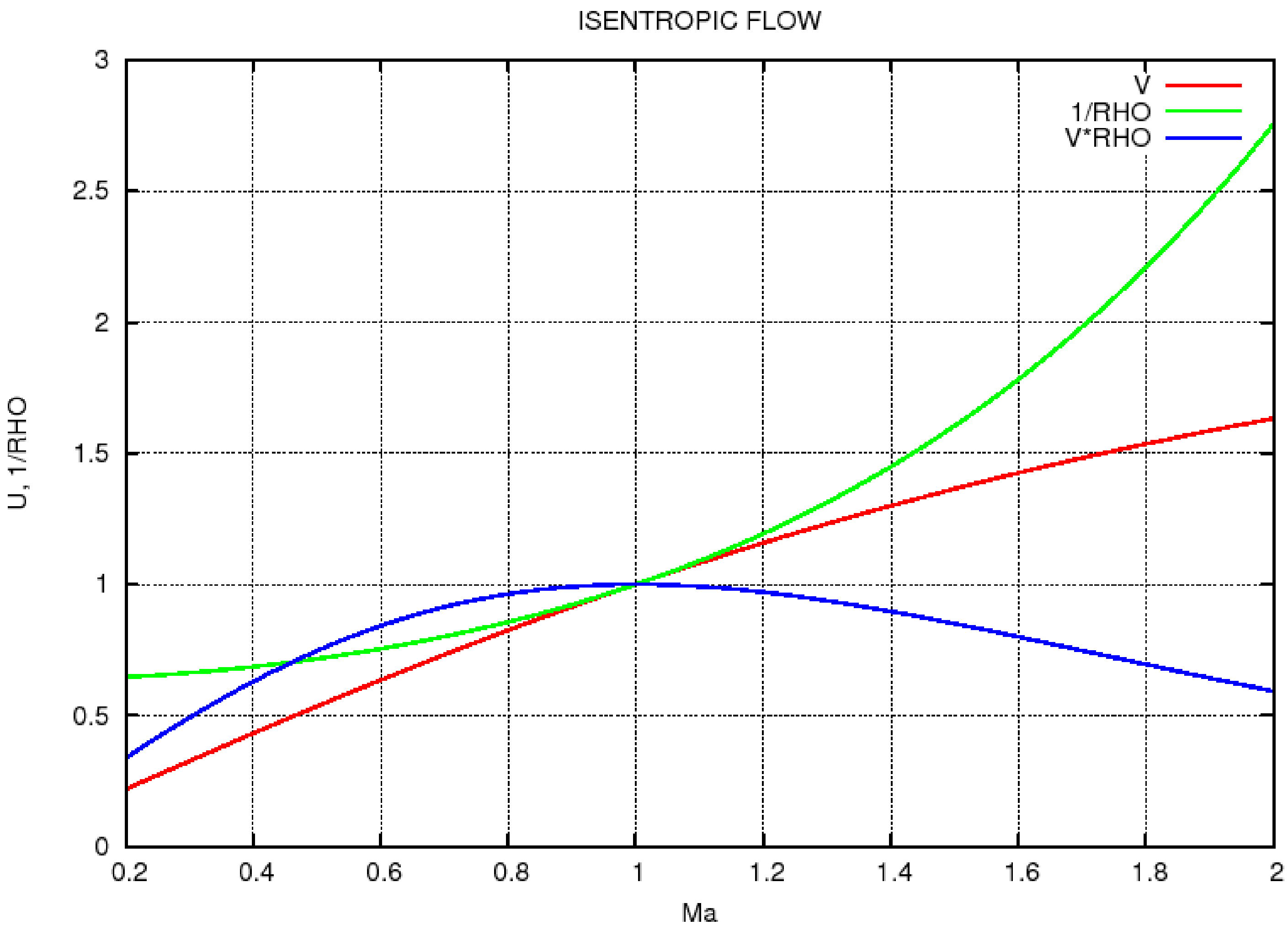
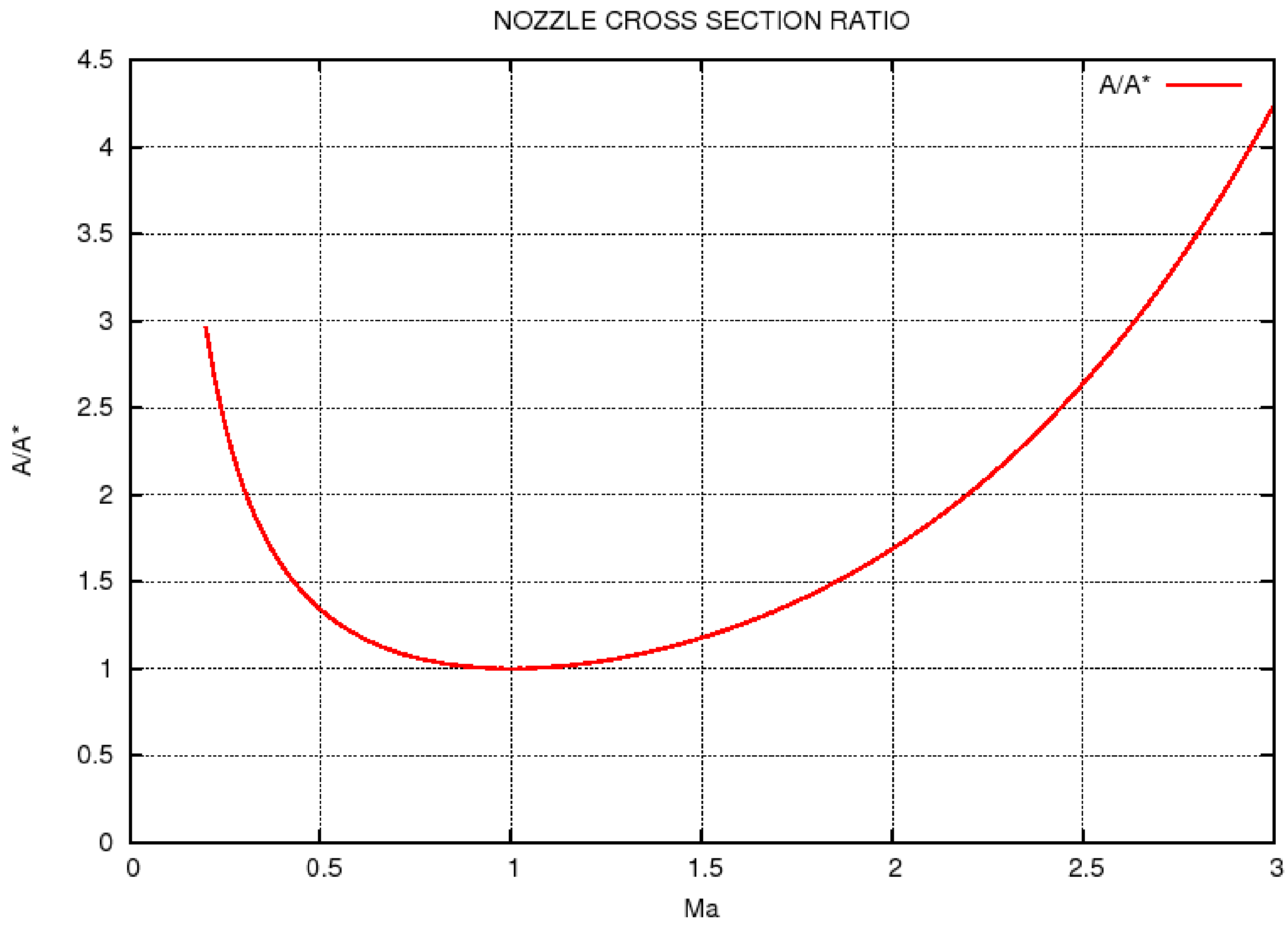

ISENTROPIC FLOW
V
1/RHO
V*RHO
U, 1/RHO
Ma
0
0.5
1
1.5
2
2.5
3
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2


NOZZLE CROSS SECTION RATIO
A/A*
A/A*
Ma
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
0
0.5
1
1.5
2
2.5
3

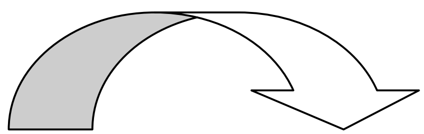
GAS DYNAMICS – NORMAL SHOCK WAVE

$$\rho_1 U_1 = \rho_2 U_2 \tag{1}$$

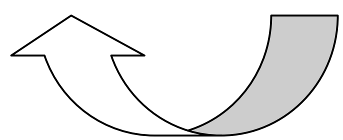
$$\rho_1 U_1^2 + p_1 = \rho_2 U_2^2 + p_2 \tag{2}$$

$$\frac{1}{2}U_1^2 + \underbrace{\frac{k}{k-1}R}_{c_p} T_1 = \frac{1}{2}U_2^2 + \underbrace{\frac{k}{k-1}R}_{c_p} T_2 \tag{3}$$

(2)/(1)



$$U_1 + \frac{p_1}{\rho_1 U_1} = U_2 + \frac{p_2}{\rho_2 U_2}$$



$$\boldsymbol{U_2} - \boldsymbol{U_1} = \frac{\boldsymbol{p_1}}{\rho_1 \boldsymbol{U_1}} - \frac{\boldsymbol{p_2}}{\rho_2 \boldsymbol{U_2}}$$

*(u1+u2)

$$\boldsymbol{U_2^2} - \boldsymbol{U_1^2} = \frac{\boldsymbol{p_1}}{\rho_1 \boldsymbol{U_1}}\boldsymbol{U_1} + \frac{\boldsymbol{p_1}}{\underbrace{\rho_1 \boldsymbol{U_1}}_{\rho_2 U_2}}\boldsymbol{U_2} - \frac{\boldsymbol{p_2}}{\underbrace{\rho_2 \boldsymbol{U_2}}_{\rho_1 U_1}}\boldsymbol{U_1} - \frac{\boldsymbol{p_2}}{\rho_2 \boldsymbol{U_2}}\boldsymbol{U_2} =$$

$$\frac{\boldsymbol{p_1}}{\rho_1} + \frac{\boldsymbol{p_1}}{\rho_2} - \frac{\boldsymbol{p_2}}{\rho_1} - \frac{\boldsymbol{p_2}}{\rho_2} = \left(\boldsymbol{p_1} - \boldsymbol{p_2}\right)\left(\frac{\mathbf{1}}{\rho_1} + \frac{\mathbf{1}}{\rho_2}\right)$$

*1/2 →(3)

$$\frac{1}{2}U_2^2 - \frac{1}{2}U_1^2 = \frac{k}{k-1}\left(\underbrace{RT_1}_{p_1/\rho_1} - \underbrace{RT_2}_{p_2/\rho_2}\right) =$$

$$\frac{k}{k-1}\left(\frac{p_1}{\rho_1} - \frac{p_2}{\rho_2}\right) = \frac{1}{2}\left(p_1 - p_2\right)\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right) \qquad *2\rho_1\rho_2$$

$$\frac{2k}{k-1}\left(\rho_2 p_1 - \rho_1 p_2\right) = \left(p_1 - p_2\right)\left(\rho_2 + \rho_1\right) \qquad /\rho_1 p_1$$

$$\frac{2k}{k-1}\left(\frac{\rho_2}{\rho_1} - \frac{p_2}{p_1}\right) = \left(1 - \frac{p_2}{p_1}\right)\left(\frac{\rho_2}{\rho_1} + 1\right)$$

$$\frac{2k}{k-1}\frac{\rho_2}{\rho_1} - \left(\frac{\rho_2}{\rho_1} + 1\right) = \frac{2k}{k-1}\frac{p_2}{p_1} - \frac{p_2}{p_1}\left(\frac{\rho_2}{\rho_1} + 1\right)$$

$$\frac{p_2}{p_1}\left(\frac{2k}{k-1}-1-\frac{\rho_2}{\rho_1}\right)=\frac{\rho_2}{\rho_1}\left(\frac{2k}{k-1}-1\right)-1$$

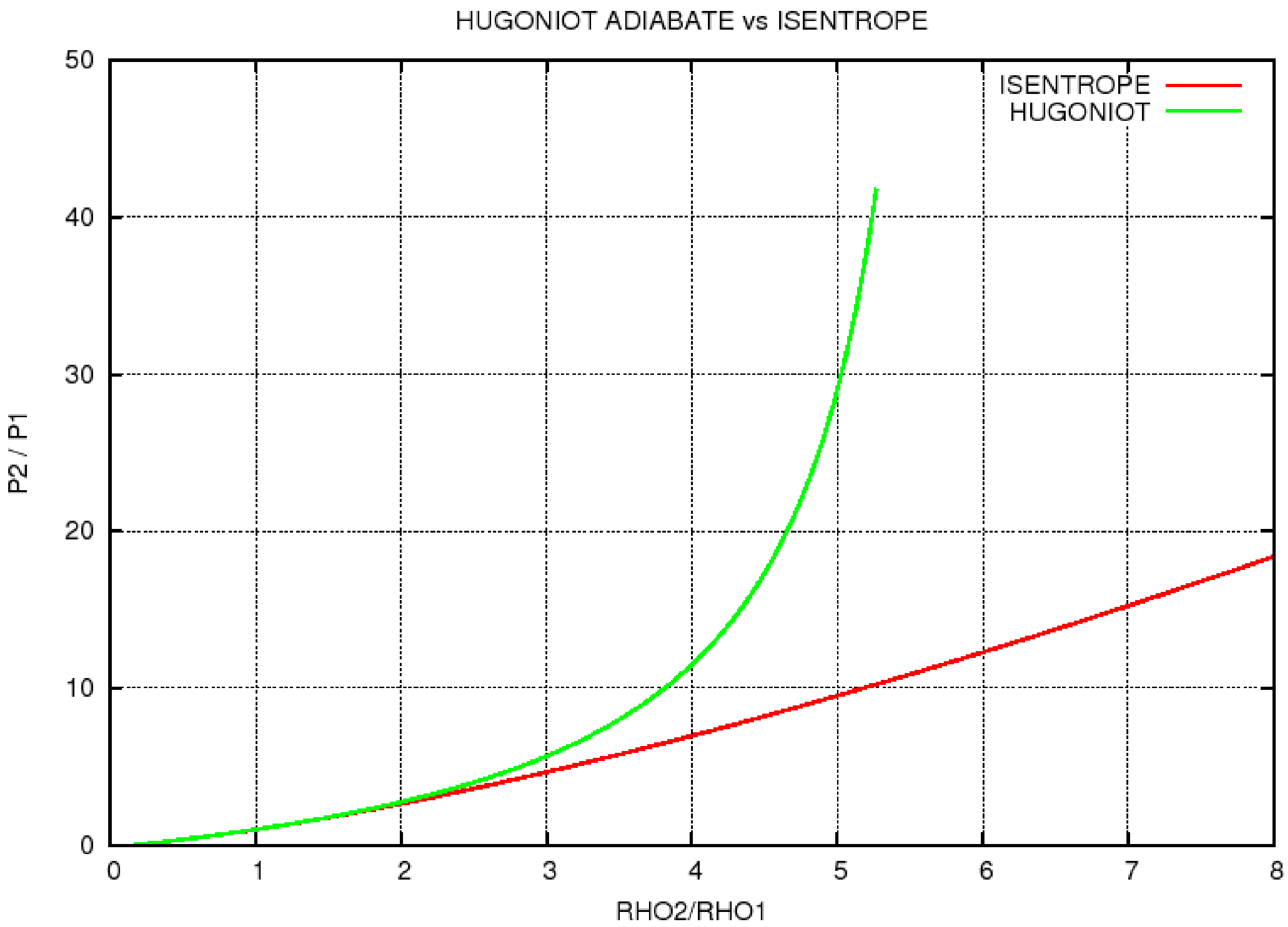
$$\frac{p_2}{p_1}=\left(\frac{k+1}{k-1}\frac{\rho_2}{\rho_1}-1\right)\Big/\left(\frac{k+1}{k-1}-\frac{\rho_2}{\rho_1}\right)$$

OR:

$$\frac{\rho_2}{\rho_1}=\left(1+\frac{k+1}{k-1}\frac{p_2}{p_1}\right)\Big/\left(\frac{k+1}{k-1}+\frac{p_2}{p_1}\right)=\frac{U_1}{U_2}$$

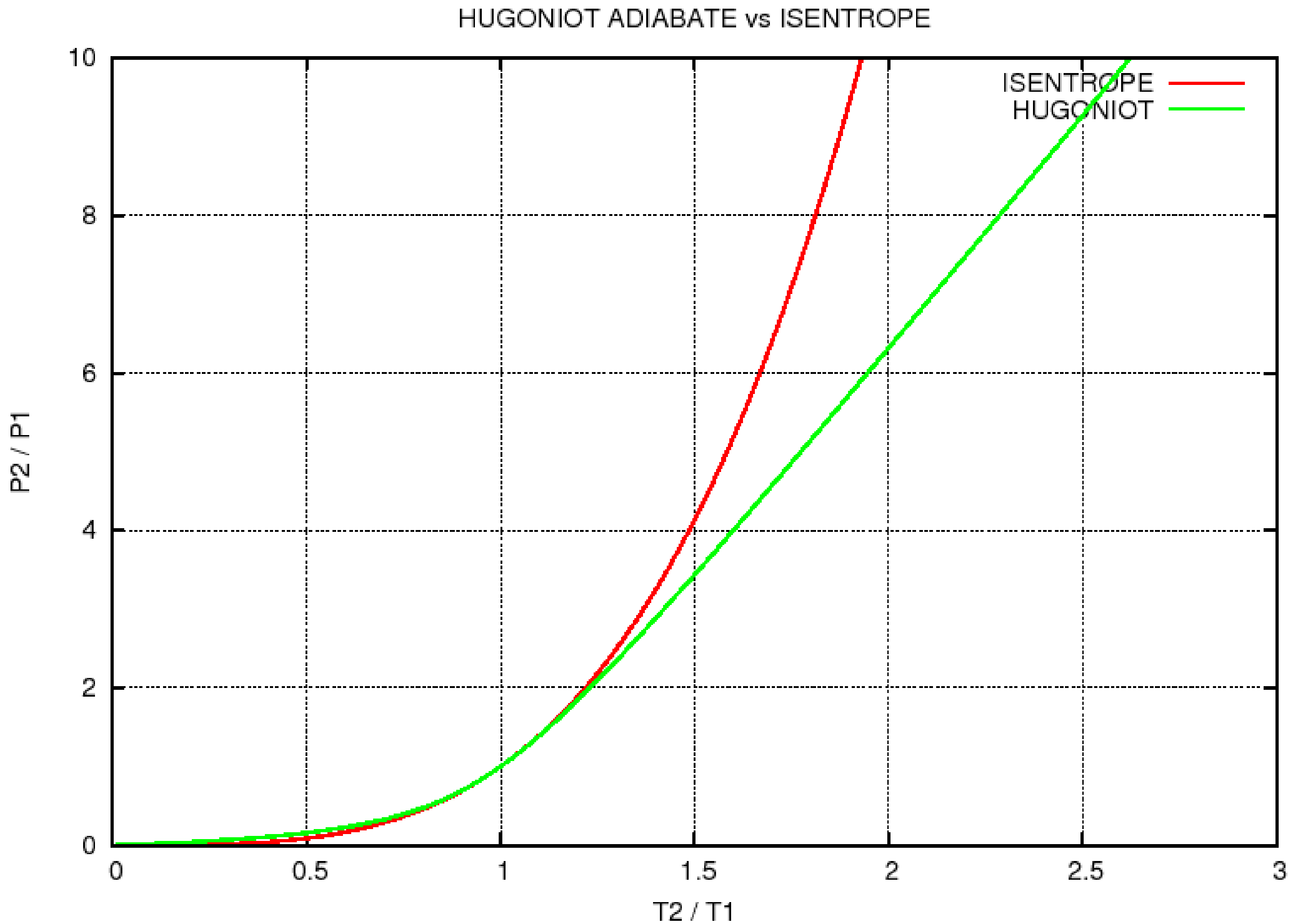
Rankinea-Hugoniot Eq.

Air, k=1.4



HUGONIOT ADIABATE vs ISENTROPE
ISENTROPE
HUGONIOT
P2 / P1
RHO2/RHO1
0
10
20
30
40
50
0
1
2
3
4
5
6
7
8

Air, k=1.4



HUGONIOT ADIABATE vs ISENTROPE
ISENTROPE
HUGONIOT
P2 / P1
T2 / T1
0
2
4
6
8
10
0
0.5
1
1.5
2
2.5
3

$$U_2 - U_1 = \frac{p_1}{\rho_1 U_1} - \frac{p_2}{\rho_2 U_2}$$

$$\left| \frac{1}{2}U^2 + \frac{k}{k-1}RT = \frac{1}{2}U^2 + \frac{k}{k-1}\frac{p}{\rho} = \frac{1}{2}U^{*2} + \frac{a^{*2}}{k-1} = \frac{(k+1)}{2(k-1)}a^{*2} \right.$$

$$\left. \frac{p}{\rho} = \frac{(k+1)}{2k}a^{*2} - \frac{(k-1)}{2k}U^2 \right|$$

$$U_1 - U_2 = \frac{(k+1)}{2k}a^{*2}\left(\frac{1}{U_2} - \frac{1}{U_1}\right) - \frac{(k-1)}{2k}U_2 + \frac{(k-1)}{2k}U_1 =$$

$$U_1 - U_2 = (U_1 - U_2)\left[\frac{(k-1)}{2k} + \frac{(k+1)}{2k}\frac{a^{*2}}{U_2 U_1}\right]$$

$$\left(U_1 - U_2\right)\left[\frac{\left(k-1\right)}{2k} - 1 + \frac{\left(k+1\right)}{2k}\frac{a^{*2}}{U_2 U_1}\right] = 0$$

$$\frac{\left(k+1\right)}{2k}\left(U_1 - U_2\right)\left[\frac{a^{*2}}{U_2 U_1} - 1\right] = 0$$

$$\left(U_1 - U_2\right)\left[\frac{a^{*2}}{U_2 U_1} - 1\right] = 0$$

$1:\quad U_2 = U_1$

$2:\quad U_2 \cdot U_1 = a^{*2}$

$2a:\quad U_1 > a^* \rightarrow Ma_1 > 1,\quad Ma_2 < 1$

$2b:\quad U_1 < a^* \rightarrow Ma_1 < 1,\quad Ma_2 > 1$

$Ma_2 = f(Ma_1)$

$$Ma_2^2 = \frac{U_2^2}{a_2^2} = \left(a^{*2}/U_1\right)^2 / \left(\frac{k+1}{2}a^{*2} - \frac{k-1}{2}U_2^2\right) =$$

$$\left(a^{*2}/U_1\right)^2 / \left(\frac{k+1}{2}a^{*2} - \frac{k-1}{2}\left(a^{*2}/U_1\right)^2\right) =$$

$$\left|\frac{k-1}{2}U^2 + a^2 = \frac{k+1}{2}a^{*2} \rightarrow a^{*2} = \frac{k-1}{k+1}U^2 + \frac{2}{k+1}a^2\right|$$

$$= \left(\frac{k-1}{k+1}U_1^2 + \frac{2}{k+1}a_1^2\right) / \left(\frac{k+1}{2}U_1^2 - \frac{k-1}{2}\left(\frac{k-1}{k+1}U_1^2 + \frac{2}{k+1}a_1^2\right)^2\right) = \ldots$$

$$Ma_2^2 = \frac{(k-1)\,Ma_1^2 + 2}{2k\,Ma_1^2 - (k-1)}$$

$p_2/p_1=f(Ma_1)$

$$\rho_1 U_1^2 + p_1 = \rho_2 U_2^2 + p_2$$

$$\left| \frac{p}{\rho} = RT \rightarrow \rho = \frac{p}{RT} = p\frac{k}{kRT} = \frac{k}{a^2}p \right|$$

$$U_1^2 \frac{k}{a_1^2} p_1 + p_1 = U_2^2 \frac{k}{a_2^2} p_2 + p_2$$

$$p_1\left(1 + k\,Ma_1^2\right) = p_2\left(1 + k\,Ma_2^2\right)$$

$$\frac{p_2}{p_1} = \frac{1 + k\,Ma_1^2}{1 + k\,Ma_2^2} = \ldots = \frac{2k}{k+1}Ma_1^2 - \frac{k-1}{k+1} = \frac{2k}{k+1}\left(Ma_1^2 - 1\right) + 1$$

$\rho_1/\rho_2=f(Ma_1)$

$$\frac{\rho_1}{\rho_2}=\frac{U_2}{U_1}=\frac{a^{*2}}{U_1^2}$$

$$\left|a^{*2}=\frac{2}{k+1}a_1^2+\frac{k-1}{k+1}U_1^2\right|$$

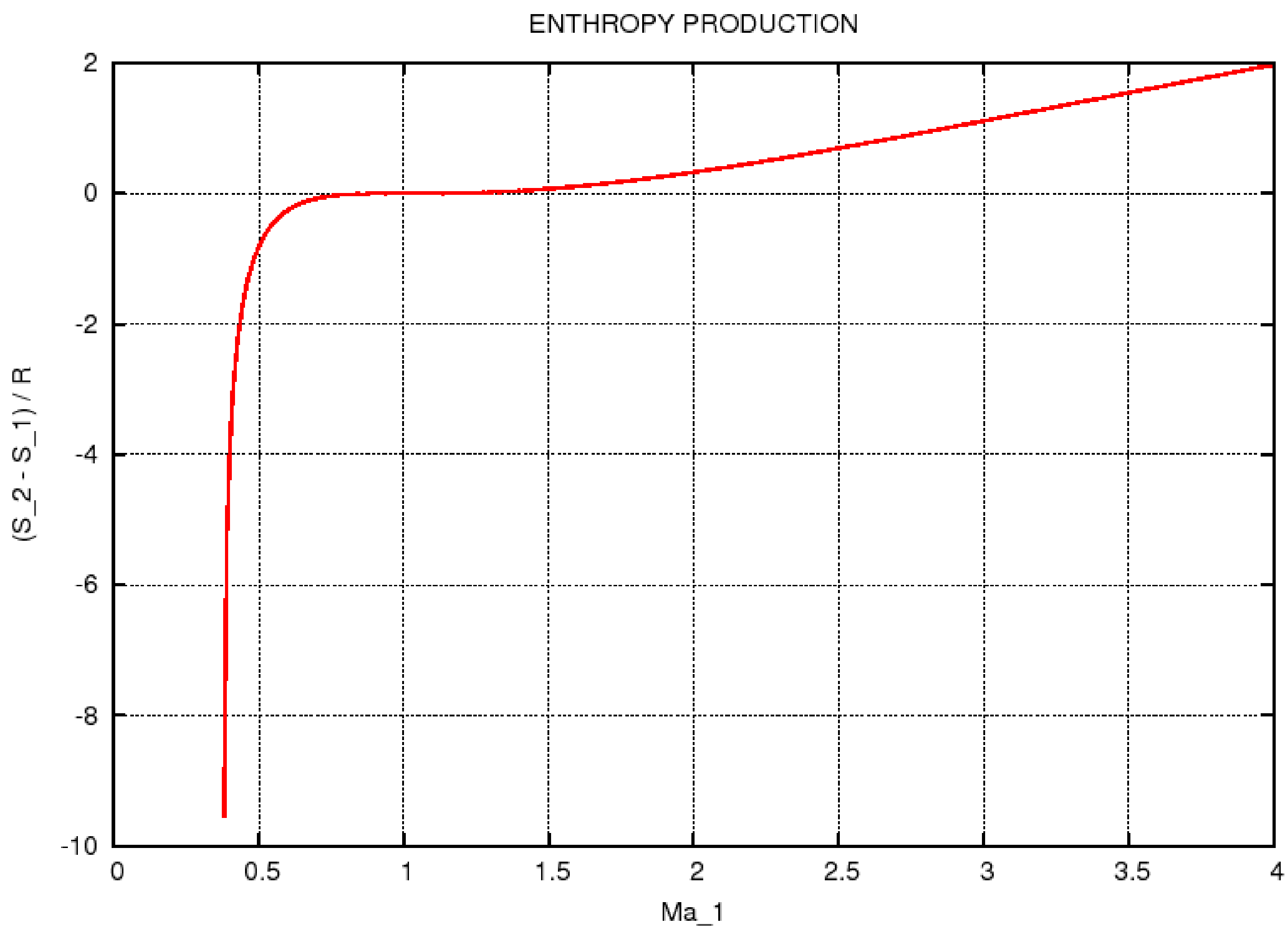
$$\frac{\rho_1}{\rho_2}=\frac{U_2}{U_1}=\left(\frac{2}{k+1}a_1^2+\frac{k-1}{k+1}U_1^2\right)/U_1^2$$

$$\frac{\rho_1}{\rho_2}=\frac{2+(k-1)Ma_1^2}{(k+1)Ma_1^2}=\frac{(k-1)\left(Ma_1^2-1\right)+k+1}{(k+1)Ma_1^2}$$

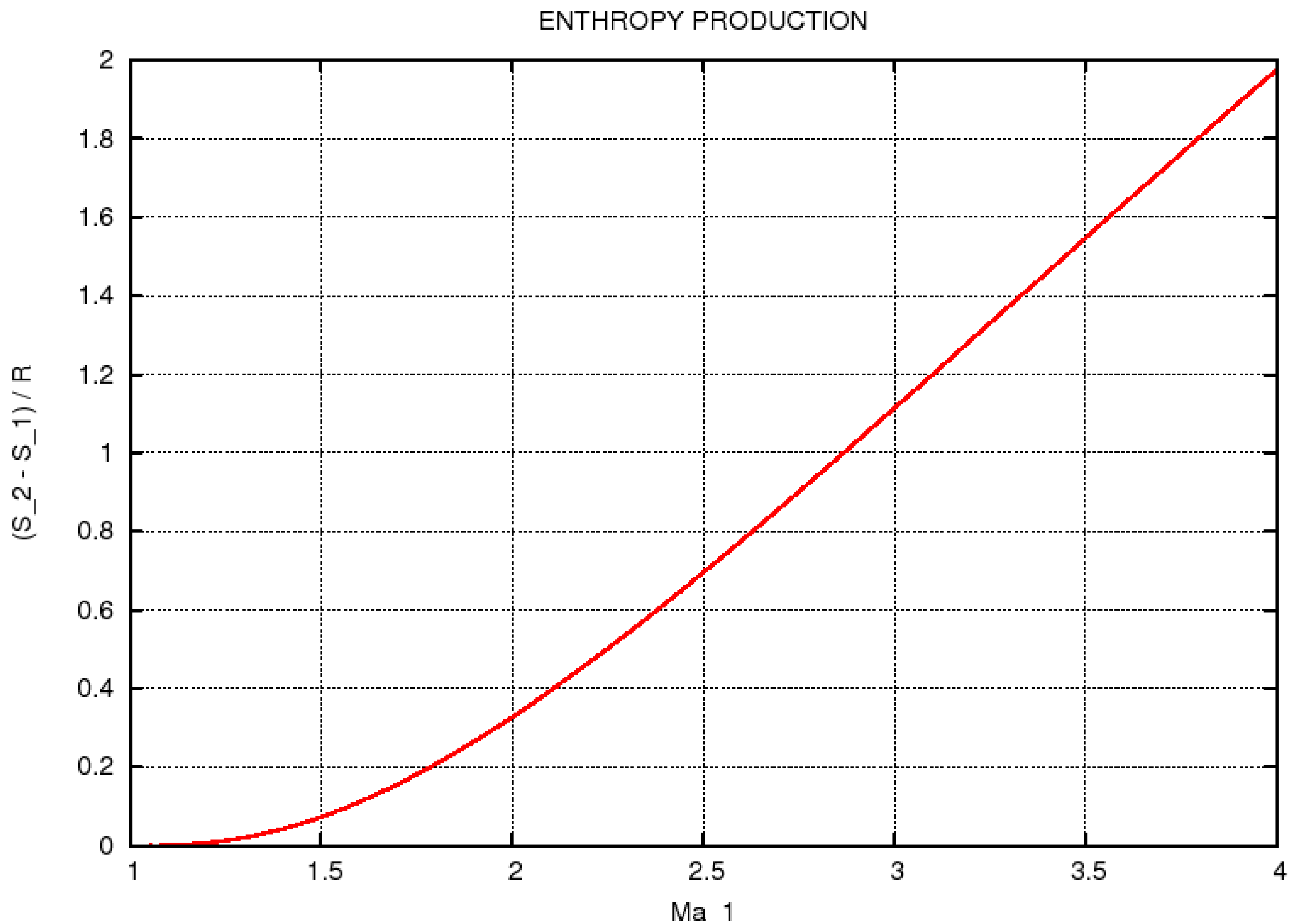
$$S_2 - S_1 = c_v \ln\left(\frac{p_2}{\rho_2^k} / \frac{p_1}{\rho_1^k} \right) = c_v \ln\left(\frac{p_2}{p_1} \cdot \left(\frac{\rho_1}{\rho_2} \right)^k \right)$$

$$S_2 - S_1 = c_v \ln\left(\left(\frac{2k}{k+1}\left(Ma_1^2 - 1\right) + 1 \right) \cdot \left(\frac{\left(k-1\right)\left(Ma_1^2 - 1\right) + k + 1}{\left(k+1\right) Ma_1^2} \right)^k \right)$$

$$S_2 - S_1 = c_v \frac{2}{3} \frac{k\left(k-1\right)}{\left(k+1\right)^2}\left(Ma_1^2 - 1\right)^3 + \cdots = \frac{2}{3} \frac{kR}{\left(k+1\right)^2}\left(Ma_1^2 - 1\right)^3 + \cdots$$



ENTHROPY PRODUCTION
(S_2 - S_1) / R
Ma_1
2
0
-2
-4
-6
-8
-10
0
0.5
1
1.5
2
2.5
3
3.5
4


ENTHROPY PRODUCTION
(S_2 - S_1) / R
Ma_1
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
1
1.5
2
2.5
3
3.5
4

$$1^{O} \xrightarrow{\textit{isentrope}} 1 \;\Big\|\; 2 \xrightarrow{\textit{isentrope}} 2^{O}$$

shock

$\Delta S > 0$

$$\textit{izentr.} \rightarrow \quad \frac{p}{\rho^{k}} = const \qquad \frac{T}{\rho^{k-1}} = const \qquad \frac{T^{k}}{p_2^{k-1}} = const$$

$$S_2 - S_1 = c_v \ln\left(\frac{T_2^{k}}{p_2^{k-1}} / \frac{T_1^{k}}{p_1^{k-1}} \right) = c_v \ln\left(\frac{T_{O2}^{k}}{p_{O2}^{k-1}} / \frac{T_{O1}^{k}}{p_{O1}^{k-1}} \right) =$$

$$\left| T_{O2}^{k} = T_{O1}^{k} \right| = c_v \ln\left(\frac{p_{O1}}{p_{O2}} \right)^{k-1} = (k-1) c_v \ln\left(\frac{p_{O1}}{p_{O2}} \right)$$

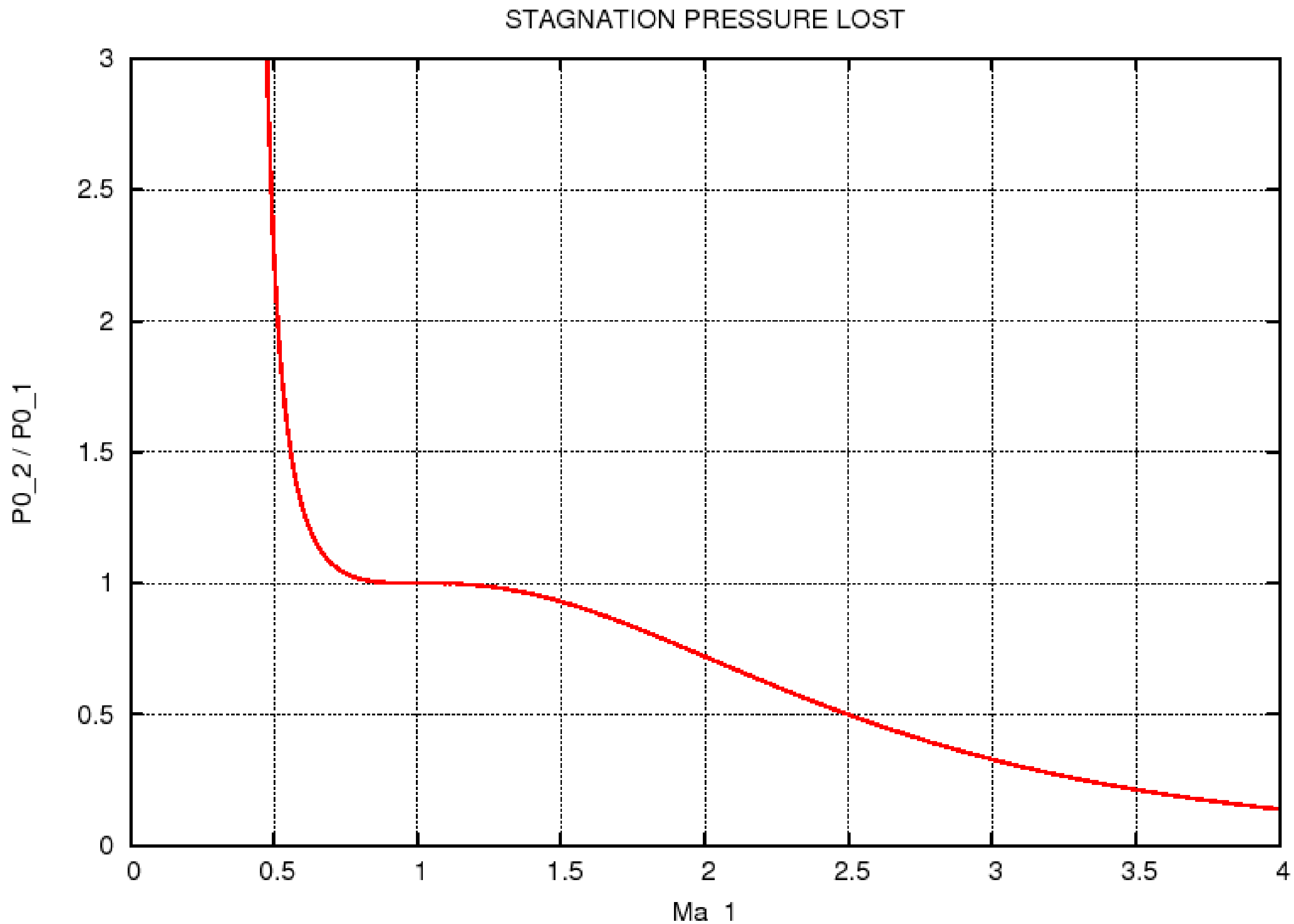
$$S_2 - S_1 = R \cdot \ln\left(\frac{p_{O1}}{p_{O2}} \right)$$

$$R \cdot \ln\left(\frac{p_{O1}}{p_{O2}}\right)_1 = \frac{R}{k-1}\ln\left(\left(\frac{2k}{k+1}\left(Ma_1^2-1\right)+1\right)\cdot\left(\frac{\left(k-1\right)\left(Ma_1^2-1\right)+k+1}{\left(k+1\right)Ma_1^2}\right)^k\right)$$

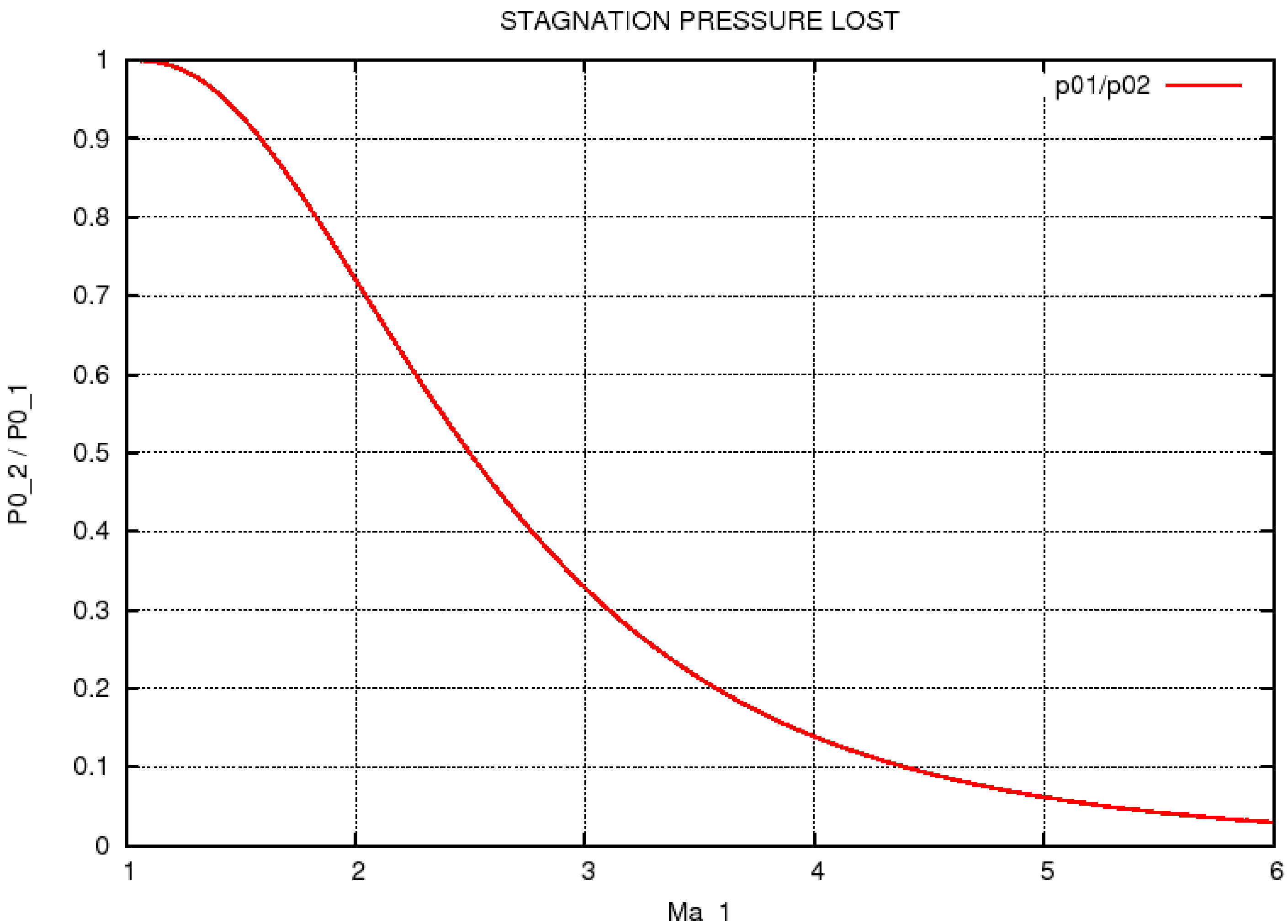
$$\frac{p_{O2}}{p_{O1}} = \left(\frac{2k}{k+1}\left(Ma_1^2-1\right)+1\right)^{-1/(k-1)} \cdot \left(\frac{\left(k-1\right)\left(Ma_1^2-1\right)+k+1}{\left(k+1\right)Ma_1^2}\right)^{-k/(k-1)}$$

$$R \cdot \ln\left(\frac{p_{O1}}{p_{O2}}\right) = \frac{2}{3}\frac{kR}{\left(k+1\right)^2}\left(Ma_1^2-1\right)^3 + \cdots$$

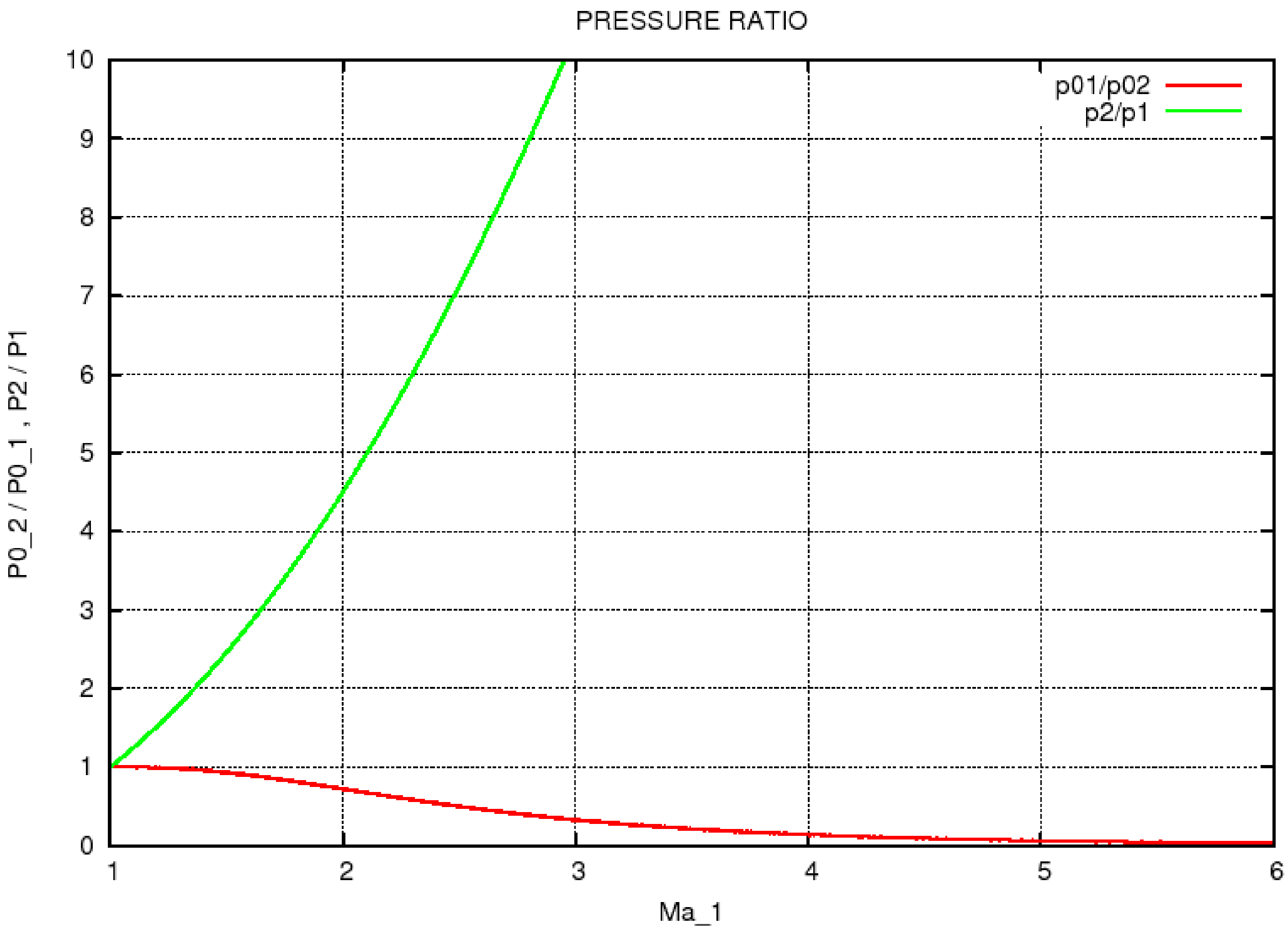
$$\frac{p_{O2}}{p_{O1}} = \exp\left(-\frac{2}{3}\frac{k}{\left(k+1\right)^2}\left(Ma_1^2-1\right)^3 + \cdots\right)$$

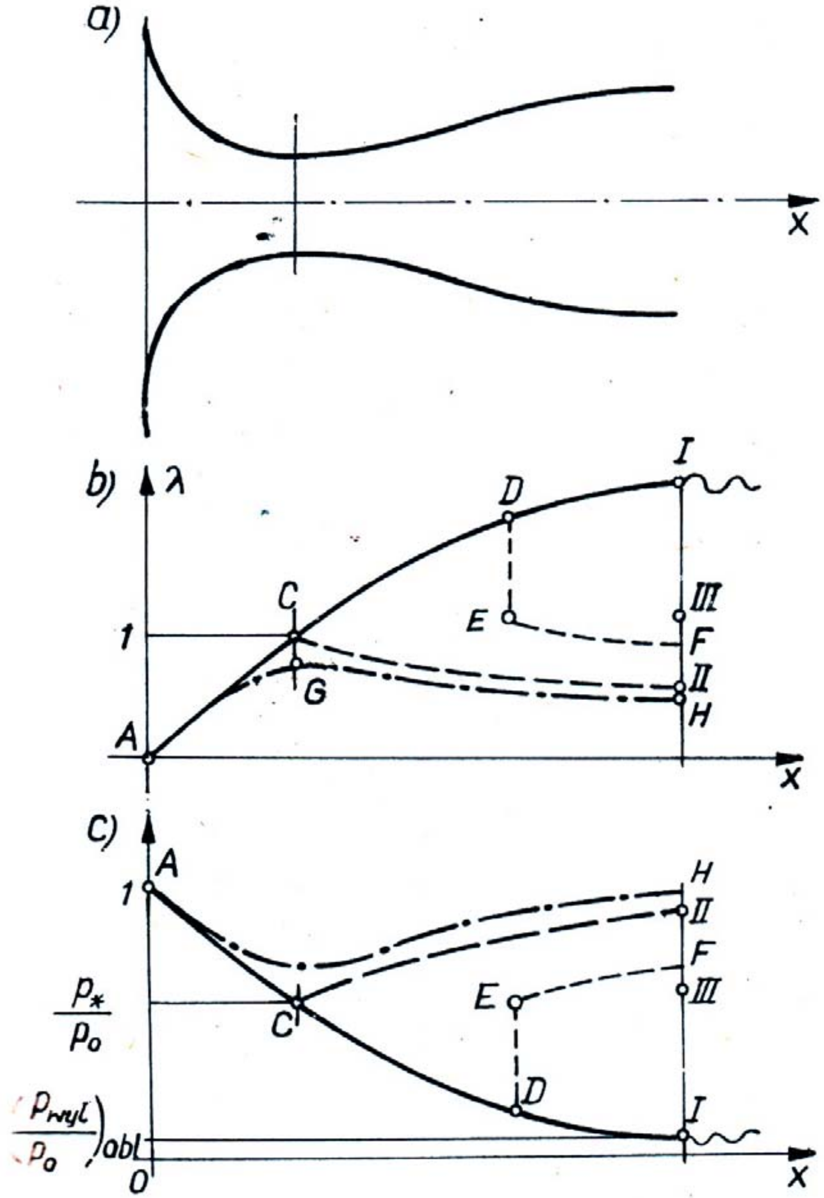


STAGNATION PRESSURE LOST
P0_2 / P0_1
Ma_1
0
0.5
1
1.5
2
2.5
3
3.5
4

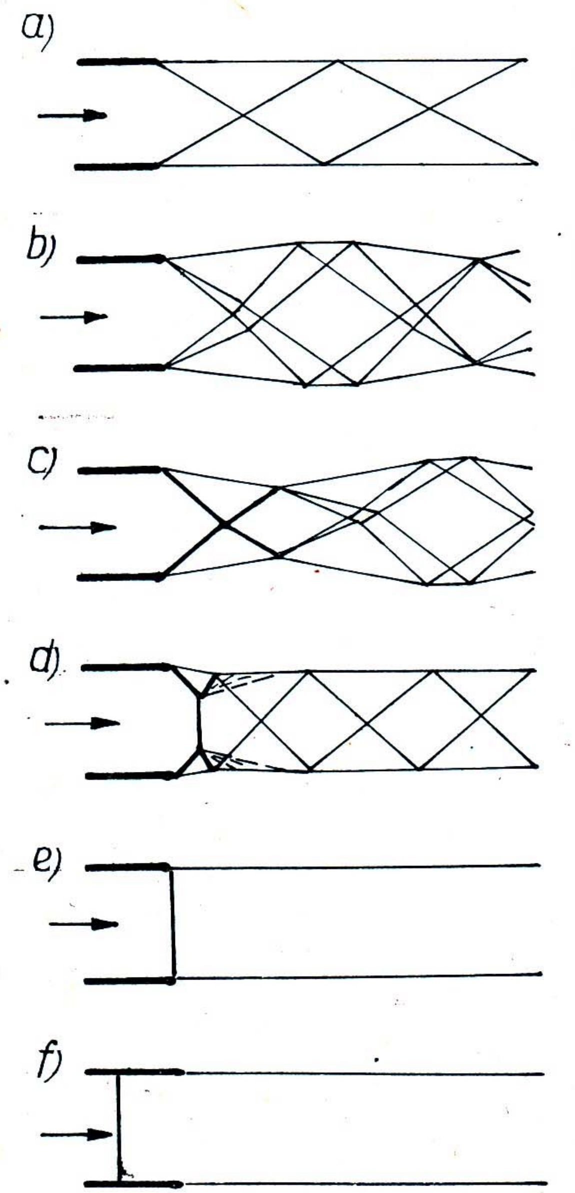


STAGNATION PRESSURE LOST
p01/p02
P0_2 / P0_1
Ma_1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
1
2
3
4
5
6

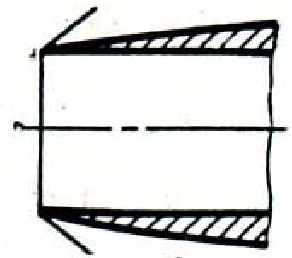

PRESSURE RATIO
p01/p02
p2/p1
P0_2 / P0_1 , P2 / P1
Ma_1
0
1
2
3
4
5
6
7
8
9
10
1
2
3
4
5
6



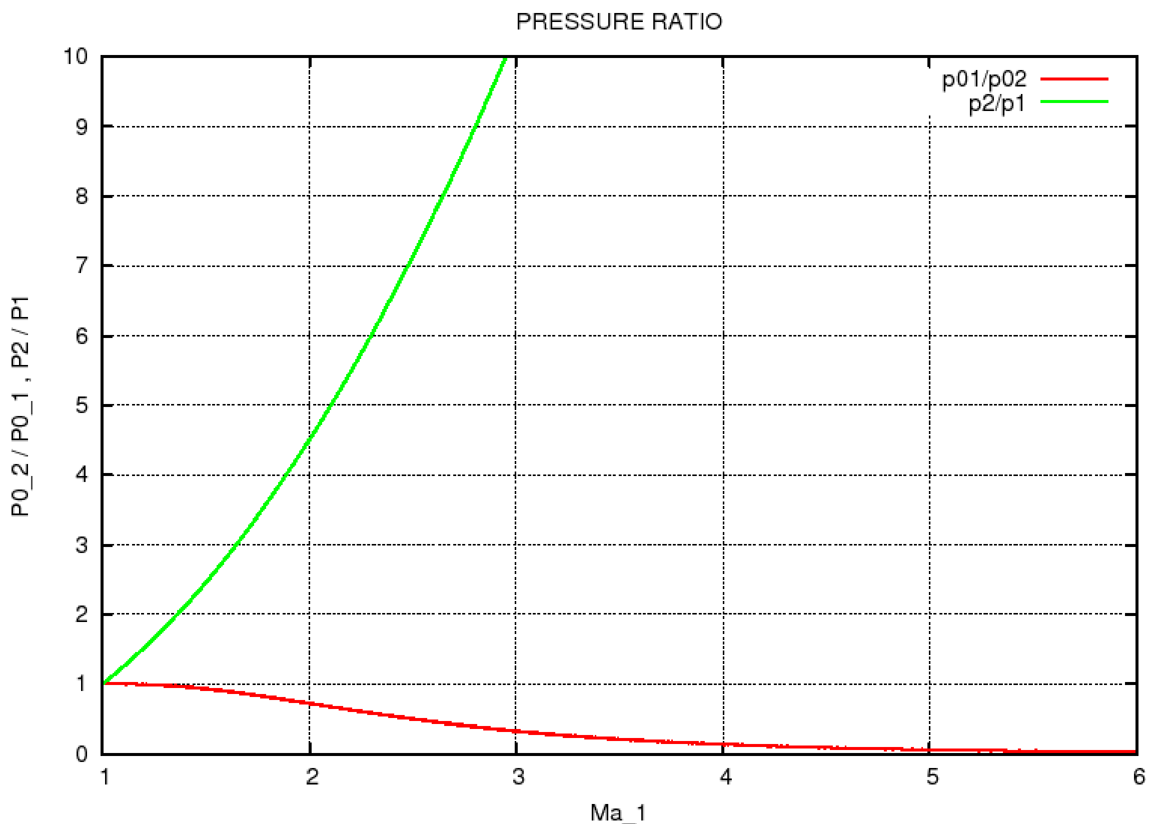
a)
x
b)
λ
I
D
C
1
E
III
F
II
G
H
A
x
c)
A
H
1
II
F
$\frac{p_*}{p_0}$
E
III
C
D
I
$\left(\frac{p_{wyl}}{p_0}\right)_{obl}$
0
x

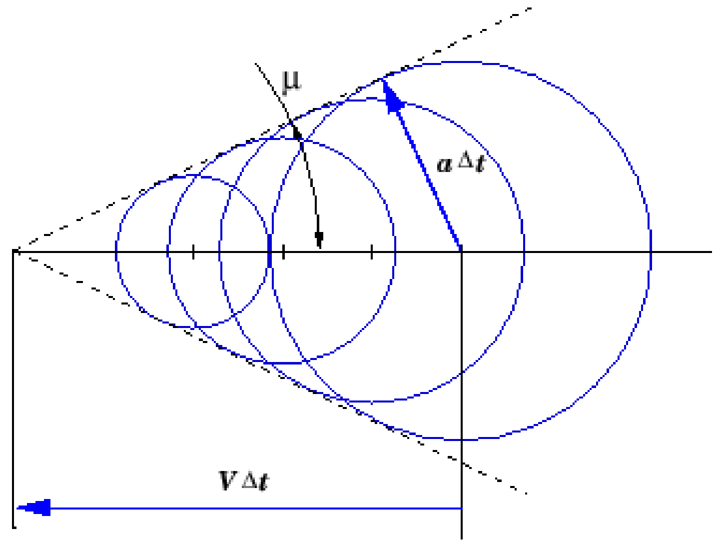


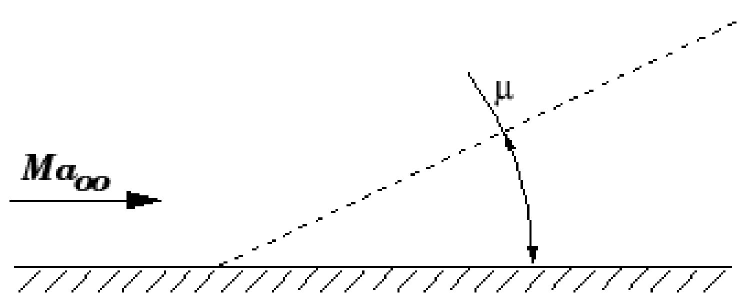
a)
b)
c)
d)
e)
f)



1

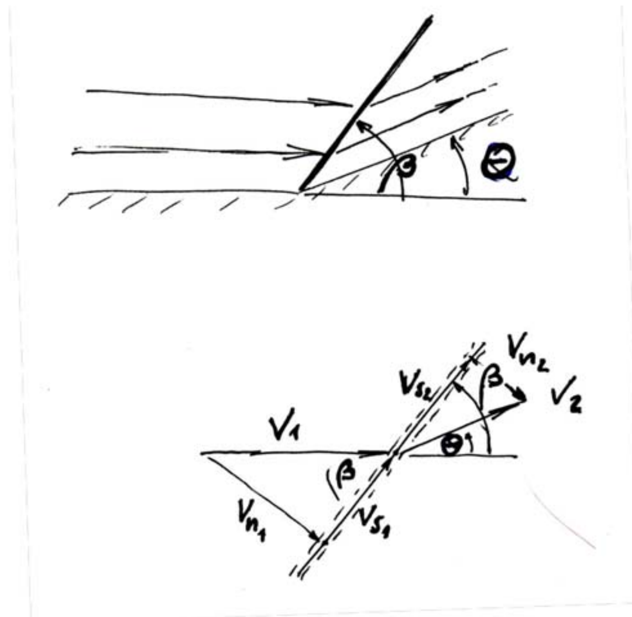






$$\sin(\mu) = a / U = \frac{1}{U / a} = \frac{1}{Ma}$$

GAS DYNAMCS – OBLIQUE SHOCK WAVE



$$\rho_1 V_{n1} = \rho_2 V_{n2} \tag{1}$$

$$\rho_1 V_{n1}^2 + p_1 = \rho_2 V_{n2}^2 + p_2 \tag{2a}$$

$$\rho_1 V_{n1} V_{s1} = \rho_2 V_{n2} V_{s2} \tag{2b}$$

$$\frac{1}{2}\left(V_{n1}^2 + V_{s1}^2\right) + \frac{a_1^2}{k-1} = \frac{1}{2}\left(V_{n2}^2 + V_{s2}^2\right) + \frac{a_2^2}{k-1} = \frac{a_0^2}{k-1} \tag{3}$$

$$\rho_1 V_{n1} V_{s1} = \rho_2 V_{n2} V_{s2}$$

$$\rho_1 V_{n1} = \rho_2 V_{n2} \quad \rightarrow \quad V_{s1} \equiv V_{s2} = V_s \tag{2b1}$$

$$\rho_1 V_{n1} = \rho_2 V_{n2} \tag{1}$$

$$\rho_1 V_{n1}^2 + p_1 = \rho_2 V_{n2}^2 + p_2 \tag{2a}$$

$$\frac{1}{2} V_{n1}^2 + \frac{a_1^2}{k-1} = \frac{1}{2} V_{n2}^2 + \frac{a_2^2}{k-1} \quad = \quad \frac{a_0^2}{k-1} - \frac{1}{2} V_s^2 \tag{3}$$

$$\rho_1 V_{n1} V_{s1} = \rho_2 V_{n2} V_{s2}$$

$$\rho_1 V_{n1} = \rho_2 V_{n2} \quad \rightarrow \quad V_{s1} \equiv V_{s2} = V_s \tag{2b1}$$

$$\rho_1 V_{n1} = \rho_2 V_{n2} \tag{1}$$

$$\rho_1 V_{n1}^2 + p_1 = \rho_2 V_{n2}^2 + p_2 \tag{2a}$$

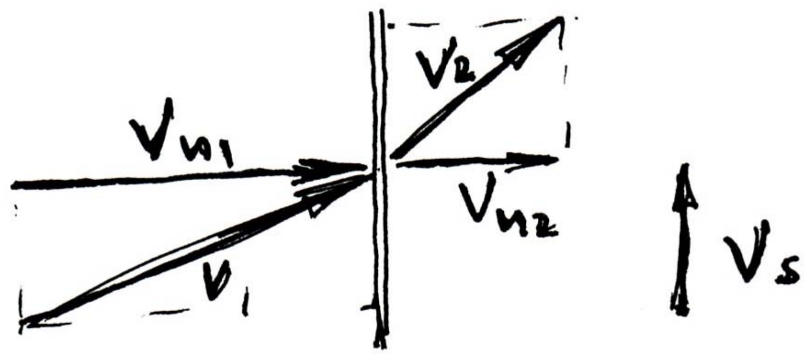
$$\frac{1}{2} V_{n1}^2 + \frac{a_1^2}{k-1} = \frac{1}{2} V_{n2}^2 + \frac{a_2^2}{k-1} \quad = \quad \frac{a_0^2}{k-1} - \frac{1}{2} V_s^2 = \frac{a_0^{'2}}{k-1} \tag{3}$$

$$\rho_1 U_1 = \rho_2 U_2 \tag{1}$$

$$\rho_1 U_1^2 + p_1 = \rho_2 U_2^2 + p_2 \tag{2}$$

$$\frac{1}{2} U_1^2 + \frac{a_1^2}{k-1} = \frac{1}{2} U_2^2 + \frac{a_2^2}{k-1} \tag{3}$$

(NORMAL SHOCK WAVE IN $\boldsymbol{n}$ DIRECTION, NEGLECTING $\boldsymbol{V_s}$



$$V_{n2} \cdot V_{n1} = a^{*'2} = a^{*2} - \frac{k-1}{k+1} V_s^2$$

$$V_1 \cdot V_2 = \sqrt{V_{n1}^2 + V_s^2}\sqrt{V_{n2}^2 + V_s^2} =$$

$$\sqrt{\left(V_{n1}V_{n2} + V_s^2\right)^2 + \left(V_{n1} - V_{n2}\right)^2 V_s^2} \geq V_{n1}V_{n2} + V_s^2 =$$

$$a^{*2} - \frac{k-1}{k+1}V_s^2 + V_s^2 = a^{*2} + \frac{2}{k+1}V_s^2$$

$$\geq a^{*2}$$

$$Ma_1 > 1 \quad ; \quad Ma_2 < 1 \quad \text{lub} \quad Ma_2 > 1$$

$$\frac{p_2}{p_1} = \frac{2k}{k+1}\left(Ma_1 \cdot \sin\beta\right)^2 - \frac{k-1}{k+1} = \frac{2k}{k+1}\left(\left(Ma_1 \cdot \sin\beta\right)^2 - 1\right) + 1$$

$$\frac{\rho_1}{\rho_2} = \frac{2 + (k-1)\left(Ma_1 \cdot \sin\beta\right)^2}{(k+1)\left(Ma_1 \cdot \sin\beta\right)^2} = \frac{(k-1)\left(\left(Ma_1 \cdot \sin\beta\right)^2 - 1\right) + k + 1}{(k+1)\left(Ma_1 \cdot \sin\beta\right)^2}$$

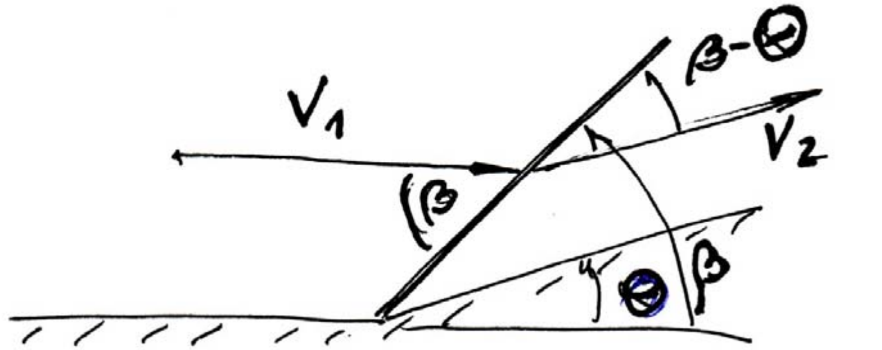
$$S_2 - S_1 = c_v \ln\left(\left(\frac{2k}{k+1}\left(\left(Ma_1 \cdot \sin\beta\right)^2 - 1\right) + 1\right) \cdot \left(\frac{(k-1)\left(\left(Ma_1 \cdot \sin\beta\right)^2 - 1\right) + k + 1}{(k+1)\left(Ma_1 \cdot \sin\beta\right)^2}\right)^k\right)$$

$$S_2 - S_1 = \frac{2}{3}\frac{kR}{(k+1)^2}\left(\left(Ma_1 \cdot \sin\beta\right)^2 - 1\right)^3 + \cdots$$

$$\frac{p_{O2}}{p_{O1}} = \left(\frac{2k}{k+1} \left(\left(Ma_1 \cdot \sin\beta \right)^2 - 1 \right) + 1 \right)^{-1/(k-1)} \cdot \left(\frac{(k-1)\left(\left(Ma_1 \cdot \sin\beta \right)^2 - 1 \right) + k + 1}{(k+1)\left(Ma_1 \cdot \sin\beta \right)^2} \right)^{-k/(k-1)}$$

$$\frac{p_{O2}}{p_{O1}} = \exp\left(-\frac{2}{3} \frac{k}{(k+1)^2} \left(\left(Ma_1 \cdot \sin\beta \right)^2 - 1 \right)^3 + \cdots \right)$$

β ?



$$V_s = V_1 \cdot \cos\beta = V_2 \cdot \cos(\beta\text{-}\theta)$$

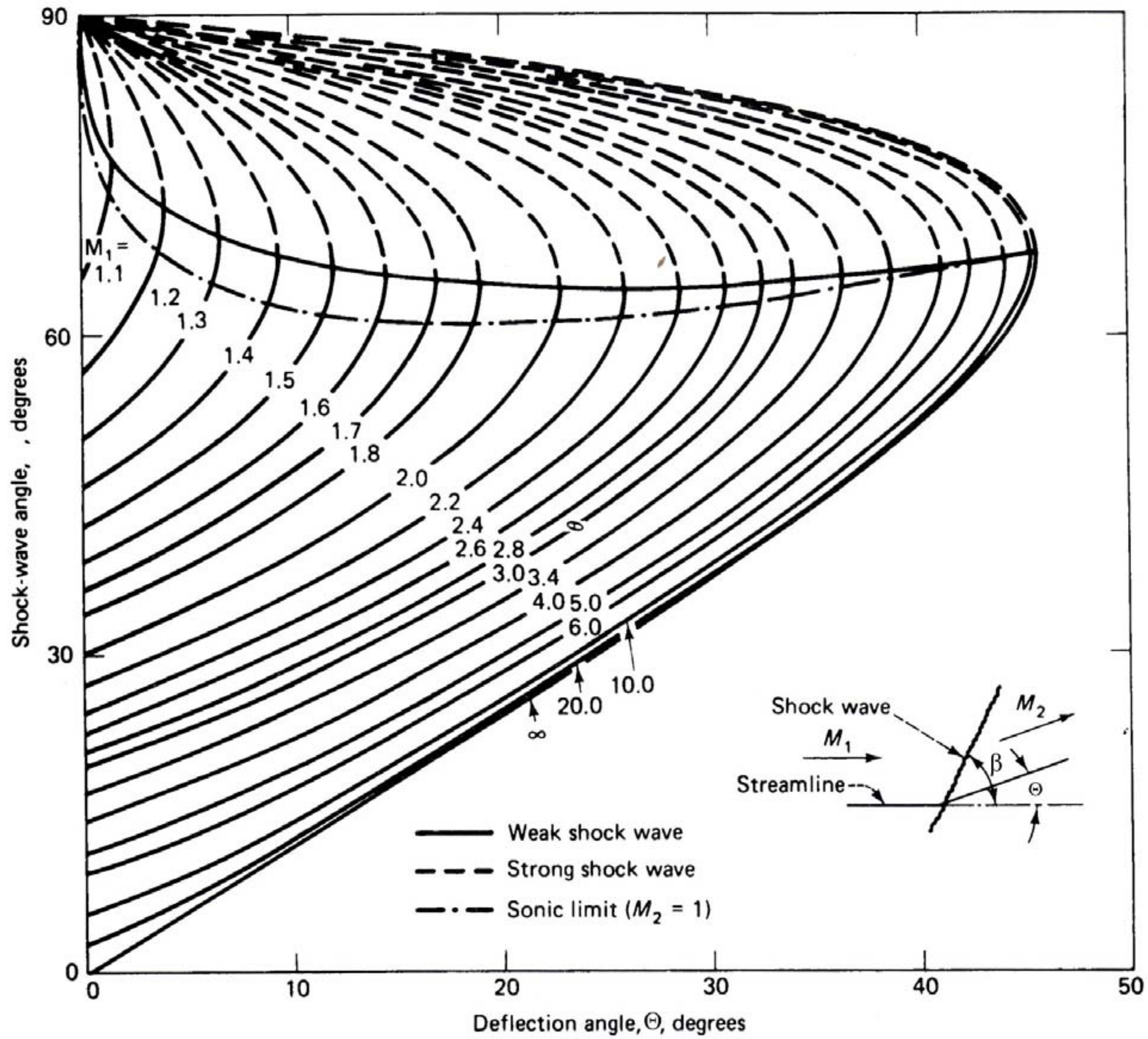
$$V_{n1} = V_1 \cdot \sin\beta$$

$$V_{n2} = V_2 \cdot \sin(\beta\text{-}\theta)$$

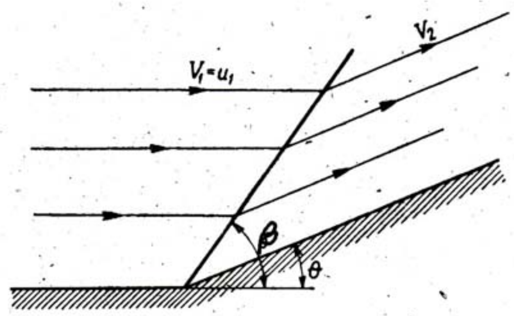
$$\tan\beta = \frac{V_{n1}}{V_s}$$

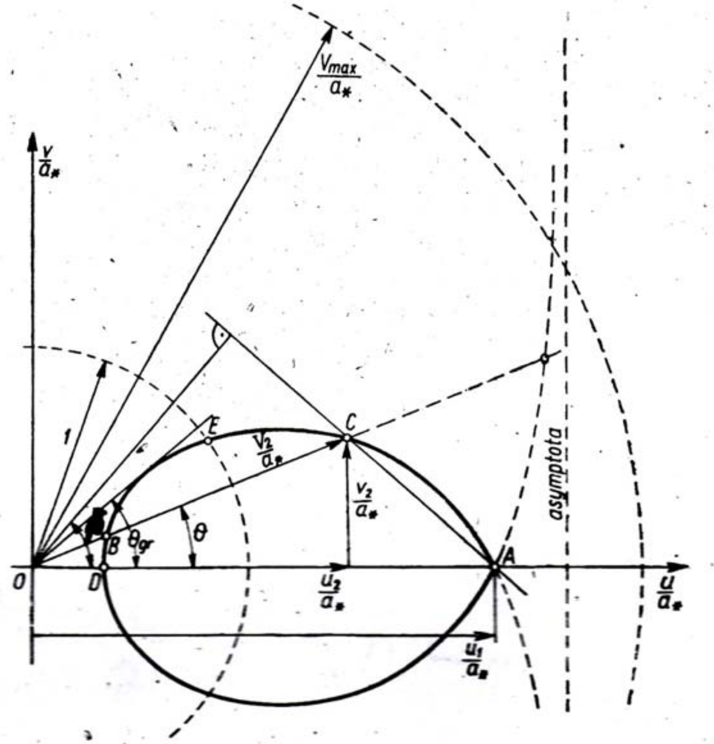
$$\tan(\beta\text{-}\theta) = \frac{V_{n2}}{V_s}$$

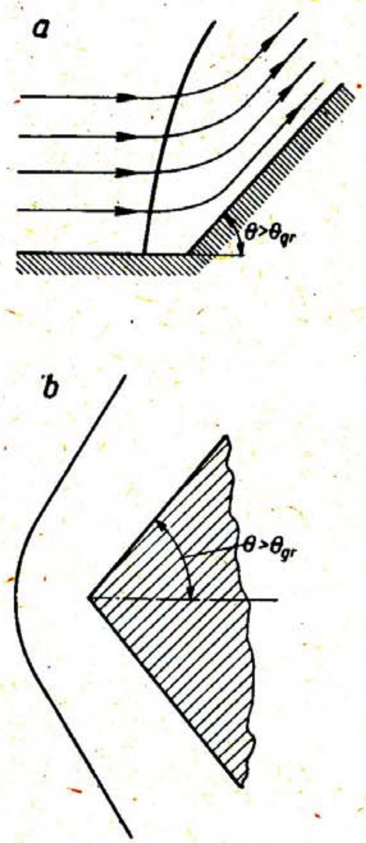
$$\frac{\tan(\beta-\theta)}{\tan\beta} = \frac{V_{n2}}{V_{n1}} = \frac{\rho_1}{\rho_2} = \frac{2+(k-1)(Ma_1\cdot\sin\beta)^2}{(k+1)(Ma_1\cdot\sin\beta)^2}$$

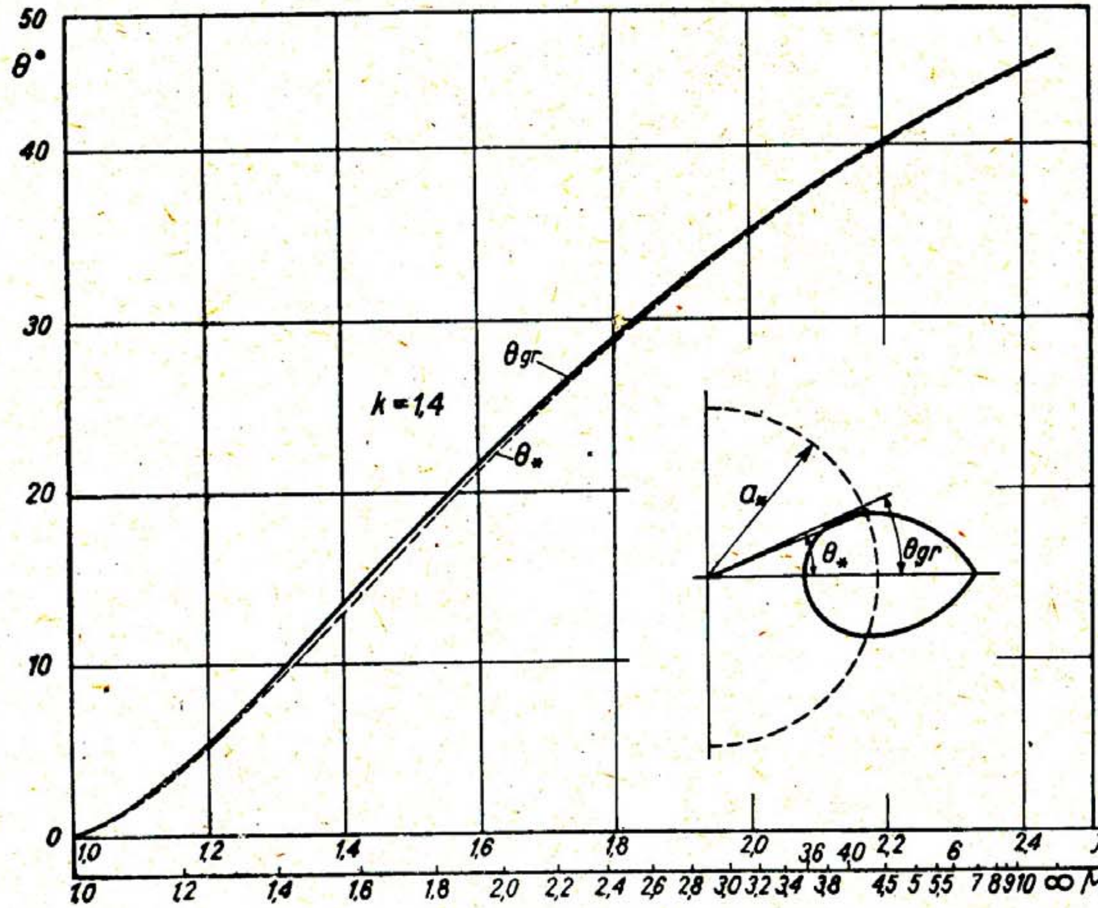


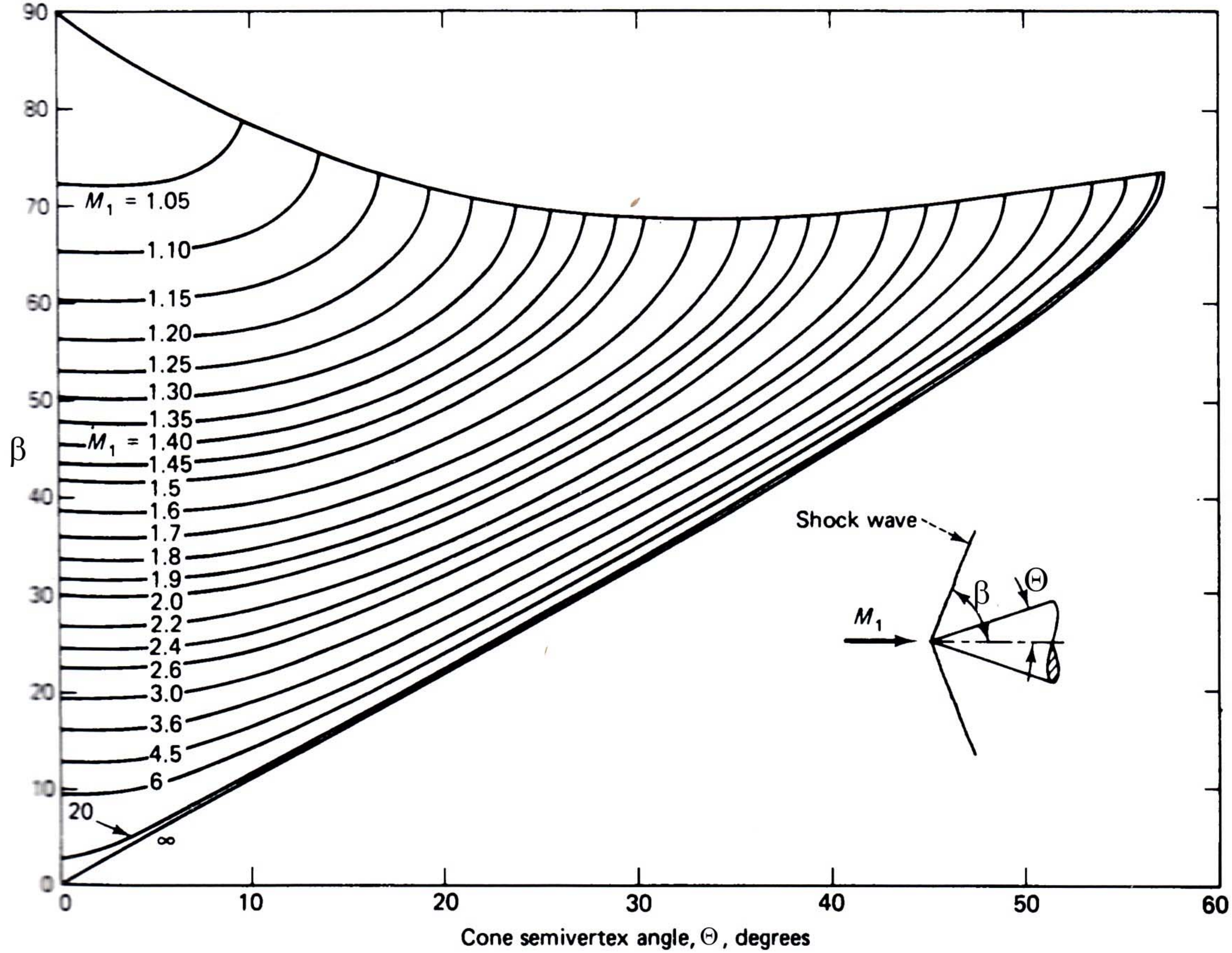
90
60
30
0
0
10
20
30
40
50
Shock-wave angle, , degrees
Deflection angle, Θ, degrees
M1 =
1.1
1.2
1.3
1.4
1.5
1.6
1.7
1.8
2.0
2.2
2.4
2.6 2.8
3.0 3.4
4.0 5.0
6.0
10.0
20.0
∞
θ
Shock wave
Streamline
M1
M2
β
Θ
Weak shock wave
Strong shock wave
Sonic limit (M2 = 1)


V1=u1
V2
β
θ


v/a*
Vmax/a*
1
E
C
V2/a*
v2/a*
asymptota
B
θgr
θ
O
D
u2/a*
A
u/a*
u1/a*


a
θ>θgr
b
θ>θgr


θ°
50
40
30
20
10
0
k=1,4
θgr
θ*
a*
θ*
θgr
1,0 1,2 1,4 1,6 1,8 2,0 2,2 2,4 λ1
1,0 1,2 1,4 1,6 1,8 2,0 2,2 2,4 2,6 2,8 3,0 3,2 3,4 3,6 3,8 4,0 4,5 5 5,5 6 7 8 9 10 ∞ M1



90
80
70
60
50
40
30
20
10
0
β
M_1 = 1.05
1.10
1.15
1.20
1.25
1.30
1.35
M_1 = 1.40
1.45
1.5
1.6
1.7
1.8
1.9
2.0
2.2
2.4
2.6
3.0
3.6
4.5
6
20
∞
Shock wave
β
Θ
M_1
0
10
20
30
40
50
60
Cone semivertex angle, Θ, degrees

$$\frac{\tan(\beta\text{-}\theta)}{\tan\beta} = \frac{V_{n2}}{V_{n1}} = \frac{\rho_1}{\rho_2} = \frac{\rho_1}{\rho_2} = \frac{2+(k-1)(Ma_1\cdot\sin\beta)^2}{(k+1)(Ma_1\cdot\sin\beta)^2}$$

/*(k+1)/2

$(Ma_1\sin\beta)^2$

$$\frac{(k+1)}{2}(Ma_1\cdot\sin\beta)^2\frac{\tan(\beta\text{-}\theta)}{\tan\beta} = \frac{(k-1)}{2}(Ma_1\cdot\sin\beta)^2+1=$$

$$\left(\frac{(k+1)}{2}-1\right)(Ma_1\cdot\sin\beta)^2+1=\frac{(k+1)}{2}(Ma_1\cdot\sin\beta)^2+1-(Ma_1\cdot\sin\beta)^2$$

$$\frac{(k+1)}{2}(Ma_1\cdot\sin\beta)^2\left(\frac{\tan(\beta\text{-}\theta)}{\tan\beta}-1\right) = 1-(Ma_1\cdot\sin\beta)^2$$

$$(Ma_1 \cdot \sin\beta)^2 - 1 \quad = \frac{(k+1)}{2} Ma_1^2 \cdot \sin^2\beta \left(1 - \frac{\tan(\beta\text{-}\theta)}{\tan\beta} \right)$$

$$(Ma_1 \cdot \sin\beta)^2 - 1 \quad = \frac{(k+1)}{2} Ma_1^2 \cdot \sin^2\beta \left(1 - \frac{\tan(\beta\text{-}\theta)}{\tan\beta} \right) =$$

$$\frac{(k+1)}{2} Ma_1^2 \left(\sin^2\beta - \frac{\sin(\beta\text{-}\theta)}{\cos(\beta\text{-}\theta)} \frac{\cos\beta}{\sin\beta} \sin^2\beta \right) = \ldots = \frac{(k+1)}{2} Ma_1^2 \frac{\sin\beta \sin\theta}{\cos(\beta\text{-}\theta)}$$

$$(Ma_1 \cdot \sin\beta)^2 - 1 = \frac{(k+1)}{2} Ma_1^2 \frac{\sin\beta \sin\theta}{\cos(\beta\text{-}\theta)}$$

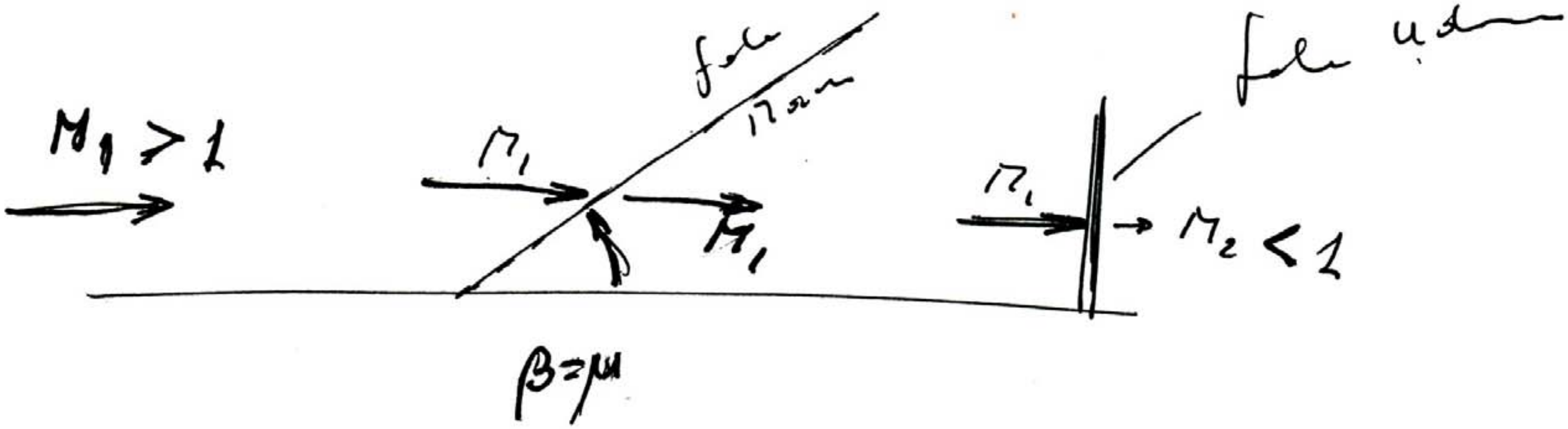
assuming: $\Theta \ll 1$

$$(Ma_1 \cdot \sin\beta)^2 - 1 \approx \frac{(k+1)}{2} Ma_1^2 \frac{\sin\beta}{\cos\beta} \theta$$

$$\left(\left(Ma_1 \cdot \sin\beta\right)^2 - 1\right)\cot\beta \approx \frac{(k+1)}{2} Ma_1^2 \cdot \theta$$

$$\theta = 0 \quad \rightarrow \quad \left(Ma_1 \cdot \sin\beta\right)^2 - 1 = 0 \quad \text{or} \quad \cot\beta = 0$$

$$\beta = \pm \sin^{-1}\left(\frac{1}{Ma_1}\right) = \mu \quad \text{or} \quad \beta = 90^o$$



$$\beta = \mu + \delta \qquad \mu = \mathbf{arcsin}\left(\frac{\mathbf{1}}{\boldsymbol{Ma_1}}\right)$$

$$\mathbf{sin}\,\mu = \frac{\mathbf{1}}{\boldsymbol{Ma_1}}$$

$$\mathbf{cos}\,\mu = \frac{\sqrt{\boldsymbol{Ma_1^2} - \mathbf{1}}}{\boldsymbol{Ma_1}}$$

$$\mathbf{sin}\,\beta = \mathbf{sin}\left(\mu + \delta\right) = \mathbf{sin}\,\beta\,\underbrace{\mathbf{cos}\,\delta}_{\sim 1} + \mathbf{cos}\,\beta\,\underbrace{\mathbf{sin}\,\delta}_{\sim\delta} \simeq \frac{\mathbf{1}}{\boldsymbol{Ma_1}} + \frac{\sqrt{\boldsymbol{Ma_1^2} - \mathbf{1}}}{\boldsymbol{Ma_1}}\delta$$

$$\mathbf{cos}\,\beta = \mathbf{cos}\left(\mu + \delta\right) = \mathbf{cos}\,\beta\,\underbrace{\mathbf{cos}\,\delta}_{\sim 1} - \mathbf{sin}\,\beta\,\underbrace{\mathbf{sin}\,\delta}_{\sim\delta} \simeq \frac{\sqrt{\boldsymbol{Ma_1^2} - \mathbf{1}}}{\boldsymbol{Ma_1}} - \frac{\mathbf{1}}{\boldsymbol{Ma_1}}\delta$$

$$\boldsymbol{ctg}\beta \approx \frac{\sqrt{\boldsymbol{Ma_1^2} - \mathbf{1}} - \delta}{\delta\sqrt{\boldsymbol{Ma_1^2} - \mathbf{1}} + \mathbf{1}} \approx \sqrt{\boldsymbol{Ma_1^2} - \mathbf{1}} - \boldsymbol{Ma_1^2} \cdot \delta$$

$$\left(\left(Ma_1 \cdot \sin\beta\right)^2 - 1\right) ctg\beta \approx \frac{(k+1)}{2} Ma_1^2 \cdot \theta$$

$$\delta \approx \ldots = \frac{k+1}{4} \frac{Ma_1^2}{Ma_1^2 - 1} \theta$$

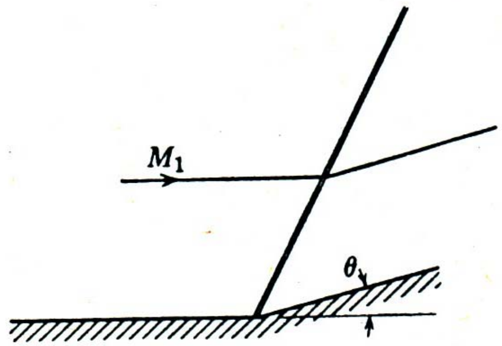
$$\beta \approx \mu + \frac{k+1}{4} \frac{Ma_1^2}{Ma_1^2 - 1} \theta$$

$$Ma_{n1}^2 = \left(Ma_1 \cdot \sin\beta\right)^2 \approx 1 + \frac{k+1}{2} \frac{Ma_1^2}{\sqrt{Ma_1^2 - 1}} \theta$$

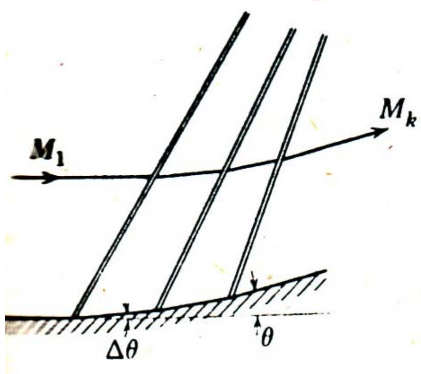
$$\Delta S \sim \left(Ma_{n1}^2 - 1\right)^3 \sim \theta^3$$

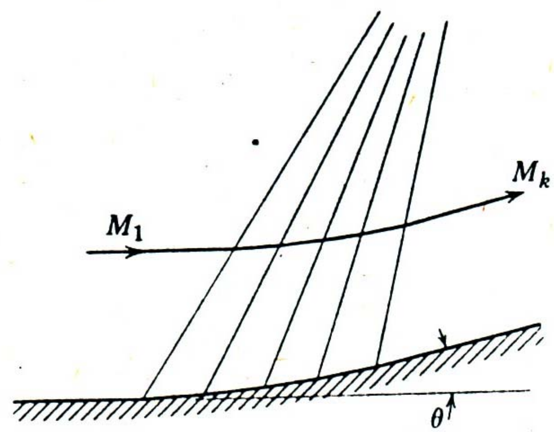
$$case: \quad init: 0 \quad final: \theta \quad \rightarrow \quad \Delta S \sim \theta^3$$

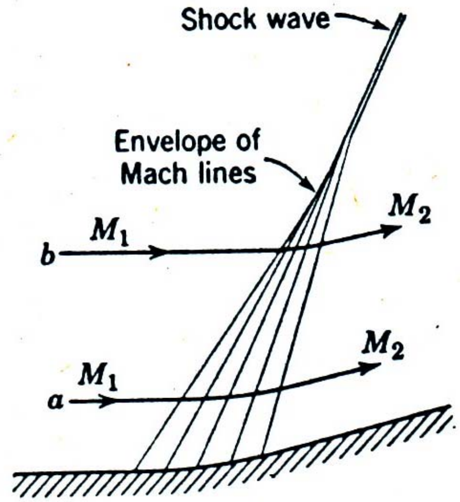
$$case: \quad init: 0 \quad final: \theta \quad step \ \Delta\theta \quad \rightarrow \quad \Delta S \sim N(\Delta\theta)^3 = N\left(\frac{\theta}{N}\right)^3 = \frac{1}{N^2}\theta^3$$



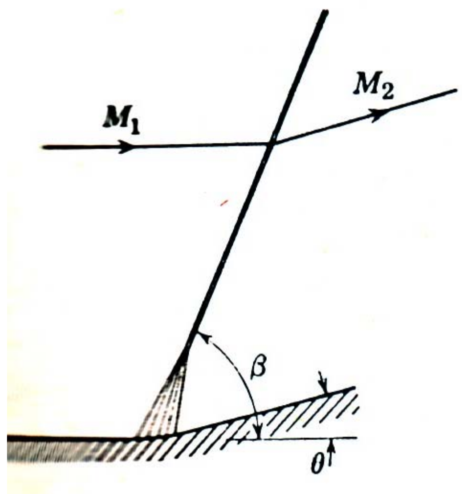
(a)



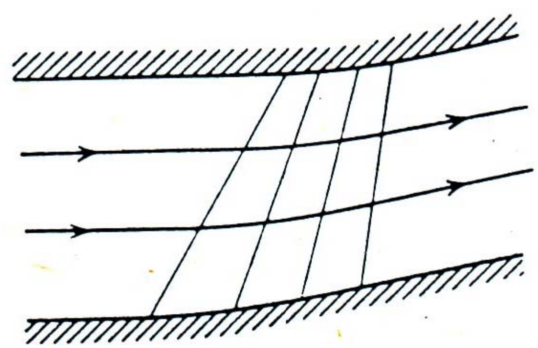



Shock wave
Envelope of Mach lines
M_1
M_2
b
M_1
M_2
a

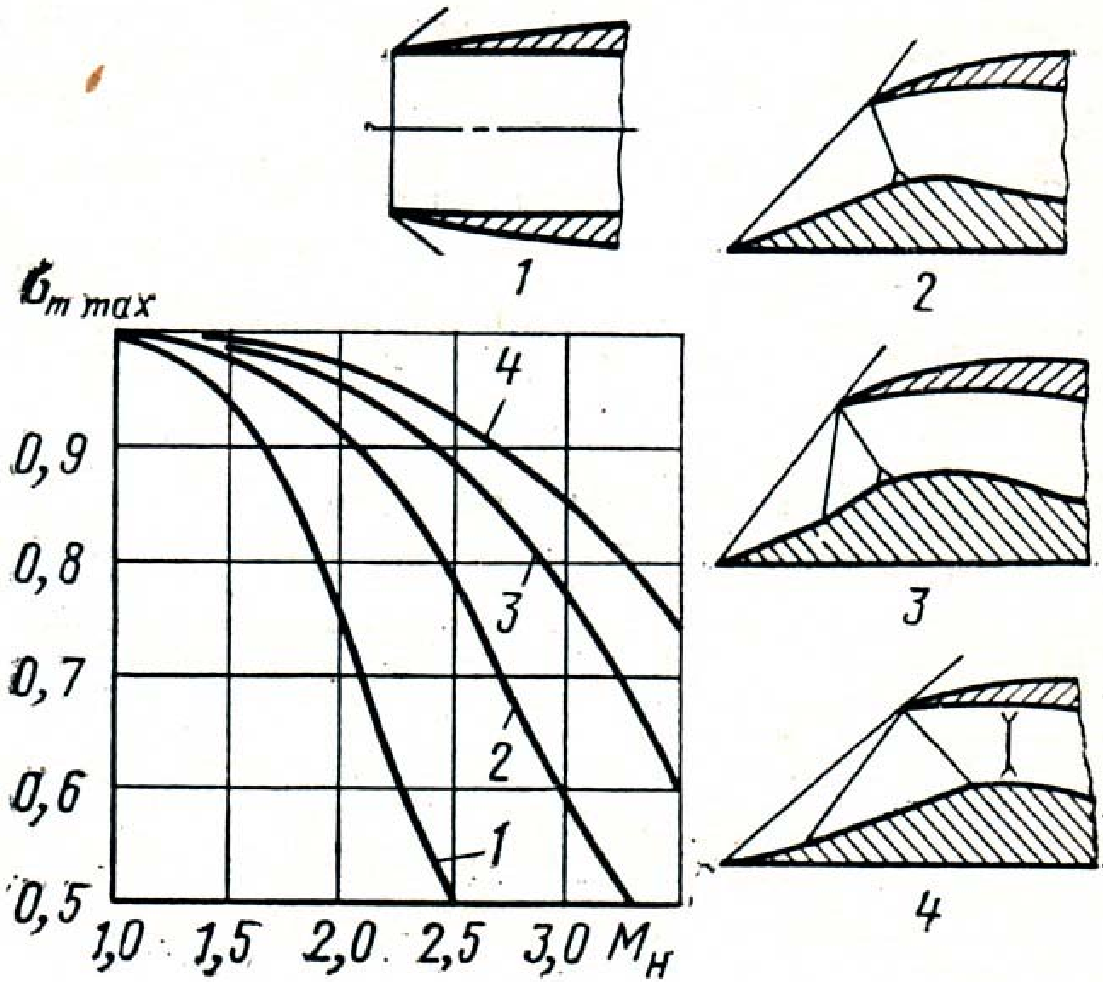
(a)


M_1
M_2
β
θ

(b)



(c)



$\sigma_{m\,max}$
0,9
0,8
0,7
0,6
0,5
1,0
1,5
2,0
2,5
3,0
M_H
1
2
3
4

$$\beta \approx \mu + \frac{k+1}{2}\frac{Ma_1^2}{\sqrt{Ma_1^2 - 1}}\theta$$

$$\frac{p_2}{p_1} = 1 + \frac{2k}{k+1}\left(\left(Ma_1 \cdot \sin\beta\right)^2 - 1\right) = \ldots \approx 1 + k\frac{Ma_1^2}{\sqrt{Ma_1^2 - 1}}\theta$$

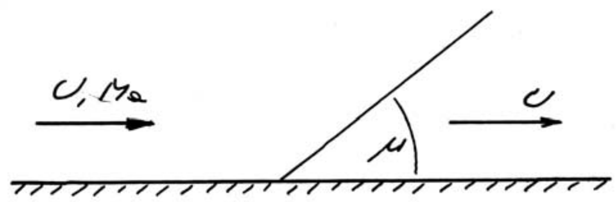
$$\frac{p_2 - p_1}{p_1} \approx k\frac{Ma_1^2}{\sqrt{Ma_1^2 - 1}}\theta$$

$$Cp = \frac{p - p_\infty}{q_\infty} = \frac{p_\infty}{\rho_\infty V_\infty^2 / 2}\frac{p - p_\infty}{p_\infty} \approx$$

$$\frac{p_\infty / \rho_\infty}{V_\infty^2 / 2} \cdot k\frac{Ma_1^2}{\sqrt{Ma_1^2 - 1}}\theta =$$

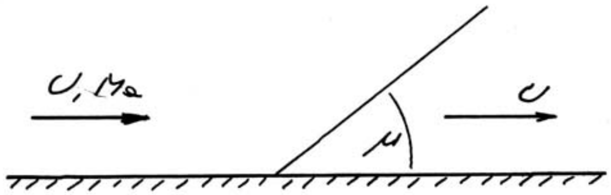
$$\frac{kRT_\infty}{V_\infty^2 / 2}\frac{Ma_1^2}{\sqrt{Ma_1^2 - 1}}\theta = \frac{2\theta}{\sqrt{Ma_1^2 - 1}}$$

SUPERSONIC EXPANSION BY TURNING (PRANDTL'A-MEYER EXPANSION)

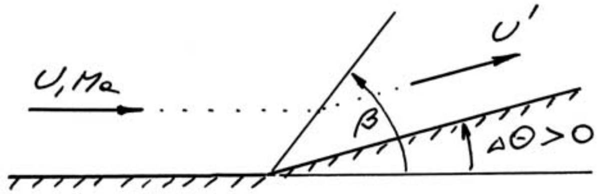


$$\mu = \mathbf{arcsin}\left(\frac{\mathbf{1}}{\boldsymbol{Ma}}\right) \qquad \Delta \boldsymbol{S} = \mathbf{0} \quad \beta = \mu$$

SUPERSONIC EXPANSION BY TURNING (PRANDTL'A-MEYER EXPANSION)

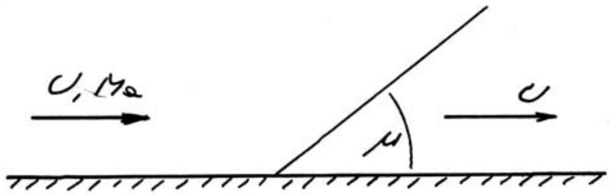


$$\mu = \mathbf{arcsin}\left(\frac{\mathbf{1}}{\boldsymbol{Ma}}\right) \qquad \Delta \boldsymbol{S} = \mathbf{0} \quad \beta = \mu$$

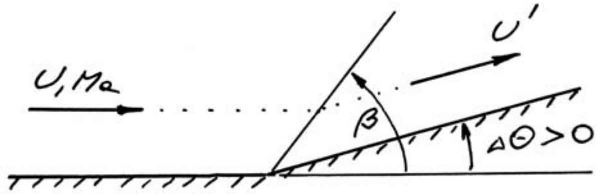


$$\Delta \boldsymbol{S} > \mathbf{0} \quad \beta > \mu$$

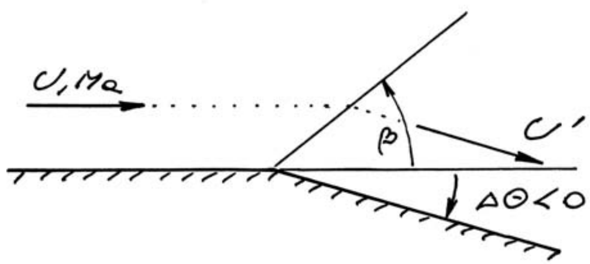
SUPERSONIC EXPANSION BY TURNING (PRANDTL'A-MEYER EXPANSION)



$$\mu = \arcsin\left(\frac{1}{Ma}\right) \qquad \Delta S = 0 \quad \beta = \mu$$

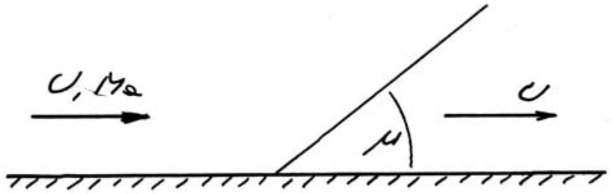


$$\Delta S > 0 \quad \beta > \mu$$

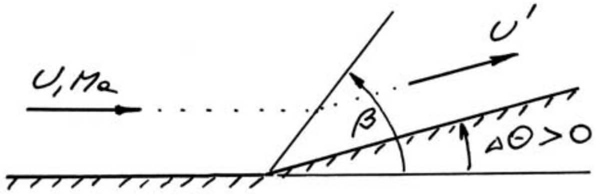


$$? \qquad \Delta S < 0 \quad \beta\,?$$

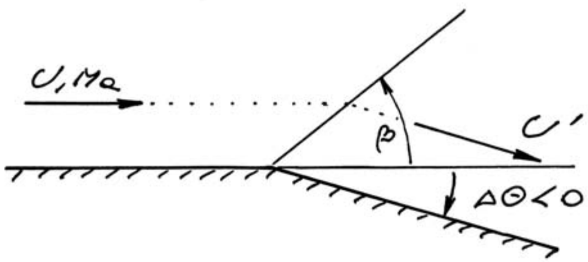
__SUPERSONIC EXPANSION BY TURNING (PRANDTL'A-MEYER EXPANSION)__



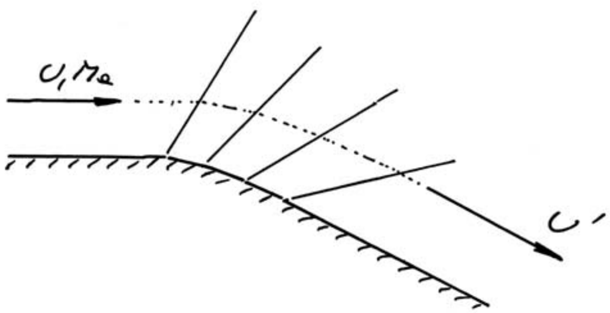
$$\mu = \arcsin\left(\frac{1}{Ma}\right) \qquad \Delta S = 0 \quad \beta = \mu$$



$$\Delta S > 0 \quad \beta > \mu$$



$$? \qquad \Delta S < 0 \quad \beta ?$$



$$\mu = \arcsin\left(\frac{1}{Ma}\right) \qquad \Delta S = 0 \quad \beta = \mu$$

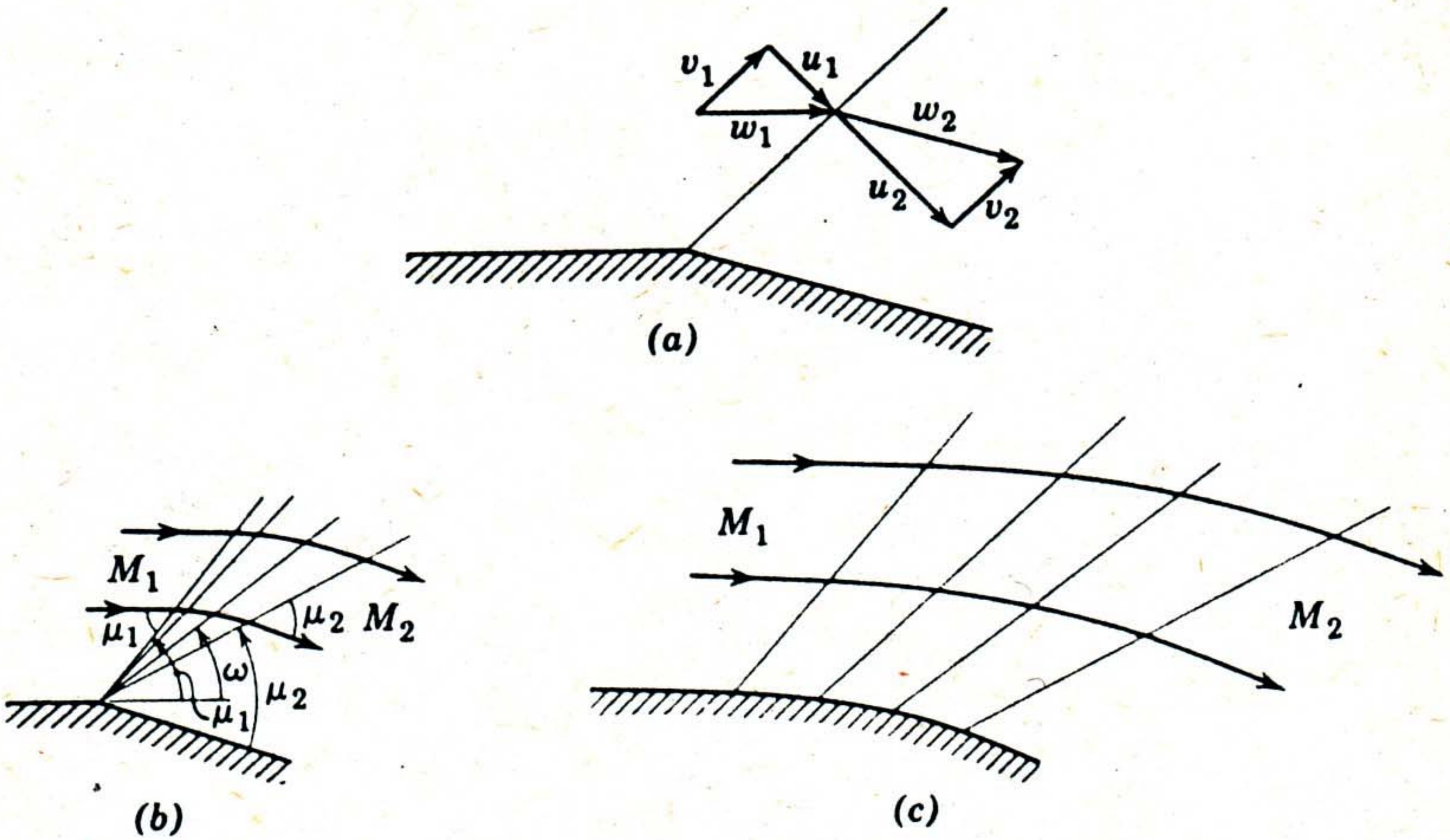


FIG. 4·8 Supersonic expansion. (*a*) Not possible on thermodynamic grounds; (*b*) centered expansion wave; (*c*) simple expansion.

$$V_s \equiv V_s' = V\cos(\beta) = V'\cos(\beta - \theta)$$

$$V' = V + \Delta V$$

$$V\cos(\beta) = (V + \Delta V)\left(\cos(\beta)\underbrace{\cos(\Delta\theta)}_{\sim 1} + \sin(\beta)\underbrace{\sin(\Delta\theta)}_{\sim \Delta\theta}\right)$$

$$\Delta V\cos(\beta) + V\sin(\beta)\Delta\theta \simeq 0$$

$$d\theta = -\frac{dV}{V} \;/\; \tan(\beta)$$

$$\Delta\theta' = -\Delta\theta$$

$$d\theta' = \frac{dV}{V} \; / \; \tan(\beta)$$

$$|\Delta\theta| \ll 1 \quad \rightarrow \quad \beta \rightarrow \mu$$

$$d\theta' = \frac{dV}{V} \; / \; \tan(\mu) = \sqrt{Ma^2 - 1}\,\frac{dV}{V}$$

$$V = a \cdot Ma = Ma \frac{a_0}{\sqrt{1 + \frac{k-1}{2} Ma^2}}$$

$$\frac{V}{a_0} = \frac{Ma}{\sqrt{1 + \frac{k-1}{2} Ma^2}}$$

$$\frac{dV}{a_0} = \frac{\sqrt{\ldots} - \frac{\frac{k-1}{2} 2Ma^2}{2\sqrt{\ldots}}}{\left(\sqrt{\ldots}\right)^2} d\,Ma = \ldots = \frac{d\,Ma}{\left(\sqrt{1 + \frac{k-1}{2} Ma^2}\right)^3}$$

$$\frac{dV}{V} = \frac{dV}{a_0} / \frac{V}{a_0} \quad = \frac{1}{1 + \frac{k-1}{2} Ma^2} \frac{d\,Ma}{Ma}$$

$$\frac{dV}{V} = \frac{1}{1+\frac{k-1}{2}Ma^2}\frac{d\,Ma}{Ma}$$

$$d\theta' = \sqrt{Ma^2-1}\,\frac{dV}{V} = \frac{\sqrt{Ma^2-1}}{1+\frac{k-1}{2}Ma^2}\frac{d\,Ma}{Ma}$$

$$\nu\left(Ma\right) = \int d\theta' = \int \frac{\sqrt{Ma^2-1}}{1+\frac{k-1}{2}Ma^2}\frac{d\,Ma}{Ma} = \ldots$$

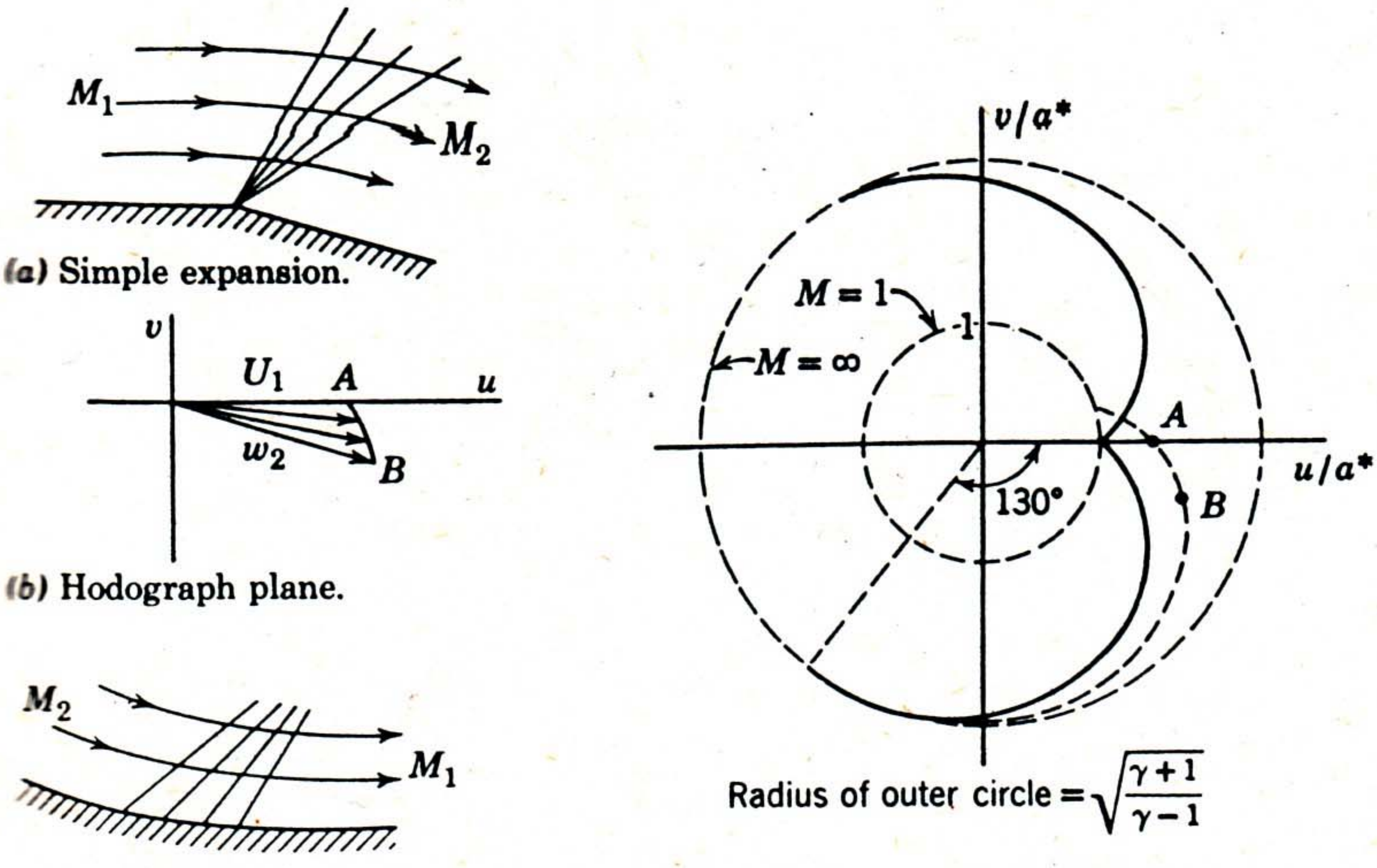
$$= \sqrt{\frac{k+1}{k-1}}\tan^{-1}\sqrt{\frac{k-1}{k+1}\left(Ma^2-1\right)} - \tan^{-1}\sqrt{Ma^2-1}$$

$$\nu\left(V/a^*\right)=\nu(\lambda)=\int d\theta'=\int\sqrt{Ma^2-1}\,\frac{dV}{V}=\ldots$$

$$=\sqrt{\frac{k+1}{k-1}}\tan^{-1}\sqrt{\frac{k-1}{k+1}\left(\frac{\lambda^2-1}{1-\frac{k-1}{k+1}\lambda^2}\right)}-\tan^{-1}\sqrt{\frac{\lambda^2-1}{1-\frac{k-1}{k+1}\lambda^2}}$$

$$\Delta\theta'=\nu(\lambda_2)-\nu(\lambda_1)\quad\rightarrow\quad\lambda_2\quad(Ma_2)$$

$$\nu(Ma_2\rightarrow\infty)=\left(\sqrt{\frac{k+1}{k-1}}-1\right)\frac{\pi}{2}\simeq 130^o$$



(a) Simple expansion.

(b) Hodograph plane.

(c) Simple compession.

(d) Prandtl–Meyer diagram.

FIG. 4·25 Representation of Prandtl-Meyer expansion in the hodograph plane.