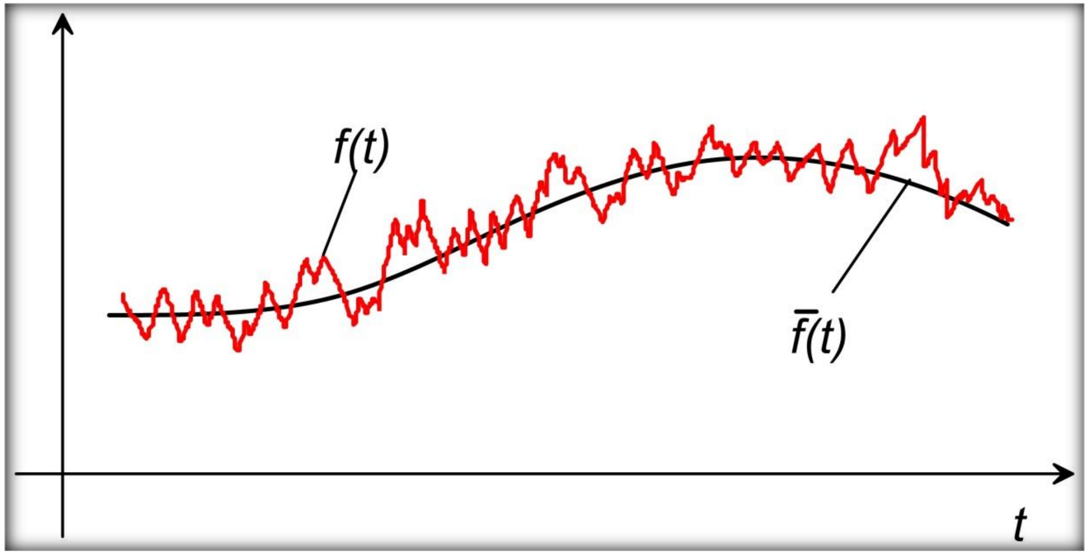
LECTURE 5

ELEMENTS OF THE BOUNDARY LAYER THEORY – PART 2

Fundamentals of turbulent flows modeling

Reynolds decomposition



$$f = \underbrace{\overline{f}}_{mean} + \underbrace{f'}_{pulsation}$$

An averaging procedure is assumed such that

$$\overline{\overline{f}} = \overline{f} \quad \Rightarrow \overline{f'} = 0$$

Reynolds averaging

Unsteady flow (slowly varying mean trend …)

$$\bar{f}(t,x)=\frac{1}{2T}\int_{t-T}^{t+T} f(\tau,\boldsymbol{x})d\tau$$

Averaging time – small when compared to the characteristic time scale of the mean tend, large when compared to the characteristic time scale of turbulent fluctuations.

Statistically steady flow …

$$\bar{f}(\boldsymbol{x})=\lim_{T\to 0}\frac{1}{2T}\int_{t-T}^{t+T} f(\tau,\boldsymbol{x})d\tau$$

The Reynolds averaging commutes with differentiation …

$$\overline{\partial_{x_k} f}(t,\boldsymbol{x}) = \frac{1}{2T}\int_{t-T}^{t+T} \partial_{x_k} f(\tau,\boldsymbol{x})d\tau = \partial_{x_k}\left[\frac{1}{2T}\int_{t-T}^{t+T} f(\tau,\boldsymbol{x})d\tau\right] = \partial_{x_k}\overline{f}(t,\boldsymbol{x})$$

$$\overline{\partial_t f}(t,\boldsymbol{x}) = \frac{1}{2T}\int_{t-T}^{t+T} \partial_t f(\tau,\boldsymbol{x})d\tau = \frac{1}{2T}\left[f(t+T,\boldsymbol{x}) - f(t-T,\boldsymbol{x})\right] =$$

$$= \frac{1}{2T}\left[\partial_t\int_0^{t+T} f(\tau,\boldsymbol{x})d\tau - \partial_t\int_0^{t-T} f(\tau,\boldsymbol{x})d\tau\right] = \partial_t\left[\frac{1}{2T}\int_{t-T}^{t+T} f(\tau,\boldsymbol{x})d\tau\right] = \partial_t\overline{f}(t,\boldsymbol{x})$$

Derivation of the (unsteady) Reynolds-averaged Navier-Stokes equations (U)RANS (only incompressible case)

We begin with …

$$\begin{cases} \frac{\partial}{\partial t}\upsilon_j + \frac{\partial}{\partial x_k}(\upsilon_j\upsilon_k) = -\frac{1}{\rho}\frac{\partial}{\partial x_j}p + \nu\frac{\partial^2}{\partial x_k \partial x_k}\upsilon_j \\ \frac{\partial}{\partial x_j}\upsilon_j = 0 \end{cases}$$

Reynolds decomposition …

$$\upsilon_k = \bar{\upsilon}_k + \upsilon'_k \quad , \quad p = \bar{p} + p'$$

After substitution the averaging is applied …

$$\frac{\partial}{\partial t}(\bar{\upsilon}_j + \upsilon'_j) + \frac{\partial}{\partial x_k}\left[(\bar{\upsilon}_j + \upsilon'_j)(\bar{\upsilon}_k + \upsilon'_k)\right] = -\frac{1}{\rho}\frac{\partial}{\partial x_j}(\bar{p} + p') + \nu\frac{\partial^2}{\partial x_k \partial x_k}(\bar{\upsilon}_j + \upsilon'_j)$$

$$\frac{\partial}{\partial t}\underbrace{\overline{(\bar{\upsilon}_j + \upsilon'_j)}}_{=\bar{\upsilon}_j} + \frac{\partial}{\partial x_k}\underbrace{\overline{(\bar{\upsilon}_j\bar{\upsilon}_k + \upsilon'_j\bar{\upsilon}_k + \upsilon'_k\bar{\upsilon}_j + \upsilon'_j\upsilon'_k)}}_{=\bar{\upsilon}_j\bar{\upsilon}_k + \overline{\upsilon'_j\upsilon'_k}} = -\frac{1}{\rho}\frac{\partial}{\partial x_j}\underbrace{\overline{(\bar{p} + p')}}_{=\bar{p}} + \nu\frac{\partial^2}{\partial x_k \partial x_k}\underbrace{\overline{(\bar{\upsilon}_j + \upsilon'_j)}}_{=\bar{\upsilon}_j}$$

We obtain ….

$$\rho\left[\frac{\partial}{\partial t}\bar{\upsilon}_j + \frac{\partial}{\partial x_k}(\bar{\upsilon}_j\bar{\upsilon}_k)\right] = -\frac{\partial}{\partial x_j}\bar{p} + \mu\frac{\partial^2}{\partial x_k \partial x_k}\bar{\upsilon}_j - \rho\frac{\partial}{\partial x_k}\overline{\upsilon'_j\upsilon'_k}$$

$$\frac{\partial}{\partial x_j}\overline{(\bar{\upsilon}_j + \upsilon'_j)} = 0 \quad \Rightarrow \quad \frac{\partial}{\partial x_j}\bar{\upsilon}_j = 0$$

The viscosity term can be written as follows (the Newtonian fluid) …

$$\mu \frac{\partial^2}{\partial x_k \partial x_k} \bar{\upsilon}_j = \frac{\partial}{\partial x_k} \left[2\mu \tfrac{1}{2} \left(\frac{\partial}{\partial x_k} \bar{\upsilon}_j + \frac{\partial}{\partial x_j} \bar{\upsilon}_k \right) \right] = \frac{\partial}{\partial x_k} 2\mu \bar{D}_{jk} = \frac{\partial}{\partial x_k} \bar{S}_{jk}$$

Define mass-specific **turbulent energy** $k = \tfrac{1}{2} \overline{\upsilon_j' \upsilon_j'}$ and the Reynolds tensor …

$$R_{jk} = -\rho \overline{\upsilon_j' \upsilon_k'} \quad \Rightarrow \quad tr\boldsymbol{R} = -\rho \overline{\upsilon_j' \upsilon_j'} = -2\rho k$$

Deviatoric part of $\boldsymbol{R}$ is the **turbulent stress tensor** …

$$T_{jk} = R_{jk} - \tfrac{1}{3} tr\boldsymbol{R}\, \delta_{jk} = -\rho \overline{\upsilon_j' \upsilon_k'} + \tfrac{2}{3} \rho k \delta_{jk}$$

Note that the trace of $\boldsymbol{T}$ is equal zero.

Two last terms in the right side of RANS Eq. can be transformed as follows

$$\mu \frac{\partial^2}{\partial x_k \partial x_k} \bar{\upsilon}_j - \rho \frac{\partial}{\partial x_k} \overline{\upsilon'_j \upsilon'_k} = \frac{\partial}{\partial x_k} (\underbrace{\bar{S}_{jk} + T_{jk}}_{S^T_{jk}} - \tfrac{2}{3} \rho k \delta_{jk}) = \frac{\partial}{\partial x_k} S^T_{jk} - \frac{\partial}{\partial x_j} (\tfrac{2}{3} \rho k)$$

Hence, the RANS equation can be written as

$$\rho \left[\frac{\partial}{\partial t} \bar{\upsilon}_j + \frac{\partial}{\partial x_k} (\bar{\upsilon}_j \bar{\upsilon}_k) \right] = - \frac{\partial}{\partial x_j} \underbrace{(\bar{p} + \tfrac{2}{3} \rho k)}_{\substack{turbulent \\ pressure}} + \frac{\partial}{\partial x_k} S^T_{jk}$$

Turbulent viscosity

There is no “natural” constitutive relation for the turbulent stress tensor – we have 6 additional unknowns and no additional equations! **The problem is unsolvable, unless it gets “closed” some way.**

Hypothesis (Boussinesq): the turbulent stress tensor can be related to the deformation rate of the mean flow in an analogous way as the “real” (molecular) stress tensor.

$$\boldsymbol{T} = 2\mu_T \bar{\boldsymbol{D}} \quad \Rightarrow \quad \boldsymbol{S}^T = 2(\mu + \mu_T)\bar{\boldsymbol{D}}$$

REMARKS:

- Deformation rate tensor $\bar{\boldsymbol{D}}$ **is defined for the mean flow**
- Postulated formula is mathematically consistent since both tensors $\boldsymbol{T}$ and $\bar{\boldsymbol{D}}$ have the zero trace.
- The quantity μ_T - called a **turbulent viscosity** – is not a physical property of the fluid! It characterizes the flow and as such it is – generally - dependent on both location and time.

Conclusion: we still need a method of evaluation of the turbulent viscosity in terms of a mean flow **!** Such methods are called **closure models** (e.g., Spalart-Almaras, $k-\varepsilon$ and many others).

Turbulent boundary layer (TBL)

Assumptions:

1. External flow - 2D
2. mean flow inside TBL – 2D
3. The Prandtl's assumptions as to the relative magnitude of different terms in the governing equations are still applicable.

Reynolds-averaged equation of motion in a streamwise x direction (traditional notation):

$$\rho\left(\bar{u}\frac{\partial}{\partial x}\bar{u}+\bar{\upsilon}\frac{\partial}{\partial y}\bar{u}\right)=-\frac{\partial}{\partial x}\bar{p}+\underbrace{\frac{\partial}{\partial x}\left(\mu\frac{\partial}{\partial x}\bar{u}-\rho\overline{u'^2}\right)}_{I}+\underbrace{\frac{\partial}{\partial y}\left(\mu\frac{\partial}{\partial y}\bar{u}-\rho\overline{u'\upsilon'}\right)}_{II}+\underbrace{\frac{\partial}{\partial z}\left(-\rho\overline{u'\upsilon'}\right)}_{III}$$

As sume that $II \gg I, III$ which leads to the simplified form

$$\rho\left(\bar{u}\frac{\partial}{\partial x}\bar{u}+\bar{\upsilon}\frac{\partial}{\partial y}\bar{u}\right)=-\frac{\partial}{\partial x}\bar{p}+\frac{\partial}{\partial y}\left(\mu\frac{\partial}{\partial y}\bar{u}-\rho\overline{u'\upsilon'}\right)$$

We also have continuity equation for the mean flow $\frac{\partial}{\partial x}\bar{u}+\frac{\partial}{\partial y}\bar{\upsilon}=0$

Let us introduce the turbulent viscosity

$$-\rho\overline{u'\upsilon'} = \rho\nu_T \frac{\partial}{\partial y}\bar{u}$$

$$\mu_T$$

The "turbulent" Prandtl Equation can be written as follows

$$\boxed{\rho(\bar{u}\frac{\partial}{\partial x}\bar{u} + \bar{\upsilon}\frac{\partial}{\partial y}\bar{u}) = -\frac{\partial}{\partial x}\bar{p} + \frac{\partial}{\partial y}[(\mu + \mu_T)\frac{\partial}{\partial y}\bar{u}]}$$

While approaching the wall, turbulent pulsations diminish to vanish completely at the very wall. Hence, the turbulent viscosity near the wall is small. At the wall, $\mu_T = 0$ and the shear stress (friction) is equal

$$\tau_w = \mu\frac{\partial}{\partial y}\bar{u}\Big|_{y=0}$$

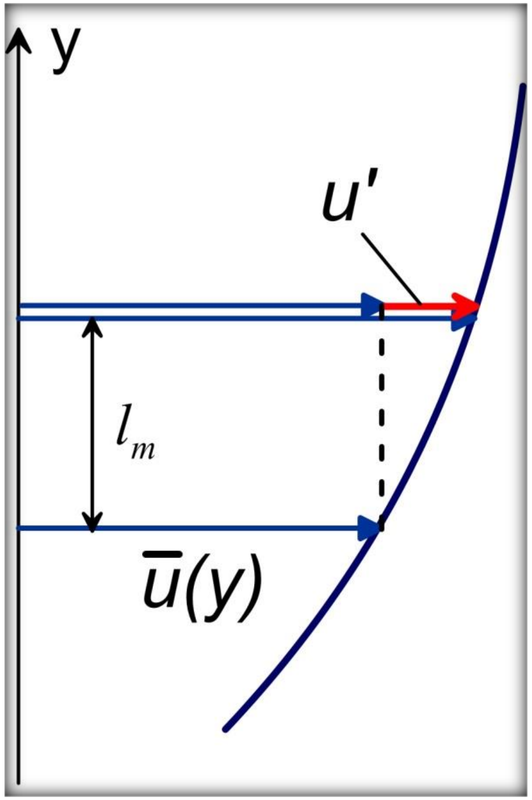
Conclusion: Von Karman equation for the TBL and the LBL is formally the same

$$\boxed{\frac{d}{dx}[\bar{U}^2(x)\bar{\theta}(x)] + \bar{U}(x)\bar{U}'(x)\bar{\delta}_*(x) = \frac{1}{\rho}\tau_w}$$

or

$$\boxed{\frac{d}{dx}\bar{\theta} + (2 + \bar{H})\frac{\bar{U}'}{\bar{U}}\bar{\theta} = \frac{C_f}{2}}$$

Mixing length hypothesis (Prandtl)



By analogy to the kinetic theory, Prandtl proposed a simple model relating the turbulent viscosity to the mean flow in the TBL.

Assume that a certain small parcel of a fluid is shifted vertically by turbulent fluctuations and the average distance of this shift is equal l_m ($l_m \ll \delta$). Assume that during such shift this parcel retains its horizontal velocity. Hence, such re-location will cause appearance of a fluctuation in the wall-parallel velocity, accordingly to the formula

$$u' = \bar{u}(y + l_m) - \bar{u}(y) \approx \frac{\partial \bar{u}}{\partial y} l_m$$

Prandtl assumed also that $\upsilon' \simeq u' \quad \Rightarrow \quad \upsilon' \simeq \frac{\partial \bar{u}}{\partial y} l_m$

which lead to the following expression $-\overline{u'\upsilon'} \simeq l_m^2 \left| \frac{\partial \bar{u}}{\partial y} \right| \frac{\partial \bar{u}}{\partial y}$

Hence, the (kinematic) turbulent viscosity is proportional to the wall-normal mean velocity gradient and the square of the mixing length

$$\nu_T \sim l_m^2 \left| \frac{\partial \bar{u}}{\partial y} \right|$$

But what is the mixing length in the TBL? Prandtl assumed that in the region near the wall the **mixing length rises proportionally to the distance from the wall**, i.e., $l_m \sim y$.

In follows that

$$\nu_T \sim y^2 \left| \frac{\partial \bar{u}}{\partial y} \right|$$

We will show that this formula leads to interesting prediction - there must be a region inside the TBL when the wall-parallel velocity profile is described by the logarithmic function.

Structure of TBL

As we already know, the turbulence dies away at direct vicinity of the wall – the flow there is essentially laminar. This region is called the **laminar (or viscous) sublayer (LS)**. The tangent stress inside the LS is basically constant with the distance from the wall and equal to the tangent stress at the wall. It follows that the velocity profile inside the LS is linear. We have

$$\tau = \mu \frac{\partial \bar{u}}{\partial y} = \tau_w \quad \Rightarrow \quad \bar{u} = (\tau_w / \mu)\, y$$

In the region over LS, the turbulent viscosity quickly increases with the distance and quite soon becomes comparable with the molecular viscosity. Further on, it increases quite rapidly and becomes dominating (it is larger by orders of magnitude than the molecular viscosity).

Accordingly to the mixing length theory, we have

$$\tau \approx y^2 \left(\frac{\partial \bar{u}}{\partial y} \right)^2 \rho$$

Since the tangent stress changes continuously, then – at the certain distance from the wall we have

$$\frac{\partial \bar{u}}{\partial y} \sim \sqrt{\frac{\tau_w}{\rho}} \frac{1}{y}$$

We can introduce the following quantity (co called friction velocity)

$$V_* = \sqrt{\frac{\tau_w}{\rho}}$$

as well as a dimensionless measure of the distance from the wall

$$y^+ = \frac{y V_*}{\nu}$$

Note that the linear velocity profile inside the LS can be now described by the formula

$$\frac{\bar{u}(y)}{V_*} = \frac{y V_*}{\nu} \equiv y^+$$

Let us integrate the expression for the velocity gradient obtained from the mixing length theory. We obtain the relation which can be written in the non-dimensional form as follows

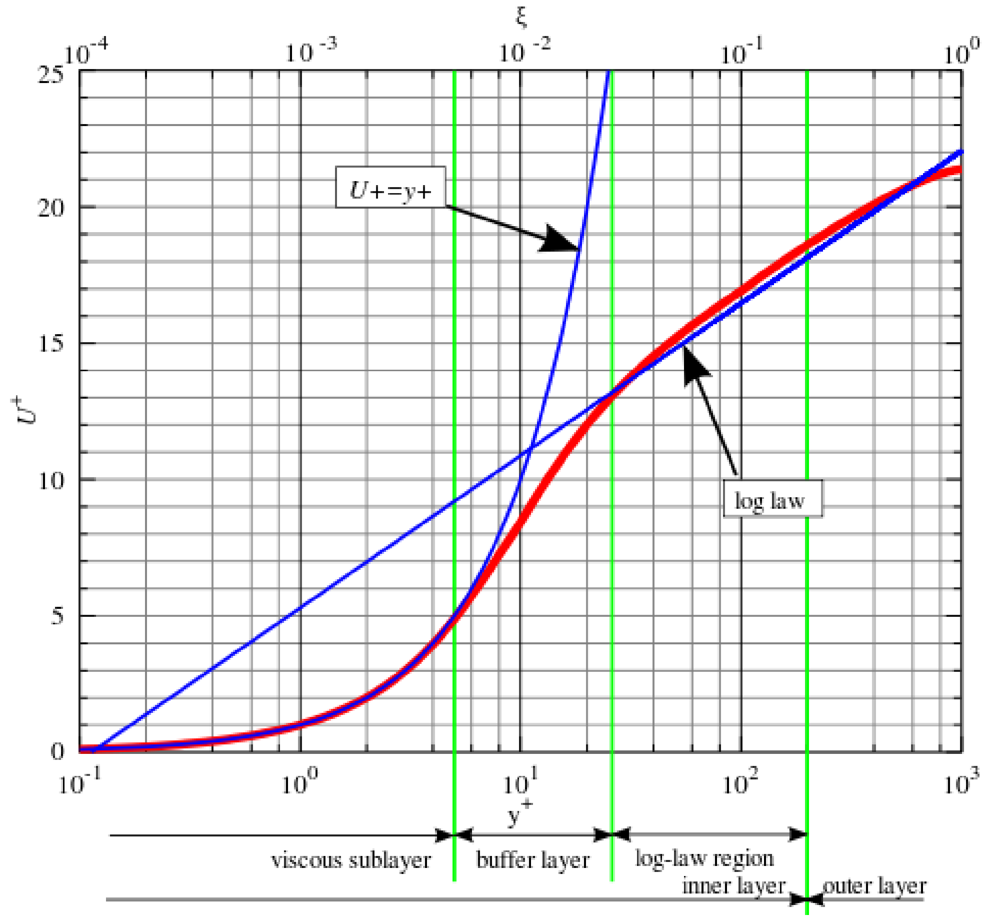
$$\frac{\bar{u}(y)}{V_*} = K \ln \frac{yV_*}{\nu} + C$$

The constants K and C have been determined experimentally. It turn out that a "typical" TBL conforms very well to the above formula and the (nearly) universal choice of these constants is possible! Usually, we write the above formula (it is referred to as the "law of the wall") in the following form

$$\frac{\bar{u}(y)}{V_*} = \frac{1}{\kappa} \ln \frac{yV_*}{\nu} + C$$

where the (universal) values are $\kappa = 0.41$ (The Karman constant) and $C \approx 5.25$.

The following figure shows the actual distribution of the non-dimensional velocity plotted against the dimensionless coordinate y^+ …

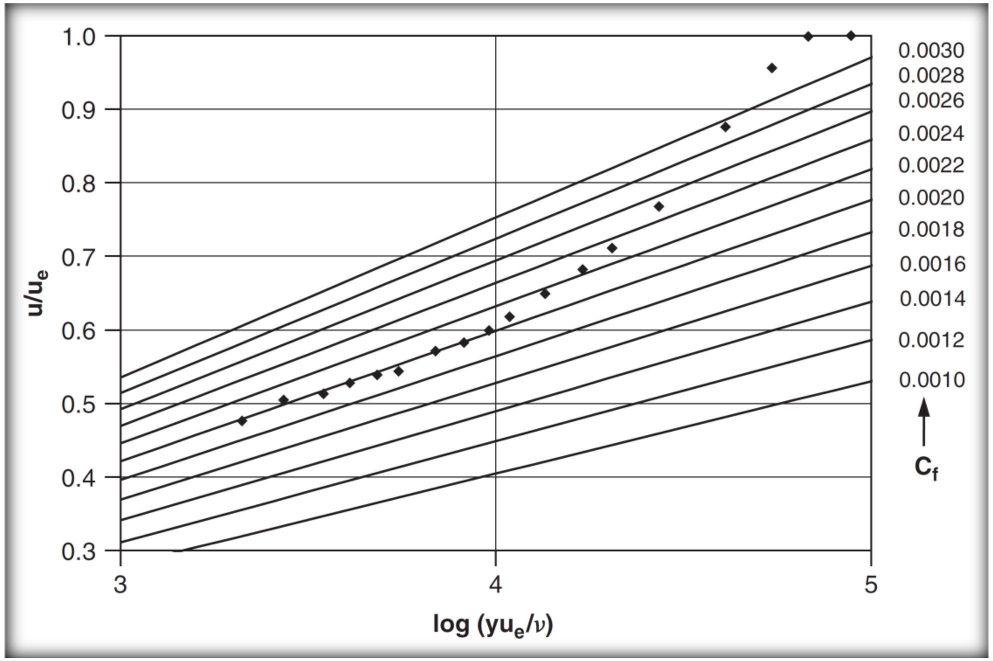


- Viscous sublayer $y^{+} \leq 5$ (viscous effect are dominating)
- Buffer layer $5 < y^{+} < 30 \div 50$ (molecular and turbulent viscosities are comparable in size)
- Logarithmic layer $50 < y^{+} < 150 \div 200$ (turbulent viscosity dominates, the wall is "sensed" by the fluid)

All together 15-20% of the total TBL thickness. The remaining part is called the **outer layer.** For larger wall distances inside OL, the influence of the wall gradually diminishes.

Application of the law of the wall to determination of the wall shear-stress

Note that direct measurement of the wall-normal gradient of the velocity at the wall is very difficult (if possible) since the viscous sublayer is extremely thin (roughly tens of microns, which is a fraction of one percent of the total TBL thickness)



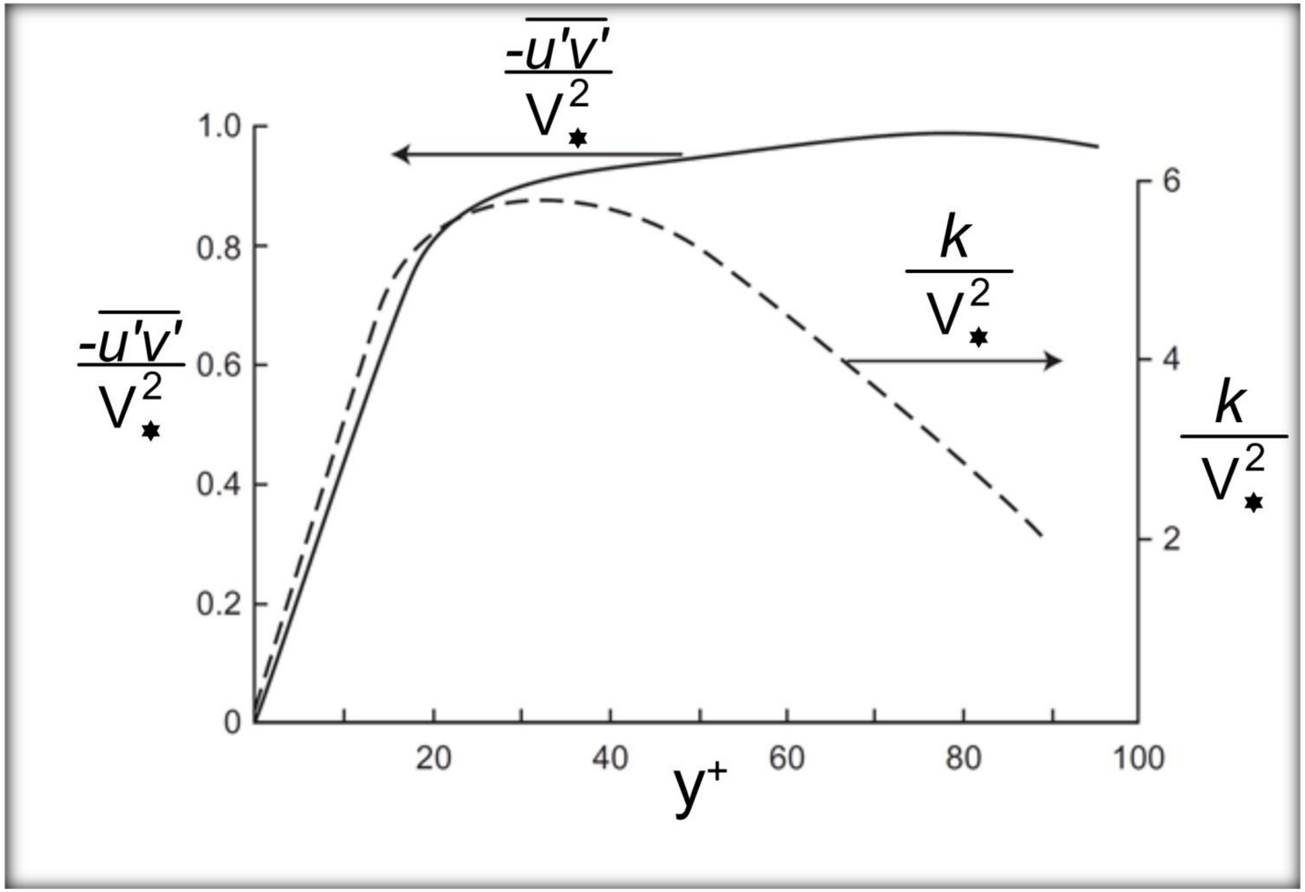
Indirect method

$$\frac{\bar{u}}{U} = \frac{V_*}{U}\frac{1}{\kappa}\ln\left(\frac{yU}{\nu}\right) + \frac{V_*}{U}\frac{1}{\kappa}\ln\left(\frac{V_*}{U}\right) + \frac{V_*}{U}C$$

$$C_f = \frac{2\tau_w}{\rho U^2} = 2\left(\frac{V_*}{U}\right)^2 \quad \Rightarrow \quad \frac{V_*}{U} = \sqrt{\tfrac{1}{2}C_f}$$

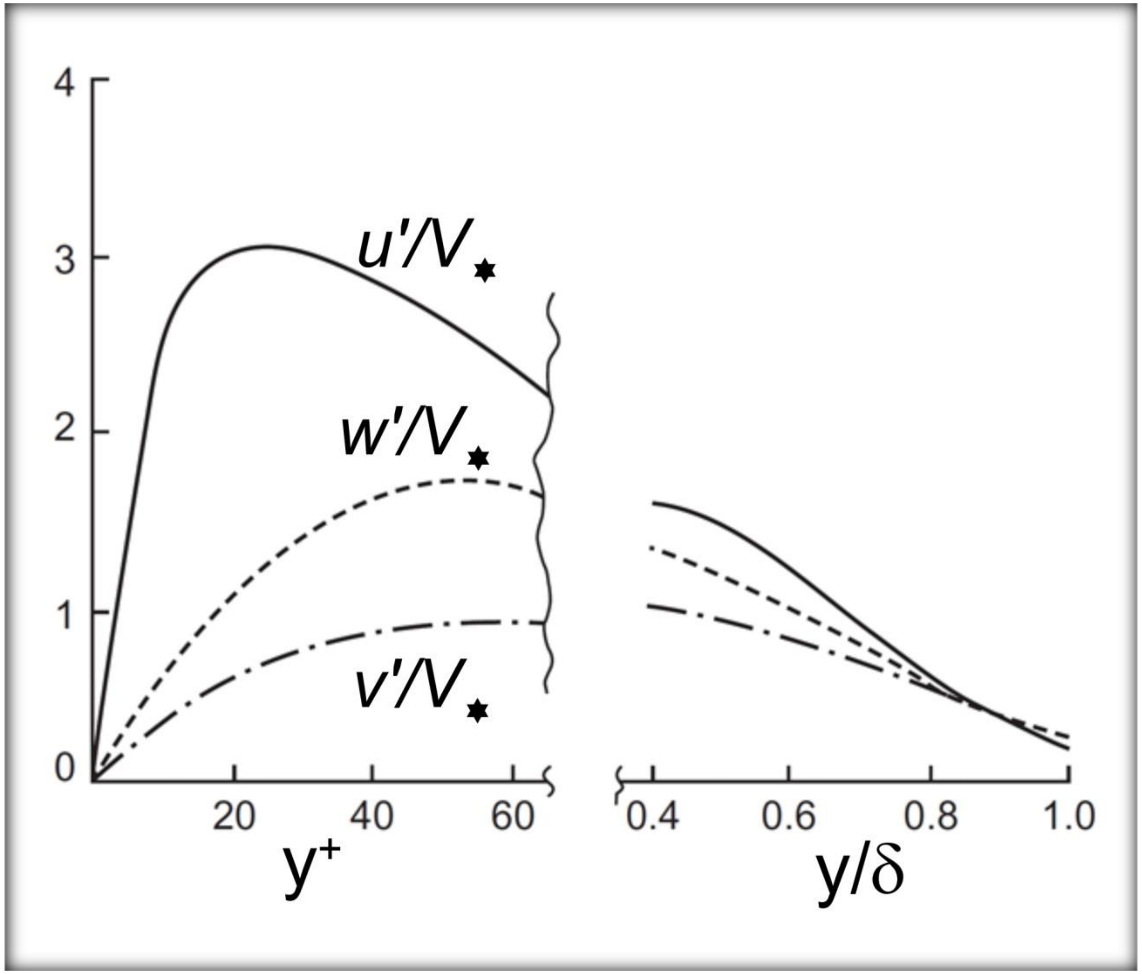
$$\frac{\bar{u}}{U} = \underbrace{\sqrt{\tfrac{1}{2}C_f}\,\frac{1}{\kappa}}_{A(C_f)}\ln\left(\frac{yU}{\nu}\right) + \underbrace{\sqrt{\tfrac{1}{2}C_f}\left[\frac{1}{\kappa}\ln\left(\sqrt{\tfrac{1}{2}C_f}\right) + C\right]}_{B(C_f)}$$

Turbulent shear stress and kinetic energy profiles across the TBL



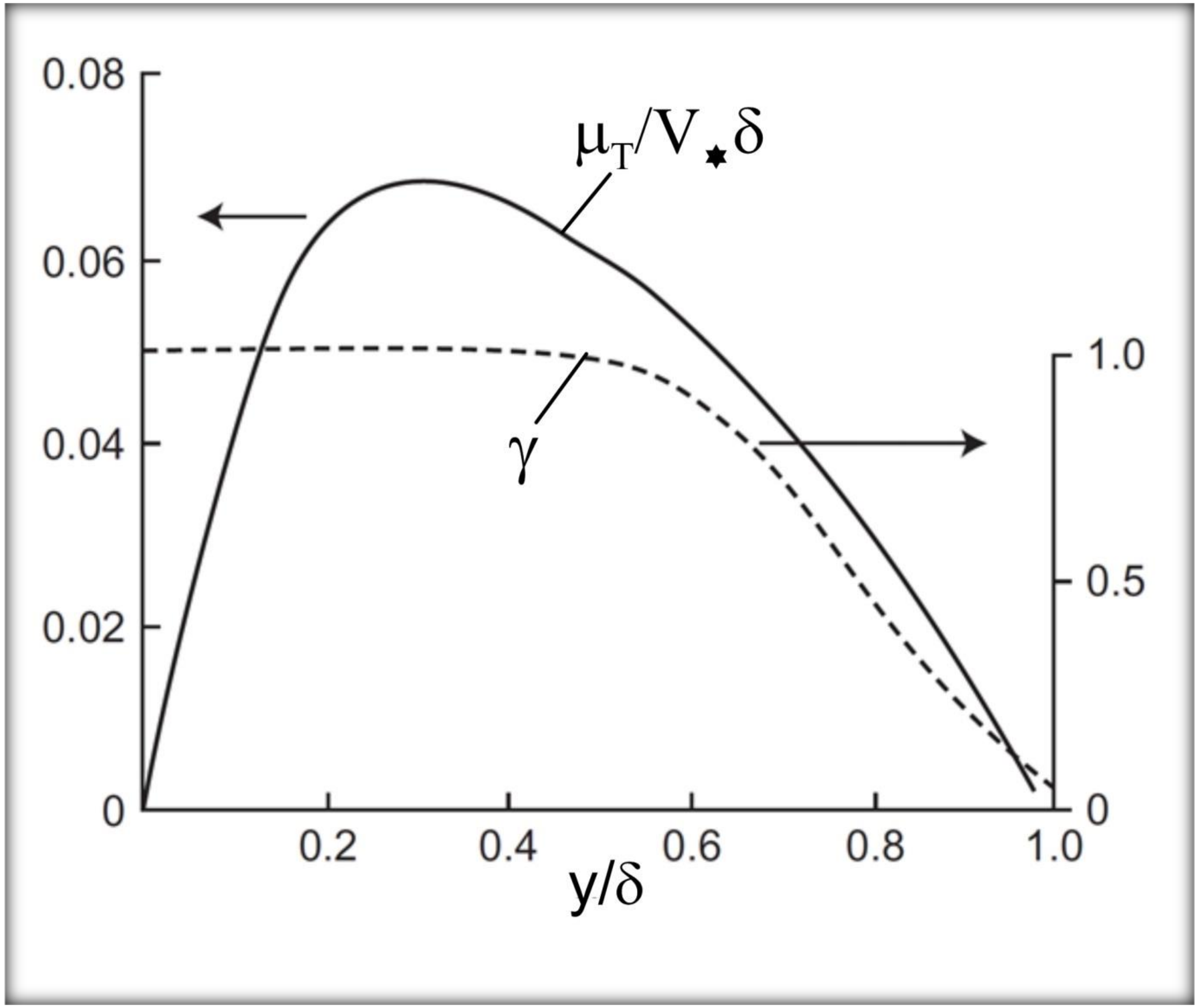
(source: E.L. Houghton at al.: Aerodynamics for Engineering Students, 6th Ed., Elsevier Ltd., 2013)

Mean amplitudes of the velocity pulsations across the TWP



(source: E.L. Houghton at al.: Aerodynamics for Engineering Students, 6th Ed., Elsevier Ltd., 2013)

Turbulent viscosity and the intermittency factor plotted across the TBL



(source: E.L. Houghton at al.: Aerodynamics for Engineering Students, 6th Ed., Elsevier Ltd., 2013)

TBL on the flat plate

Although real lifting surfaces are not flat, one can acquire some useful quantitative characteristics of TBLs by analyzing the particular case – the TBL over the flat plate (zero pressure gradient).

Result 1 – the 1/7 law (Prandtl).

The velocity profile of the TBL over the flat plate is quite reasonably approximated by the following simple power law (see "Aerodynamics for Engineering Students" by E.L. Houghton at al. for derivation, page 522)

$$\frac{u}{U_\infty} = \left(\frac{y}{\delta}\right)^{1/7}$$

Note:

- this formula cannot be applied at the very wall (singularity in the drivative!)
- this formula is applicable for the local Reynolds $\mathrm{Re}_x < 10^7$

Result 2: Wall shear stress and the local friction coefficient can be calculated from the formulae (see again the book AforES)

$$\tau_w \approx 0.0234 \rho U_\infty^{7/4} \left(\nu/\delta\right)^{1/4} \quad , \quad C_f = \frac{\tau_w}{\frac{1}{2}\rho U_\infty^2} = 0.0468 \left(\frac{\nu}{U_\infty \delta}\right)^{1/4} = \frac{0.0468}{\mathrm{Re}_\delta^{1/4}}$$

Result 3: Rate of thickening of the TBL

From the von Karman equation written for the TBL with zero pressure gradient we get

$$\frac{d\theta}{dx} = \frac{C_f}{2}$$

Assuming the 1/7, one can calculate the momentum thickness θ as a fraction of the total TBL thickness δ, namely

$$\theta = \delta \int_0^1 \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right) d\,\bar{y} = \delta \int_0^1 \bar{y}^{1/7}\left(1 - \bar{y}^{1/7}\right) d\,\bar{y} = \tfrac{7}{72}\delta$$

Thus

$$\frac{d\delta}{dx} = \frac{72C_f}{14} = 0.2406\left(\frac{\nu}{U_\infty \delta}\right)^{1/4}$$

Assuming that the thickness of the TBL at $x = 0$ is zero, we get the result

$$\delta(x) \approx 0.383\left(\frac{\nu}{U_\infty}\right)^{1/5} x^{4/5}$$

or, equivalently

$$\frac{\delta(x)}{x} \approx 0.383\left(\frac{\nu}{U_\infty x}\right)^{1/5} = \frac{0.383}{\mathrm{Re}_x^{1/5}}$$

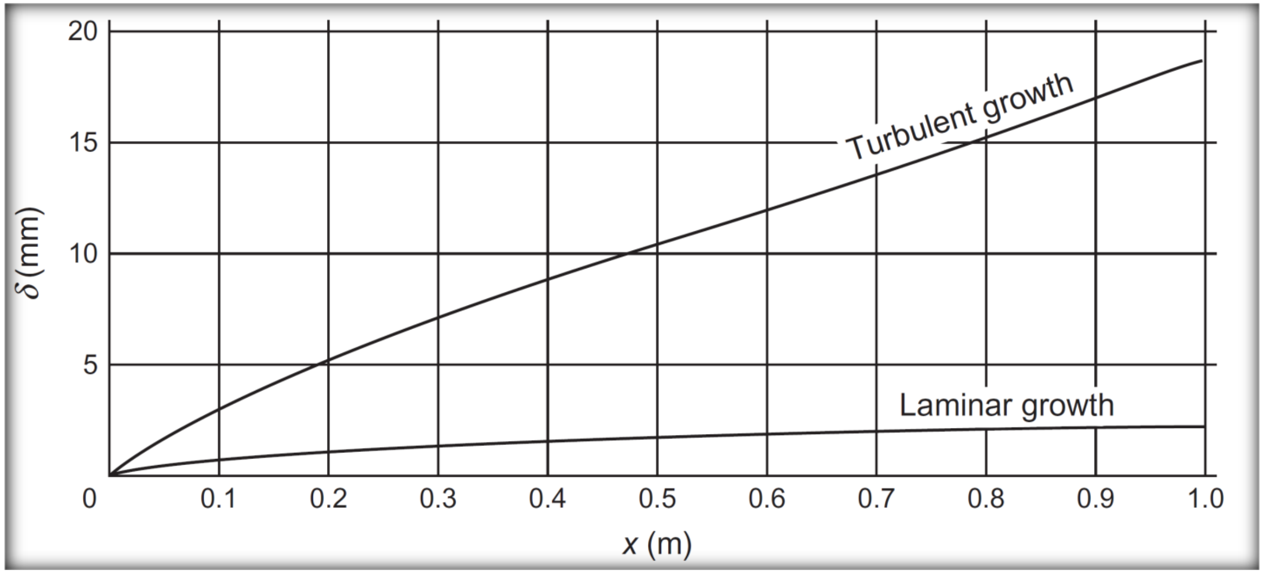
Using the 1/7 law we can also calculate the displacement thickness

$$\delta_* = 0.125\delta$$

Note that the shape factor $H \approx\sim 1.3$ **is much smaller than for the LBL** (Blasius), i.e., around 2.6.

Finally, we obtain the following formulae

$$\frac{\delta_*(x)}{x} = \frac{0.0479}{\mathrm{Re}_x^{1/5}} \quad , \quad \frac{\theta(x)}{x} = \frac{0.0372}{\mathrm{Re}_x^{1/5}}$$



Comparison of the rate of thickening of LBL and TBL on the flat plate (sorce: E.L. Houghton at al.: Aerodynamics for Engineering Students, 6th Ed., Elsevier Ltd., 2013)

Result 4: Local and total coefficient of a friction drag

Substitution of the formula for the TBL thickness to the expression for the local friction coefficient yields

$$C_f = \frac{0.0595}{\mathrm{Re}_x^{1/5}}$$

Hence, the wall shear stress varies along the wall as

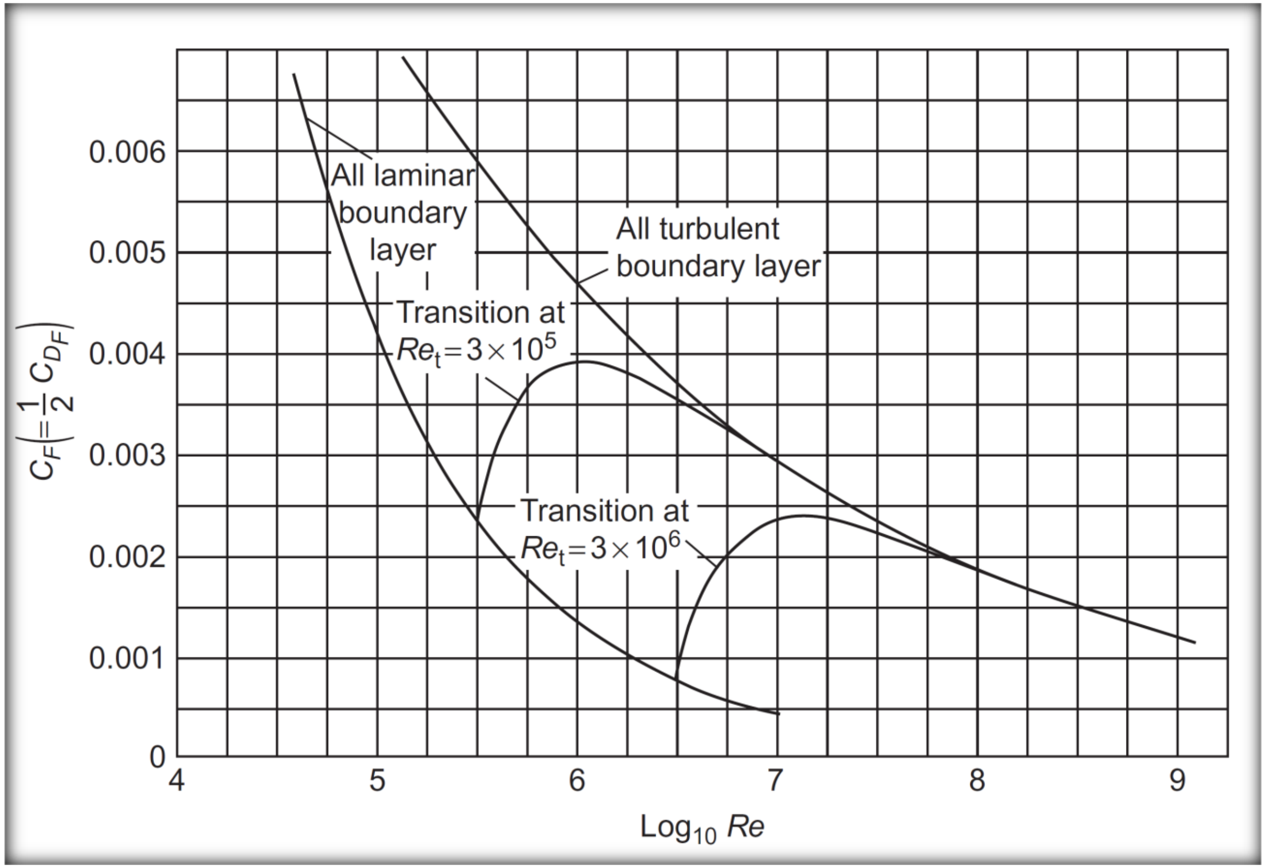
$$\tau_w = 0.0298 \rho \nu^{1/5} U_\infty^{9/5} x^{-1/5}$$

The coefficient of the total friction drag can be computed as follows (here L is the streamwise length of the plate)

$$C_D = \frac{1}{\frac{1}{2}\rho U_\infty^2 L} \int_0^L \tau_w(x) dx = \frac{1}{\frac{1}{2}\rho U_\infty^2 L} 0.0298 \rho \nu^{1/5} U_\infty^{9/5} \int_0^L x^{-1/5} dx =$$

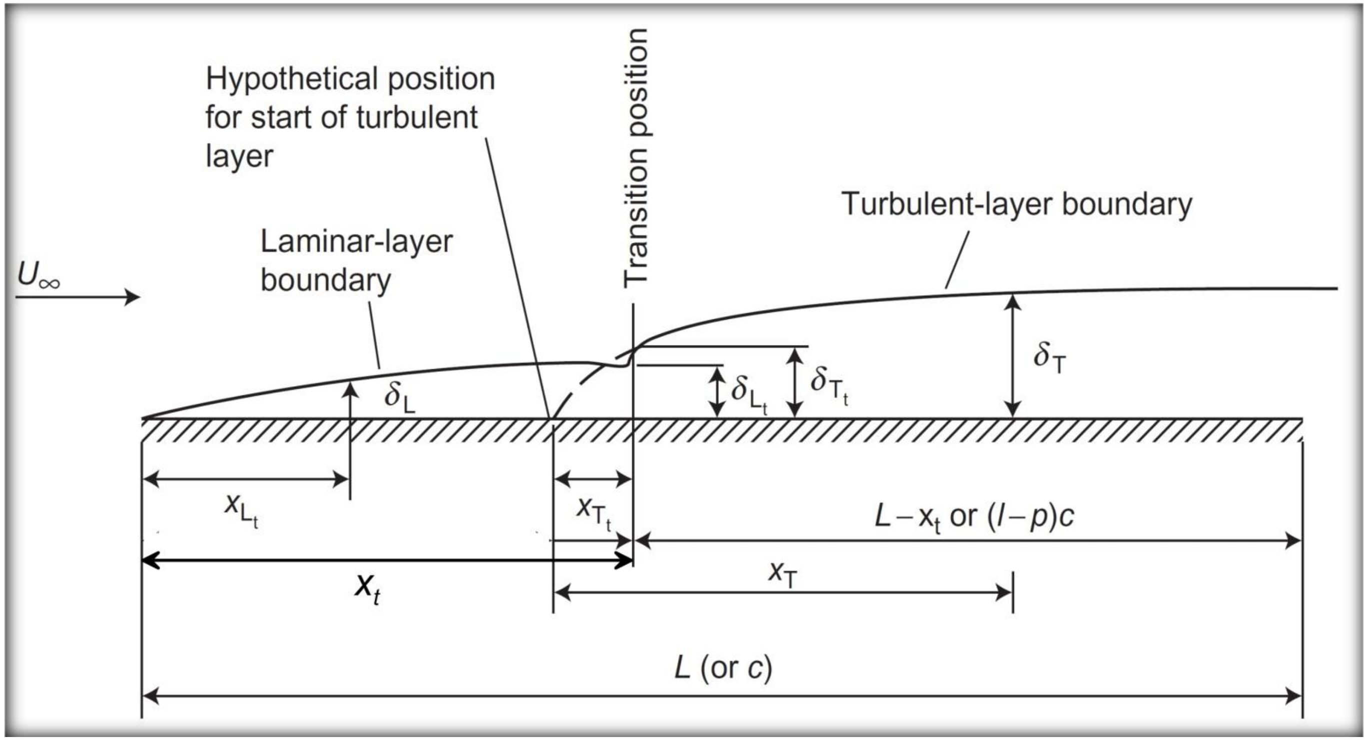
$$= \frac{0.0596}{L} \frac{\nu^{1/5}}{U_\infty^{1/5}} \frac{5}{4} L^{4/5} = 0.0745 \left(\frac{\nu}{U_\infty L} \right)^{1/5} = \frac{0.0745}{\mathrm{Re}_L^{1/5}}$$

REMARK: the above formula describes the value of the friction drag coefficient in the situation when the boundary later is the turbulent one from the very beginning of the plate (leading edge).



Local friction coefficient plotted against the local Reynolds number (źródło: E.L. Houghton at al.: Aerodynamics for Engineering Students, 6th Ed., Elsevier Ltd., 2013)

Engineering approach to the friction drag mixed BLs



Mixed BL on the flat plate (from E.L. Houghton at al.: Aerodynamics for Engineering Students, 6th Ed., Elsevier Ltd., 2013)

Hypothetical beginning of the TBL: $\delta_{T_t} = \dfrac{0.383 x_{T_t}}{\sqrt[5]{(\mathrm{Re}_x)_{T_t}}} = 0.383\left(\dfrac{\nu}{U_\infty}\right)^{1/5} x_{T_t}^{4/5} \Rightarrow x_{T_t}$

Distance from the leading edge to the transition point: $x_t = \frac{\nu}{U_\infty} \mathrm{Re}_t$

Momentum thickness of the laminar BL at the transition point $x = x_t$:

$$\theta_{L_t} = 0.646 \frac{x_t}{\sqrt{\mathrm{Re}_t}} = 0.646 \left(\frac{\nu}{U_\infty} \right)^{1/2} x_t^{1/2} = 0.646 \frac{\nu}{U_\infty} \sqrt{\mathrm{Re}_t}$$

Momentum thickness of the TBL at the transition point $x = x_t$: $\theta_{T_t} = 0.037 \frac{x_{T_t}}{(\mathrm{Re}_x)_{T_t}^{1/5}}$

Key point: the momentum thickness must change continuously in the transition region – otherwise the wall shear stress at the transition point is not well defined (see the Karman Eq.!).

Hence, $\theta_{L_t} = \theta_{T_t}$, meaning that

$$0.646 \sqrt{\frac{\nu x_t}{U_\infty}} = 0.037 \frac{(x_{T_t})^{4/5} \nu^{1/5}}{U^{1/5}} \quad \Rightarrow \quad x_{T_t} = 35.5 \frac{\nu}{U_\infty} \mathrm{Re}_t^{5/8}$$

“Effective” length of the TBL is $L - x_t + x_{T_t}$

It follows from the Karman Eq. written for the BL with zero pressure gradient that the total friction force developed on the wall segment [a,b] is equal

$$\tau_w = \rho U_\infty^2 \theta' \;\Rightarrow\; D_f = \int_a^b \tau_w dx = \rho U_\infty^2 [\theta(b) - \theta(a)] = \rho U_\infty^2 \,\Delta\theta\Big|_{[a,b]}$$

Note that the increments of the momentum thickness of the LBL along the segment $[0, x_t]$ and the momentum thickness of the TBL along the segment $[x_t - x_{T_t}, x_t]$ are equal. Thus, **in both cases the friction force is the same!**

In other words: the total friction drag of the whole mixed BL is equal to the friction drag of the TBL developing along the segment $[x_t - x_{T_t}, L]$.

We can write

$$D_f = \tfrac{1}{2} \rho U_\infty^2 \cdot 0.0595 \left(\frac{\nu}{U_\infty} \right)^{1/5} 1.25 (L - x_t + x_{T_t})^{4/5}$$

The (global) friction drag coefficient is

$$C_{D_f} = \frac{D_f}{\frac{1}{2}\rho U_\infty^2 L} = 0.0744\left(\frac{\nu}{U_\infty}\right)^{1/5} \frac{\left(L - x_t + x_{T_t}\right)^{4/5}}{L} =$$

$$= 0.0744\frac{\nu}{U_\infty L}\left(\frac{U_\infty L}{\nu} - \frac{U_\infty x_t}{\nu} + \frac{U_\infty x_{T_t}}{\nu}\right)^{4/5} =$$

$$= \frac{0.0744}{\mathrm{Re}}(\mathrm{Re} - \mathrm{Re}_t + 35.5\,\mathrm{Re}_t^{5/8})^{4/5}$$

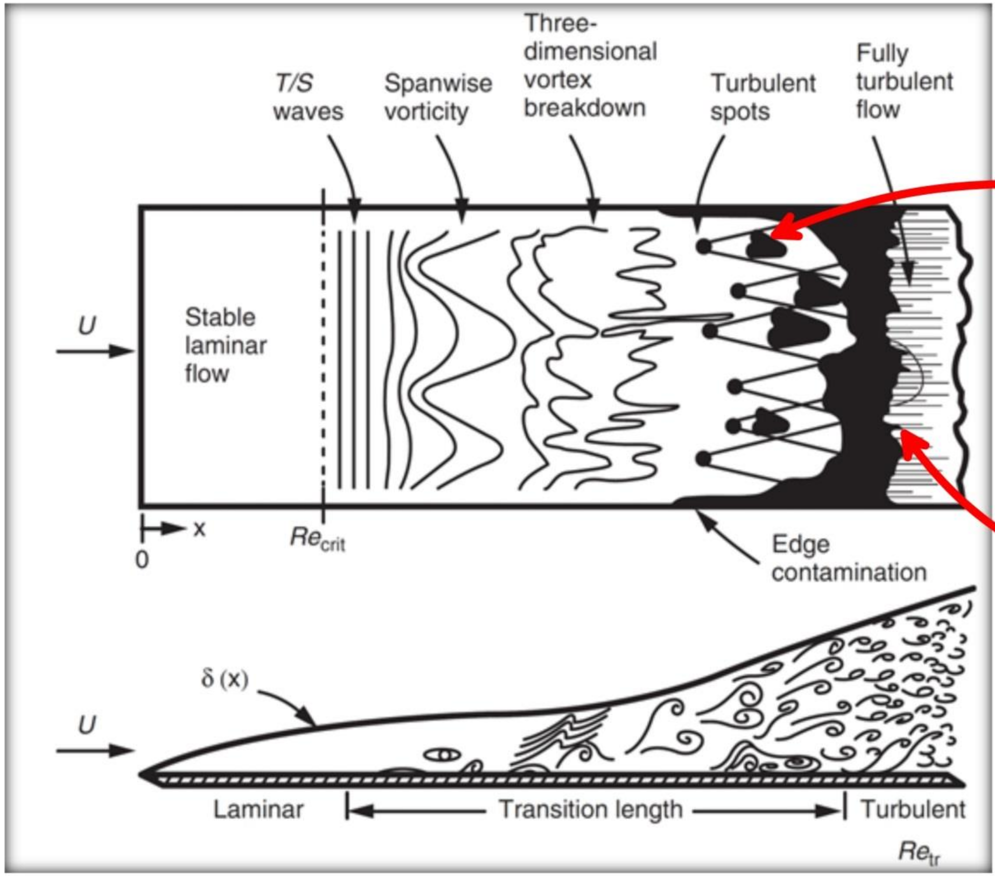
Note: The obtained formula is valid only if $\mathrm{Re} > \mathrm{Re}_t$. Otherwise, the boundary layer is entirely laminar and the global friction coefficient is given by the formula derived earlier, namely

$$C_{D_f} = \frac{2.586}{\sqrt{\mathrm{Re}}}$$

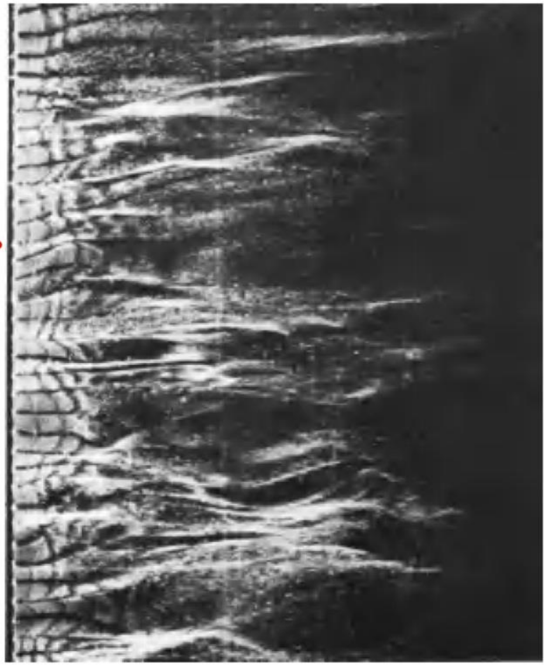
<u>Exercise:</u> Using the flat-plate LBL and TBL formulae for displacement thickness and assuming that $H_{LBL} \approx 2.6$, $H_{TBL} \approx 1.3$, show that the BL's thickness (say, δ_{99}) increase in the transition region by approx. 40%.

LAMINAR-TURBULENT TRANSITION IN THE BLS

General structure of the transition region (natural transition)




turbulent spot


Streamwise streaks in the viscous sublayer

Transition scenarios

1. Natural transition (low level of background turbulence, smooth wall)

Exponential growth of 2D linearly unstable wave-like disturbances (Tollmien-Schlichting waves), secondary instability leading to 3D disturbance field in the form of streamwise streaks and vortices, development and breakdown of hairpin vortices followed generation of the scattered pattern of the wall turbulent spots, merging of spots into fully developed BL.

2. Bypass transition (high level of background turbulence, wall roughness/waviness)

Small disturbances "absorbed into" the LBL are rapidly amplified in the non-modal (algebraic growth) to the level when nonlinearity takes over. Further development mostly similar to the natural conditions.

3. Forced transition

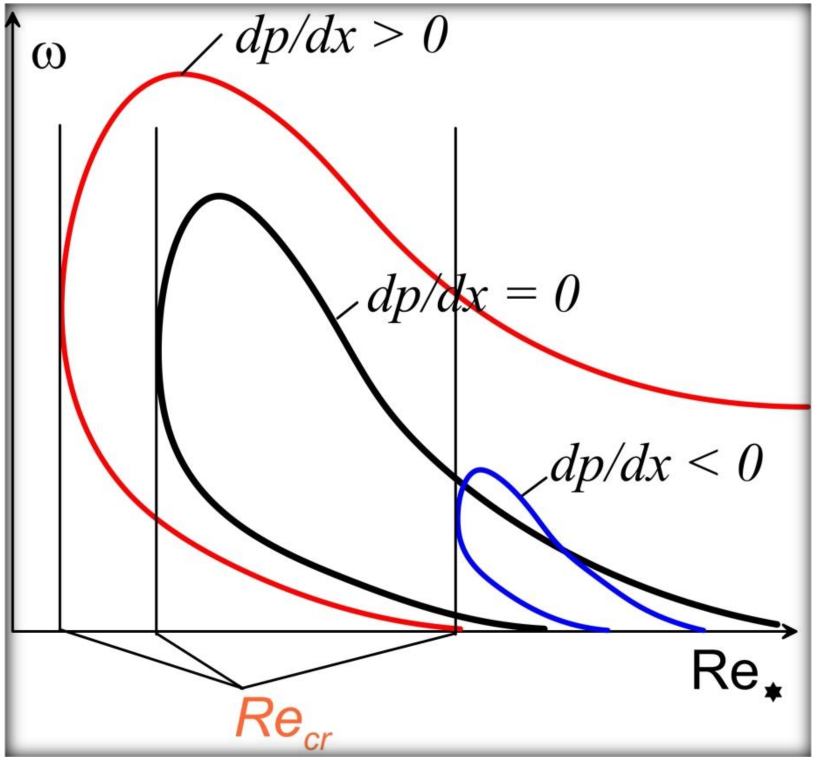
Laminar BL is disrupted by strong nonlinear interaction with vortex structures generated in wakes behind the localized obstacles attached to the wall (for purpose - like the vortex generators, or accidentally - like surface contaminations, e.g., dead insects).

PRIMARY MECHANISMS OF DISTURBANCE GROWTH

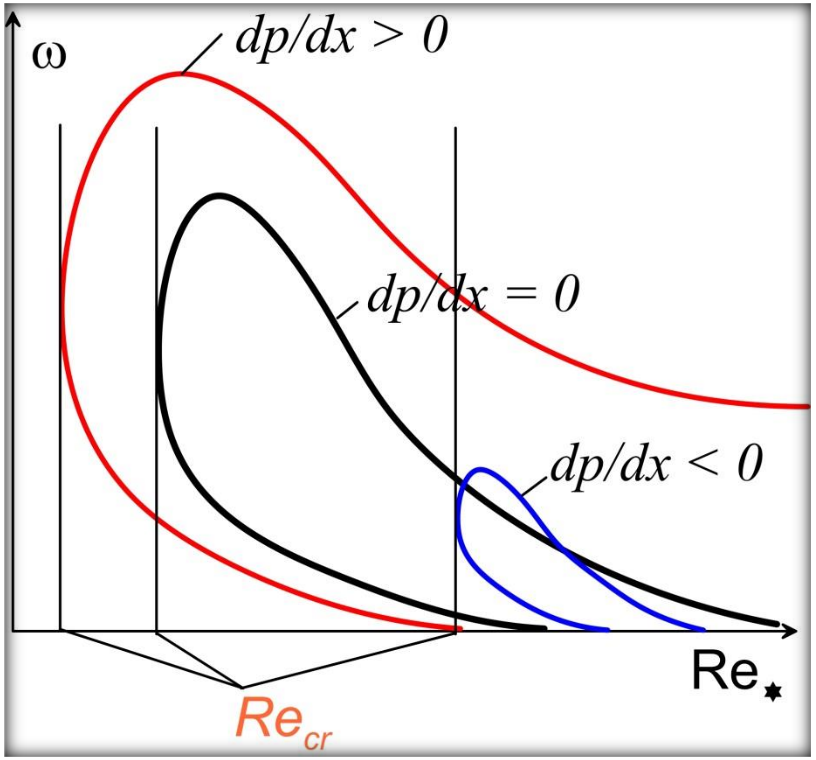
Modal mechanism – exponential growth of the linearly unstable modes of small disturbances in the flow. This is the basic mechanism of initial disturbance growth in the natural transition scenario.

$$[u', \upsilon', w', p'](t, x, y, z) = \Re\mathfrak{e}\{[A_u, A_\upsilon, A_w, A_p](y)\exp[i(\alpha x + \beta z - \omega t)]\}$$

$$\omega, \beta \in R \quad , \quad \alpha = \alpha_r + i\alpha_i \in C \Rightarrow \text{unstable if } \alpha_i < 0$$



- Stability region – inside the loop of the neutral stability curve (NSC).
- When $dp/dx \leq 0$ and $\mathrm{Re}_* \to \infty$ (the limit if vanishing viscosity) both branches of the NSC approach asymptotically the *Re*-axis.
- The case $dp/dx > 0$ is different! The velocity of such BL has an inflection point which renders the flow unstable even if the viscosity is absent (Reyleigh and Fjortoft instability criteria).
- No amplification is possible for sufficiently large frequencies of the modes (the BL works as a low-pass filter).



• For the unstable modes $\alpha_r > 0$, i.e., the unstable 2D modes have the form of travelling waves (the Tollmien-Schlichting waves). In the self-similar BLs (Falkner-Skan) the range of the wave number of the most unstable T-S waves is – roughly - [0.25-0.35] (the unit of length is the displacement thickness. It corresponds to the wave length in the range of 7-10 δ_{99}.

• Stability threshold of the 2D T-S waves depends strongly of the streamwise pressure gradient. Favorable gradient delays instability (increases Re_{cr}) while the adverse one promotes instability (decreases Re_{cr}).

Example: critical Reynolds number (based on the displacement thickness δ_*) for

- Blasius BL (flat plate, zero pressure gradient) is $\mathrm{Re}_{*,cr} \approx 520$.
- Falknera-Skan BL with $m = -0.075$ (adverse pressure gradient, the BL at the flat plate at the angle of attack approx. 14.6 degrees) is $\mathrm{Re}_{*,cr} \approx 130$
- Falknera-Skan BL with $m = 0.075$ (favorable pressure gradient, the BL at the flat plate at the angle of attack approx. -12.5 degrees) is $\mathrm{Re}_{*,cr} \approx 2000$

Reminder: for the Blasius BL

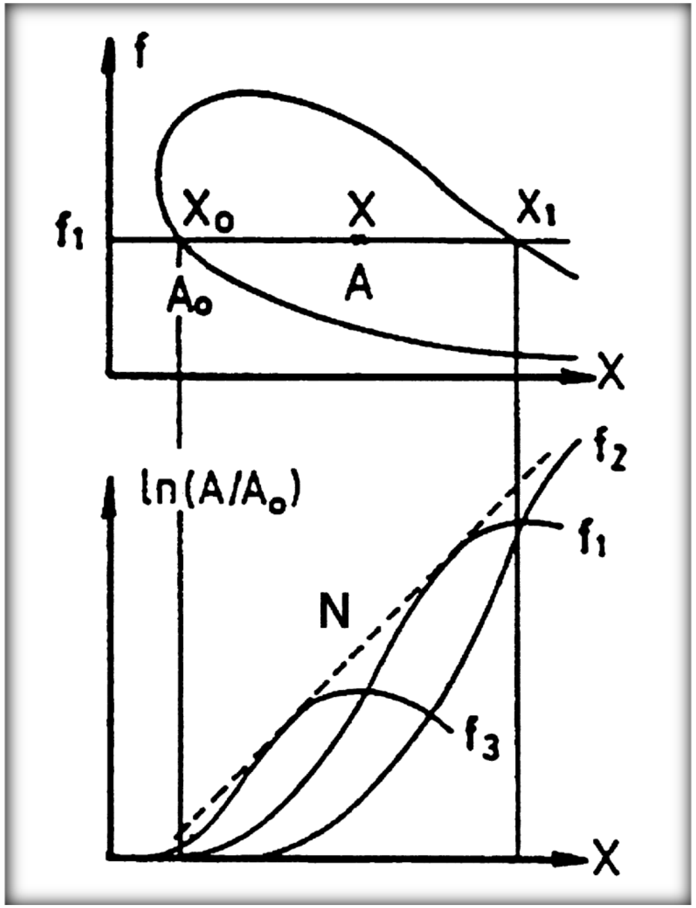
$$\mathrm{Re}_* = \frac{U_\infty \delta_*}{\nu} \approx 1.721 \frac{U_\infty}{\nu} \frac{\sqrt{\nu x}}{\sqrt{U_\infty}} = 1.721 \sqrt{\frac{U_\infty x}{\nu}} = 1.721 \sqrt{\mathrm{Re}_x}$$

Thus, $\mathrm{Re}_x \approx 0.338\,\mathrm{Re}_*^2$ and $\mathrm{Re}_{x,cr} \approx 91300$.

Actual transition point is further downstream the BL. The reason is that it takes a certain distance along the wall until the amplitude of the 2D waves grows sufficiently to trigger further stages of transition. But when they begin, the further process is very rapid - a short distance from the point when 2D finite-amplitude (saturated) T-S waves undergo a secondary instability, the sudden eruption of chaos is observed. Therefore, **it is justified to assume that the transition point is where the amplification of the initial 2D disturbances reaches a certain threshold, sufficient to sustain further development of instability towards a fully turbulent flow**.

The position of the transition "point" depends strongly of the external pressure gradient, geometry and quality of the wall surface (smoothness) and the level of the external disturbances that could be "received" by the BL. The latter problem – e.g. the mechanism by which external perturbation (turbulence, acoustic noise, wall imperfections) are "internalized" by the BL is called the "receptivity problem".

Engineering approach to prediction of the laminar-turbulent transition in the BL.



The most popular and computationally effective engineering methods of transition prediction are based on the idea of the "**e^N method**".

In the basic form of this method, the cumulative amplitude growth define as

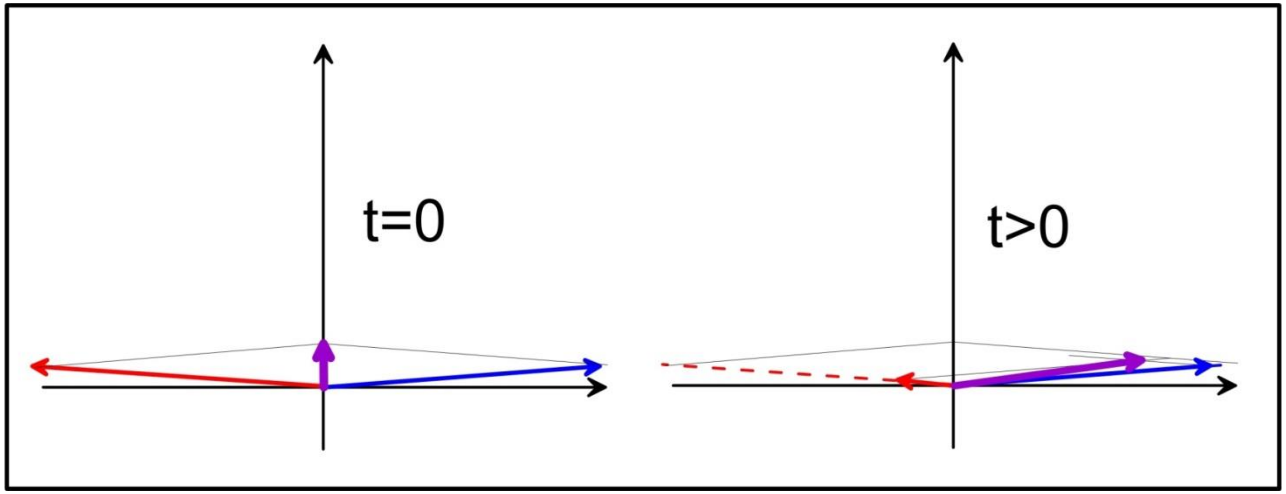
$$\ln(A/A_0) = \int_{x_0}^{x_1} (-\alpha_i)(x)dx$$

is determined for a bunch of modes with a different frequencies. As a result, one obtains a family of lines showing how the amplification factor of each mode grows downstream the BL. Then, one can define the envelope for this family, which is a line effectively parametrized by the frequency (see figure aside).

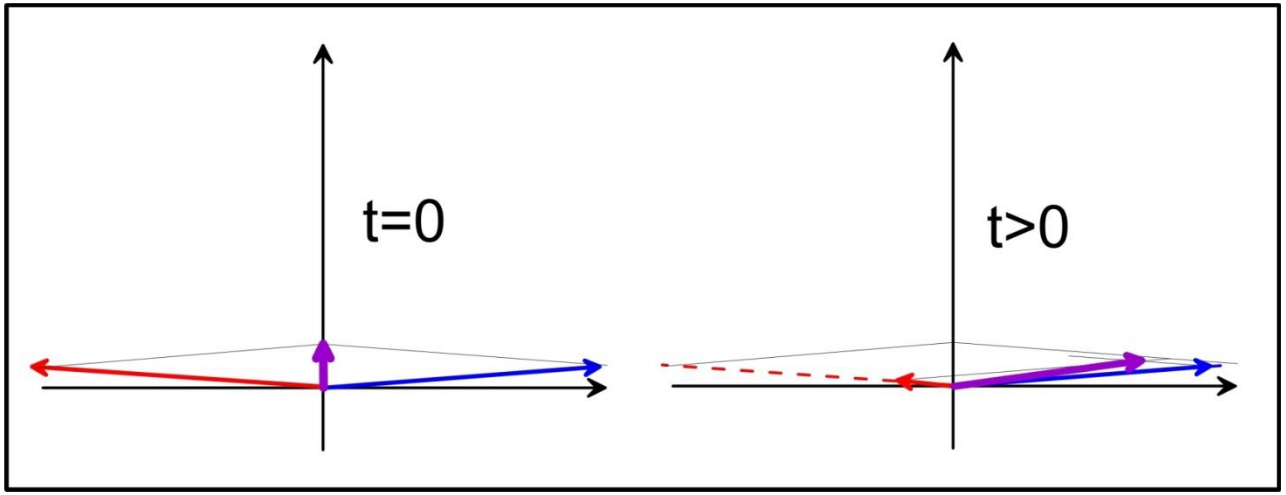
It is assumed that the transition location is where the **value at the envelope reaches the critical value , usually in the range 7-10**. In modern realizations of this method, local amplification rates are computed with the use of approximate formulae based on geometric characteristics of the BL velocity profiles. This way, time-consuming linear stability calculations are avoided.

Nonmodal (algebraic) growth

The amplification of disturbances follows from presence of "nearly parallel" modes of disturbances, which – when combined properly in the initial disturbance – may lead to disturbance amplification by several orders of magnitude, which triggers the nonlinear effects. This type of (transient) growth may occur even if all modes are linearly stable (!) and it is characterized by high amplification rates (much faster that in the modal mechanism).

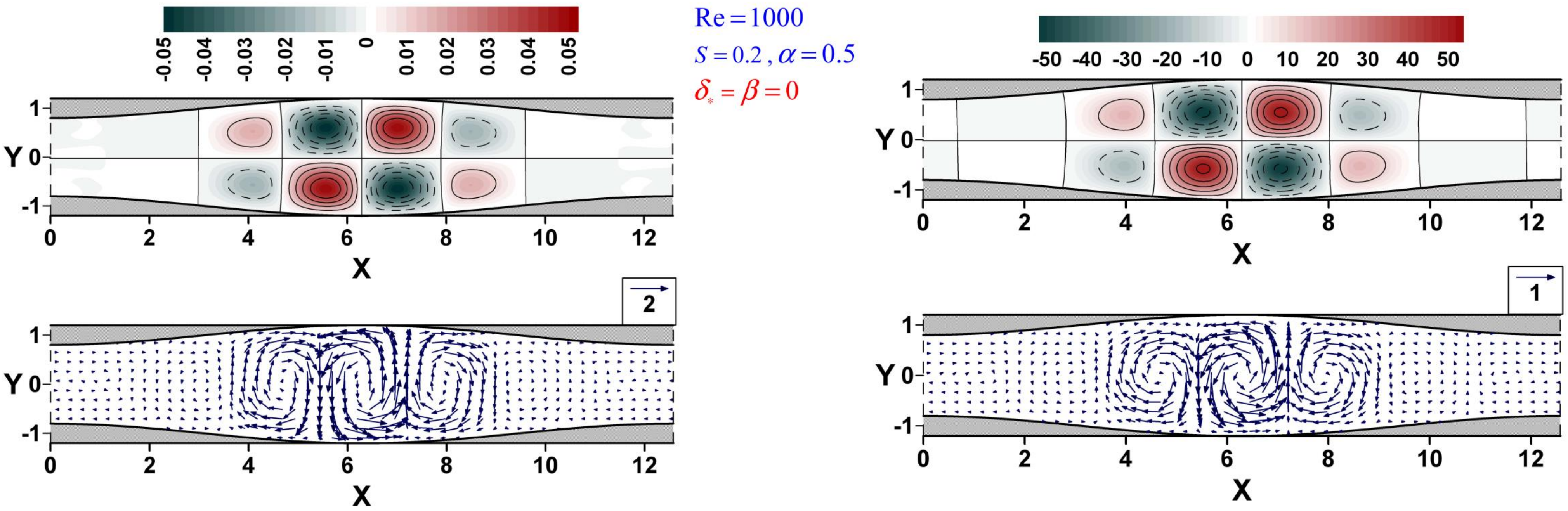


Initial stage (left) - "nearly parallel" 3D modes combined together give the initial disturbance in the form of streamwise-oriented weak vortices, generating mostly wall-normal and spanwise velocity components (υ' and w'). These modes have also strong streamwise velocity component u', however, due to cancellation, **this components in almost nonexistent in the initial disturbance**.



Final stage (right) – since the component modes are damped with different rates, the amplitude of the combined disturbance initially grows. The cancellation of the streamwise component is no longer possible – a large u' (typically, $u' \gg \upsilon', w'$) appears in the flow. Since the disturbance is spanwise-periodic, this development results in the spanwise modulation of the main (undisturbed flow), seen in the form of the "streamwise streaks".

Example: algebraic growth of small disturbances in the wavy channel



t = 0

t = 96

Streamwise (normal to the slide) component of the velocity disturbance field has been amplified by the factor of 1000!

Note: this is example of the temporal transient growth, while in the BL we deal with the spatial transient growth (more complicated issue).