PROGRAM ROZWOJOWY
POLITECHNIKI WARSZAWSKIEJ

Advanced Computational Fluid Dynamics - Training problems 2016

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Problem 1

For the system of Partial Differential Equations:

$$\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} = 0 \qquad u = \begin{bmatrix} u_1 \\ \cdots \\ u_n \end{bmatrix}, \qquad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots \\ \cdots & \cdots & \ddots & \cdots \\ a_{n1} & \cdots & \cdots & a_{nn} \end{bmatrix} \qquad (1)$$

decide for which matrix A the system is hyperbolic (and why):

$$\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} \qquad \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \qquad \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \qquad (2)$$

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \qquad \begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix} \qquad \begin{bmatrix} 3 & 1 & \\ & 3 & \\ & & 3 \end{bmatrix} \qquad \begin{bmatrix} 3 & 1 & \\ & 2 & \\ & & 3 \end{bmatrix} \qquad \mathbf{(3)}$$

Problem 2

Suppose that the matrix A is diagonalisable. Prove in elementary manner that

$$\sin(2A) = 2\cos(A)\sin(A) \qquad (4)$$

Show also that $f(A) \cdot g(A) = g(A) \cdot f(A)$ for arbitrary scalar functions $f(x), g(x)$.

Problem 3

For a given PDE system, find a solution for t>0

$$\begin{cases} \dfrac{\partial u}{\partial t} + \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix} \dfrac{\partial u}{\partial x} = 0 \\ u(x, t = 0) = \begin{bmatrix} \sin(x) \\ 0 \end{bmatrix} \end{cases} \qquad (5)$$

Problem 4

For a given nonlinear scalar equation, identify how long the solution will remain continuous and in which place discontinuity will appear

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial}{\partial x}(-2u^2) = 0 \\ u(x, t = 0) = f(x) \equiv \begin{cases} 0 & \text{dla } |x| > 1 \\ 1 + x & \text{dla } x \in \langle -1, 0 \rangle \\ 1 - x & \text{dla } x \in \langle 0, 1 \rangle \end{cases} \end{cases} \qquad (6)$$

Problem 5

For a previous equation, find the solution at time $t = 1/8$ for the different initial conditions below:

a) $f(x) = \begin{cases} 1, & |x| < 1 \\ 0, & |x| \geq 1 \end{cases}$

b) $f(x) = \begin{cases} 3, & |x| < 2 \\ -1, & |x| \geq 2 \end{cases}$

Problem 6

For a given, nonlinear Boundary Value Problem propose iterative algorithm basing on the method of quasi-linearization, i.e., Newton's method

a) $$\begin{cases} u_{xx} + (u_x)^3 - (1+u^2)u = 0 \\ u(0) = 0 \\ u(1) = 1 \end{cases}$$

b) $$\begin{cases} \operatorname{div}\big(\lambda(T)\operatorname{grad}(T)\big) = 0 \quad \text{on} \ \ \Omega \\ T|_{\partial\Omega} = g \end{cases}$$

Where $\lambda(T)$ is a known function of T only (e.g., $\lambda(T) = e^T$), while function g is known on the boundary $\partial\Omega$.

c) $$\begin{cases} u_{xx} - u = 0 \\ u(0) = 1 \\ u(2) \cdot u_x(2) = 3 \end{cases}$$

d) $$\begin{cases} u''' + uu' = 0 \\ u(0) = u'(0) = 0 \\ u'(10) = 0 \end{cases}$$

Problem 7

For a given matrix $A \in \mathbb{R}^{N\times N}$

$$A = \begin{bmatrix} 4 & 1 & & & \\ 1 & 4 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 4 & 1 \\ & & & 1 & 4 \end{bmatrix} \qquad (7)$$

a) What are eigenvalues and eigenvectors of A

b) What is the value of $\|A\|_2, \|A\|_1, \|A\|_\infty$,

c) What is the value of $\|A^{-1}\|_2$

d) Is the Jacobi iterative method convergent for the system $Au = f$ – provide the proof basing on the value of $\|\cdot\|$ for the suitable matrix.

e) How many iterations are necessary to reduce the solution error by a factor of 100 (as a function of matrix size N).

Problem 8

Suppose that the matrix A is diagonalisable and positive. Propose, basing on the Newton's method, an iterative algorithm to find $B = \sqrt{A}$ (do NOT assume that you actually know the eigenvectors and the eigenvalues). The algorithm must consist of elementary operation on A and A^{-1} (+,-,*) only.