STATIONARY GAS FLOWS IN DUCTS WITH FRICTION AND/OR HEATING/COOLING REVISITED

GENERALIZED CONSERVATION EQUATIONS

$$\rho_1 u_1 = \rho_2 u_2 \tag{1}$$

$$p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2 + p_1 f \tag{2}$$

$$i_1 + \tfrac{1}{2} u_1^2 + i_1 q = i_2 + \tfrac{1}{2} u_2^2 \tag{3}$$

EQUIVALENT FORMS

$$\frac{p_1 M_1}{\sqrt{T_1}} = \frac{p_2 M_2}{\sqrt{T_2}} \quad \Rightarrow \quad \frac{p_1}{p_2} = \frac{M_2}{M_1}\sqrt{\frac{T_1}{T_2}} \tag{4}$$

$$p_1(1 + kM_1^2 - f) = p_2(1 + kM_2^2) \quad \Rightarrow \quad \frac{p_2}{p_1} = \frac{1 + kM_1^2 - f}{1 + kM_2^2} \tag{5}$$

$$T_1(1 + \tfrac{k-1}{2} M_1^2 + q) = T_2(1 + \tfrac{k-1}{2} M_2^2) \quad \Rightarrow \quad \frac{T_2}{T_1} = \frac{1 + \frac{k-1}{2} M_1^2 + q}{1 + \frac{k-1}{2} M_2^2} \tag{6}$$

FROM (4), (5) AND (6) …

$$\frac{M_2}{M_1} = \frac{1 + kM_2^2}{1 + kM_1^2 - f}\sqrt{\frac{1 + \frac{k-1}{2}M_1^2 + q}{1 + \frac{k-1}{2}M_2^2}}$$

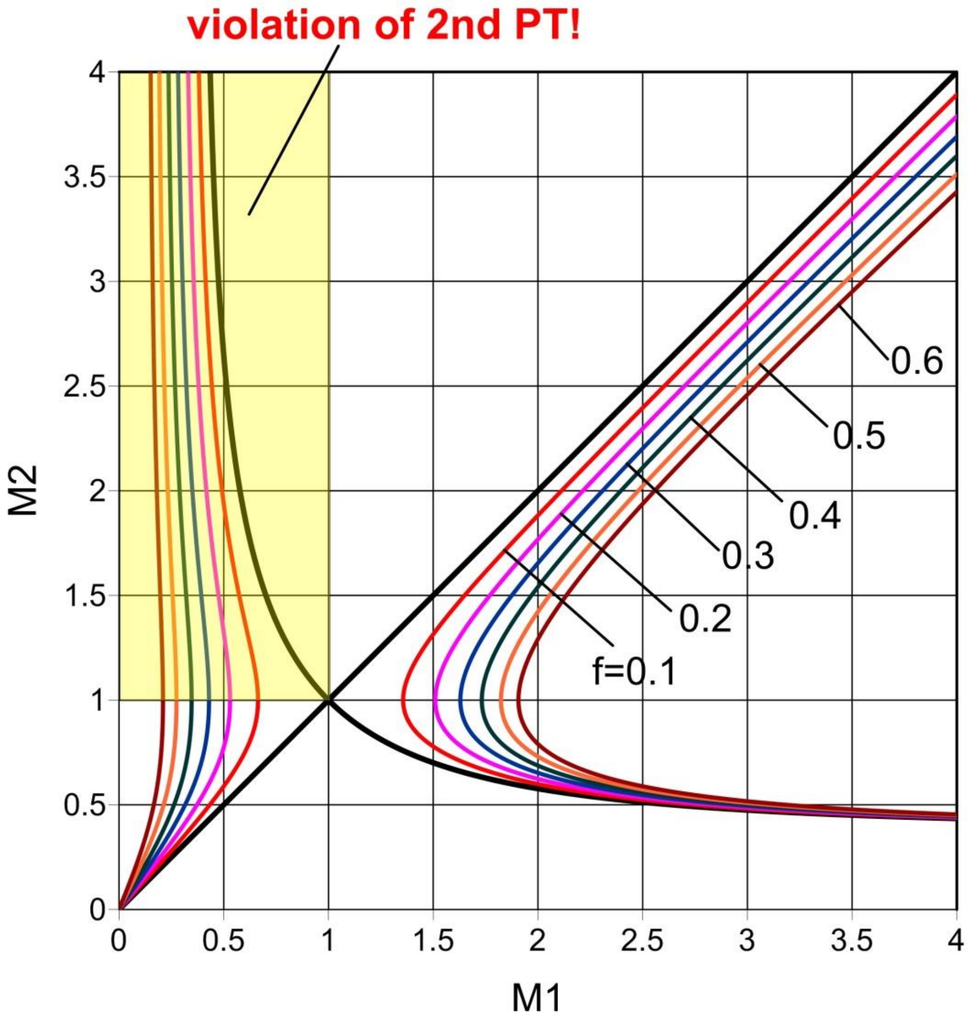
OR

$$\frac{M_2^2(1 + \frac{k-1}{2}M_2^2)}{(1 + kM_2^2)^2} = \frac{M_1^2(1 + \frac{k-1}{2}M_1^2 + q)}{(1 + kM_1^2 - f)^2}$$

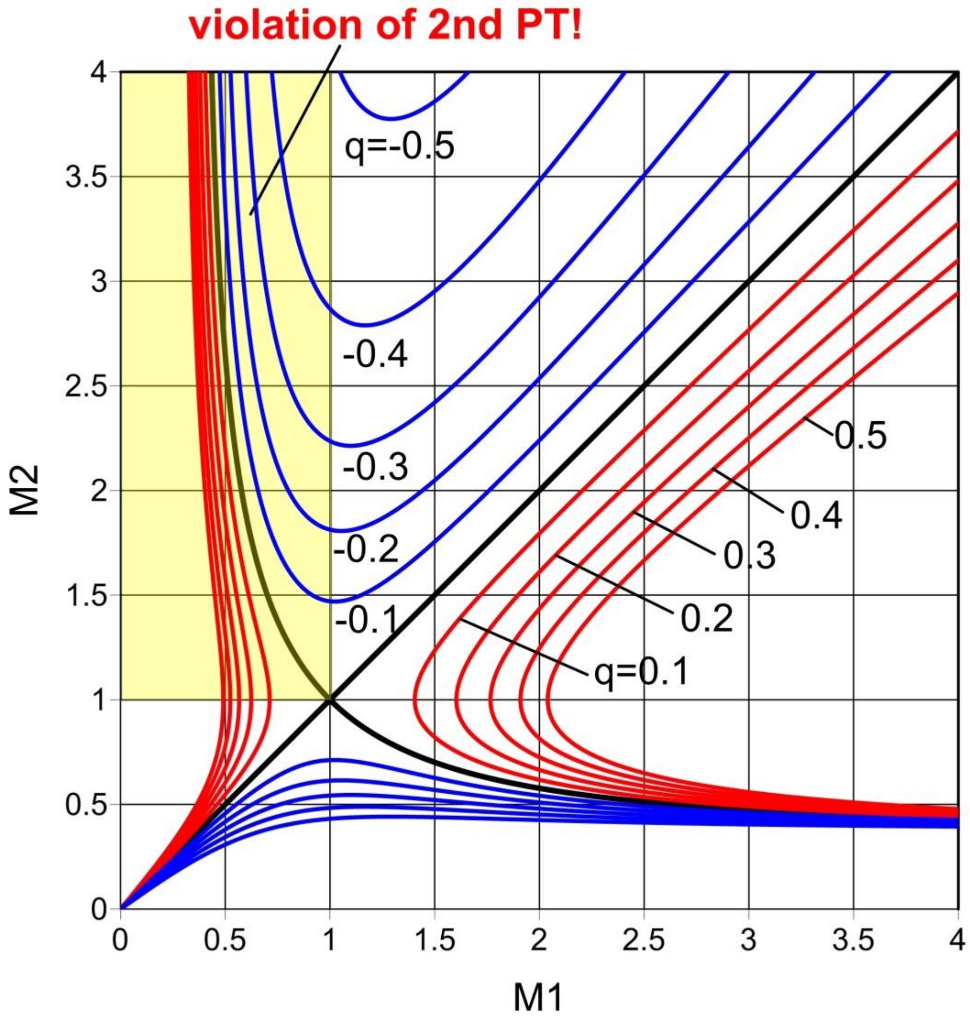
NOTE:

$f = 0, q \neq 0$ - RAYLEIGH FLOW

$f > 0, q = 0$ - FANNO FLOW



$f > 0\,, q = 0$ - FANNO FLOW



$f = 0\,, q \neq 0$ - RAYLEIGH FLOW

EFFECTS OF FRICTION (FANNO)

FROM MASS CONSERVATION $\frac{pM}{\sqrt{T}} = const \quad \Rightarrow \quad \frac{dp}{p} + \frac{dM}{M} - \frac{dT}{2T} = 0$

FROM ENERGY CONSERVATION

$$T(1 + \tfrac{k-1}{2} M^2) = const \quad \Rightarrow \quad \boxed{\frac{dT}{T} = -\frac{(k-1)MdM}{1 + \frac{k-1}{2} M^2}}$$

HENCE

$$\frac{dp}{p} = -\frac{[1 + (k-1)M^2]MdM}{M^2(1 + \frac{k-1}{2} M^2)} \quad \Rightarrow \quad MdM = -\frac{M^2(1 + \frac{k-1}{2} M^2)}{1 + (k-1)M^2} \frac{dp}{p}$$

INCREMENTAL/DIFFERENTIAL FORM OF THE LINEAR MOMENTUM EQUATION

$$p(1+kM^2-\Delta f)=(p+\Delta p)[1+k(M+\Delta M)^2]$$

DROPPING NONLINEAR TERMS ONE GETS

$$-p\Delta f=(1+kM^2)\Delta p+2kpM\Delta M$$

MEANING THAT

$$\frac{dp}{p}=-\frac{1+(k-1)M^2}{1-M^2}df$$

STARTING FROM THE BASIC FORM …

$$p+\rho u^2-p\Delta f=p+\Delta p+(\rho+\Delta\rho)(u+\Delta u)^2$$

AND DROPPING NONLINEAR TERMS ONE GETS

$$-pdf=dp+u^2d\rho+2\rho udu$$

FROM MASS CONTINUITY $\rho u = const \Rightarrow \rho du + u d\rho = 0$

IT FOLLOWS THAT $u^2 d\rho = u u d\rho = -\rho u\, du$

AFTER INSERTION

$$-p d f = dp + \rho u du \Rightarrow -d f = \frac{dp}{p} + \frac{\rho u^2}{p}\frac{du}{u}$$

$$-d f = -\frac{1+(k-1)M^2}{1-M^2} d f + k\frac{u^2}{\underbrace{\frac{kp}{\rho}}_{a^2}}\frac{du}{u} = -\frac{1+(k-1)M^2}{1-M^2} d f + kM^2\frac{du}{u}$$

FINALLY

$$\boxed{\frac{du}{u} = \frac{1}{1-M^2} d f}$$

AGAIN, FROM THE MASS CONSERVATION

$$\rho u = const \quad \Rightarrow \quad \frac{d\rho}{\rho} + \frac{du}{u} = 0$$

THUS

$$\boxed{\frac{d\rho}{\rho} = -\frac{1}{1-M^2}\, df}$$

<u>SUMMARIZING</u>

	M<1	M>1
M	↑	↓
u	↑	↓
p	↓	↑
ρ	↓	↑
T	↓	↑
p_0	↓	↓

EFFECTS OF HEATING/COOLING (RAYLEIGH)

INCREMENTAL/DIFFERENTIAL FORM OF THE ENERGY EQUATION

$$T(1+\tfrac{k-1}{2}M^2+\Delta q)=(T+\Delta T)[1+\tfrac{k-1}{2}(M+\Delta M)^2]$$

$$\Downarrow$$

$$dq=(1+\tfrac{k-1}{2}M^2)\frac{dT}{T}+(k-1)MdM$$

AGAIN, FROM MASS CONSERVATION

$$\frac{pM}{\sqrt{T}}=const \quad \Rightarrow \quad \frac{dp}{p}+\frac{dM}{M}-\frac{dT}{2T}=0$$

AGAIN, FROM LINEAR MOMENTUM EQUATION

$$p(1+kM^2)=const \quad \Rightarrow \quad \frac{dp}{p}+\frac{2kM}{1+kM^2}dM=0$$

THUS

$$\frac{dp}{p}=-\frac{2kMdM}{1+kM^2}=-\frac{2kM^2}{1+kM^2}\frac{dM}{M}$$

AND

$$MdM=\frac{M^2(1+kM^2)}{2(1-kM^2)}\frac{dT}{T}$$

AFTER INSERTION ONE GETS

$$dq=(1+\tfrac{k-1}{2}M^2)\frac{dT}{T}+(k-1)\frac{M^2(1+kM^2)}{2(1-kM^2)}\frac{dT}{T}=\frac{1-M^2}{1-kM^2}\frac{dT}{T}$$

HENCE

$$\frac{dT}{T}=\frac{1-kM^2}{1-M^2}dq$$

IT FOLLOWS THAT

$$\frac{dM}{M}=\frac{1+kM^2}{2(1-M^2)}dq$$

AND

$$\frac{dp}{p}=\frac{kM^2}{M^2-1}dq$$

INCREMENTAL/DIFFERENTIAL FORM OF THE ENERGY EQUATION AGAIN …

$$i + \tfrac{1}{2}u^2 + i\,\Delta q = i + \Delta i + \tfrac{1}{2}(u + \Delta u)^2$$

AFTER DROPPING NONLINEAR TERMS …

$$i\,dq = di + u\,du \quad \Rightarrow \quad dq = \frac{di}{i} + \frac{u}{i}du \quad \Rightarrow \quad dq = \frac{dT}{T} + \frac{u}{i}du$$

USING THE FORMULA FOR TEMPERATURE ONE GETS

$$\frac{u}{i}du = dq - \frac{dT}{T} = (1 - \frac{1 - kM^2}{1 - M^2})dq = \frac{(k-1)M^2}{1 - M^2}dq$$

NOTE THAT

$$\frac{u}{i}du = (k-1)\frac{u^2}{a^2}\frac{du}{u} = (k-1)M^2\frac{du}{u}$$

HENCE

$$\frac{du}{u}=\frac{1}{1-M^2}dq$$

$$\frac{d\rho}{\rho}=\frac{1}{M^2-1}dq$$

SUMMARIZING

FLOW WITH HEATING $(q>0)$

	M<1	M>1
M	$\uparrow$	$\downarrow$
u	$\uparrow$	$\downarrow$
p	$\downarrow$	$\uparrow$
ρ	$\downarrow$	$\uparrow$
T	$\uparrow$ if $M<1/\sqrt{k}$ $\downarrow$ if $M>1/\sqrt{k}$	$\uparrow$
p_0	$\downarrow$	$\downarrow$