Finite Difference Method (FDM)
Boundary Element Method (BEM)
and Finite Element Method (FEM)
Draft presentation for solving Poisson's equation in 2D space

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Poisson's equation is a partial differential equation with broad utility in electrostatics, mechanical engineering and theoretical physics.

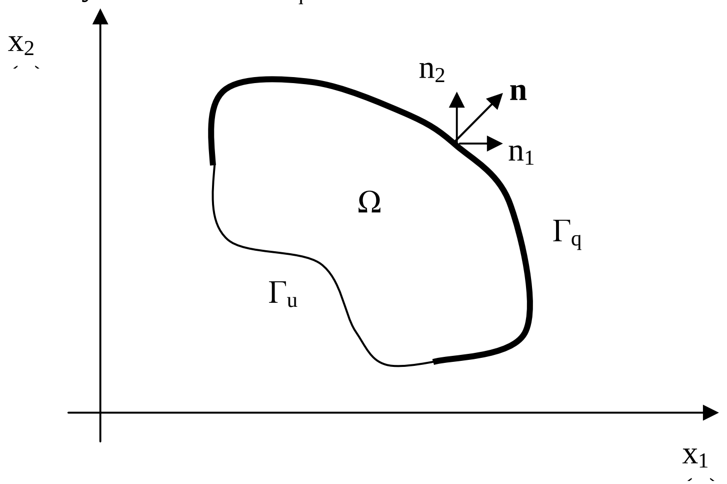
$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + f(x_1, x_2) = 0 \quad ,$$

For vanishing f, this equation becomes Laplace's equation.

We consider a Dirichlet boundary condition on Γ_u and a Neumann boundary condition on Γ_q:

$$u(\overline{x}) = u_0 \quad , \quad \overline{x} \in \Gamma_u$$

$$q(x) = \frac{\partial u(\overline{x})}{\partial n} = q_0 \quad , \quad \overline{x} \in \Gamma_q$$



where u_0 and q_0 are given functions defined on those portions of the boundary.
In some simple cases (shape of the domain Ω and boundary conditions) the Poisson equation may be solved using analytical methods.

Finite Difference Method

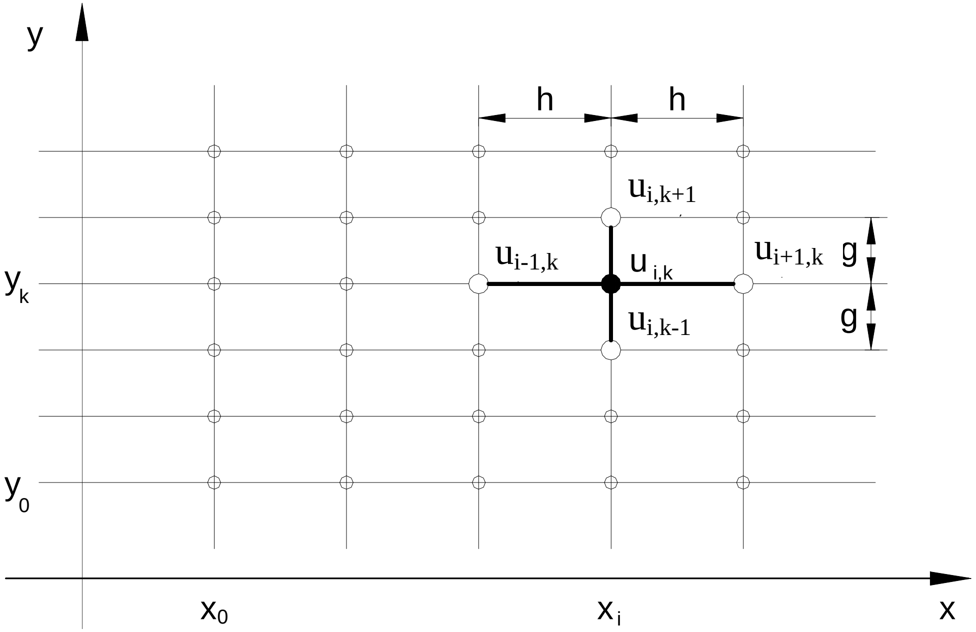
Finite-difference method approximates the solution of differential equation by replacing derivative expressions with approximately equivalent difference quotients. That is, because the first derivative of a function f (x) is, by definition,

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h},$$

then a reasonable approximation for that derivative would be to take

$$f'(a) \approx \frac{f(a+h) - f(a)}{h}$$ (difference quotient)

for some small value of h. Depending on the application, the spacing *h* may be variable or held constant.
The approximation of derivatives by finite differences plays a central role in finite difference methods
In similar way it is possible to approximate the first **partial derivatives** using **forward , backward** or central **differences**



$$x_i = x_o + ih, \quad y_k = y_o + kg, \qquad u_{i,k} = u(x_i, y_k)$$

a) $$\frac{\partial u}{\partial y} \approx \frac{\Delta u}{\Delta y} = \frac{u_{i,k+1} - u_{i,k}}{g},$$

b) $$\frac{\partial u}{\partial y} \approx \frac{\Delta u}{\Delta y} = \frac{u_{i,k} - u_{i,k-1}}{g},$$

c) $$\frac{\partial u}{\partial y} \approx \frac{\Delta u}{\Delta y} = \frac{u_{i,k+1} - u_{i,k-1}}{2g}.$$

x_o, y_o - reference point of the grid, $u(x, y)$ - unknown function

Differences corresponding to higher derivatives

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{\Delta^2 u}{\Delta x^2} = \frac{u_{i+1,k} - 2u_{i,k} + u_{i-1,k}}{h^2},$$

$$\frac{\partial^2 u}{\partial y^2} \approx \frac{\Delta^2 u}{\Delta y^2} = \frac{u_{i,k+1} - 2u_{i,k} + u_{i,k-1}}{g^2}.$$

$$\frac{\partial^4 u}{\partial x^4} \approx \frac{\Delta^4 u}{\Delta x^4} = \frac{u_{i+2,j} - 4u_{i+1,j} + 6u_{i,j} - 4u_{i-1,j} + u_{i-2,j}}{h^4}$$

Using the finite differences we can approximate the partial differential equation at any point (x_i, y_j) by an algebraic equation .
In the case of Poissons equation:

$$\frac{1}{h^2}\left(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right) + \frac{1}{g^2}\left(u_{i,j+1} - 2u_{i,j} + u_{i,j-1}\right) + f\left(x_i, y_j\right) = 0 .$$

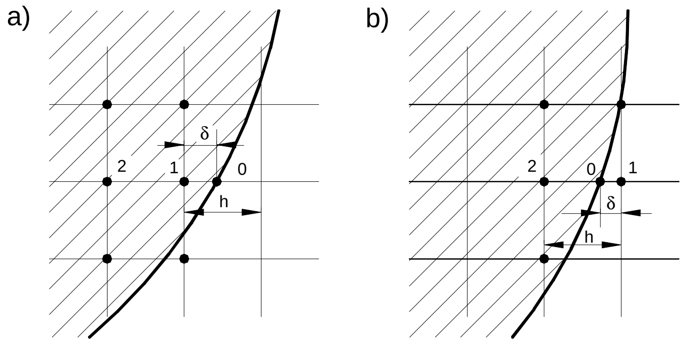
If $h = g$ i $f \equiv 0$ (Laplace equation) we get

$$u_{i,j} = \frac{u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}}{4} .$$

N grid points in the domain Ω , N equations, N unknows

$$[A]\{u\} = \{b\}$$

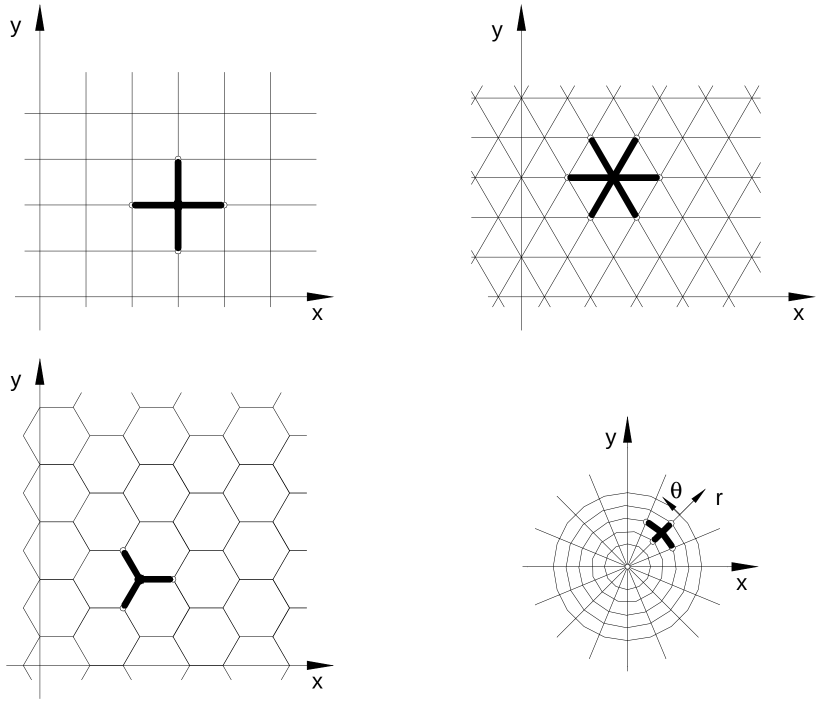
discrete form of boundary conditions



In the case of irregular boundary shape

a) assumed $u_1 = \dfrac{hu_0 + \delta u_2}{h + \delta}$ instead of $u = u_0$

b) assumed $u_1 = \dfrac{hu_0 - \delta u_2}{h - \delta}$ instead of $u = u_0$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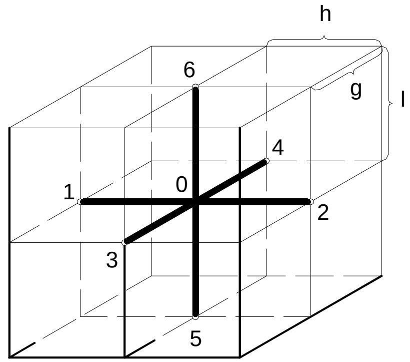
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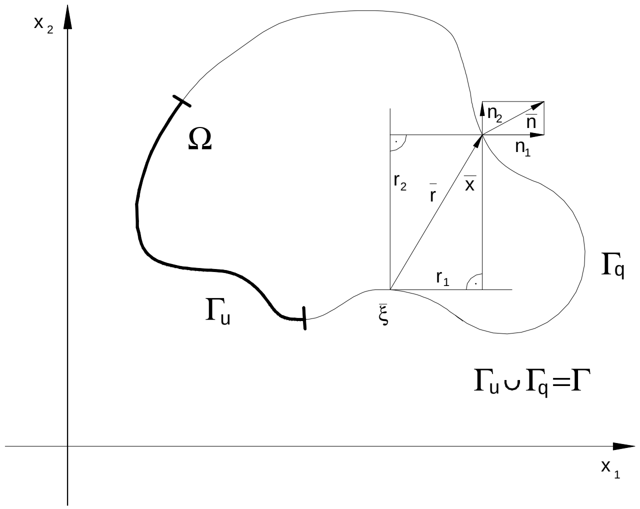
Boundary Element Method

Uses the boundary integral equation (equivalent to the Poisson's equation with the adequate b.c.)

$$c(\overline{\xi})u(\overline{\xi}) = \int_{\Gamma} u(x)q^*(\overline{\xi},\overline{x})d\Gamma(x) - \int_{\Gamma} \frac{\partial u(\overline{x})}{\partial \overline{n}} u^*(\overline{\xi},\overline{x})d\Gamma(\overline{x}) + \int_{\Omega} f(x)u^*(\overline{\xi},\overline{x})dR(\overline{x})$$

$c(\overline{\xi})$ - coefficient equal to 1 on the smooth contour, 1 inside the domain Ω



Kernel functions $u^* = (\overline{\xi},\overline{x}) = \frac{1}{2\pi}\ln\left(\frac{1}{r}\right),$

$$r = \sqrt{(x_1-\xi_1)^2+(x_2-\xi_2)^2} .$$

$$q^*(\overline{\xi},\overline{x}) = \frac{\partial u^*(\overline{\xi},\overline{x})}{\partial n} .$$

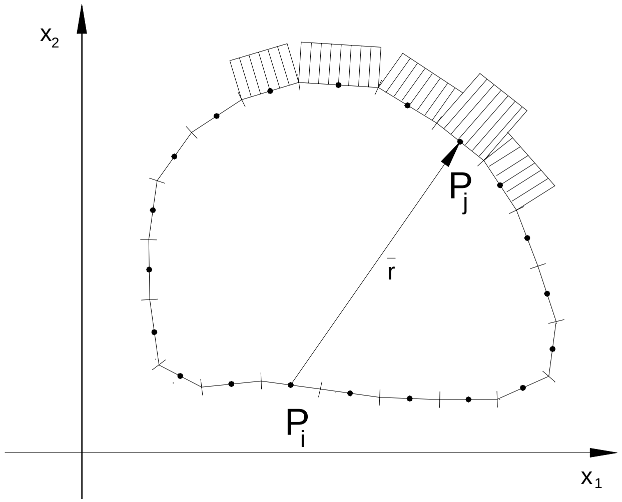
$$q^* = \frac{\partial u^*}{\partial x_1}\cdot n_1 + \frac{\partial u^*}{\partial x_2}\cdot n_2,$$

$$q^* = \frac{-(r_1\cdot n_1 + r_2\cdot n_2)}{2\pi r^2},$$

$$\frac{\partial r}{\partial x_i} = \frac{x_i-\xi_i}{r} = \frac{r_i}{r} .$$

The boundary integral equation states the relation between $u(\overline{x})$ and its derivative in normal direction $q(\overline{x}) = \frac{\partial u(\overline{x})}{\partial \overline{n}}$ on the boundary Γ.

The numerical approach



1. **Discretization of the boundary (LE boundary elements)**
2. **Approximation of** $u(\overline{x})$ and $q(\overline{x})$ on the boundary

(e.g. $u(P_i)$, $q(P_i)$ constant on boundary elements)

3 . **Building the set of linear equations**

$$\frac{1}{2}u(P_i)=\sum_{j=1}^{LE}\int_{\Gamma_j}u^*(P_i,\overline{x})q(P_j)d\Gamma_j-\sum_{j=1}^{LE}\int_{\Gamma_j}q^*(P_i,\overline{x})u(P_j)d\Gamma_j$$

$$+\int_{\Omega}f(\overline{x})u^*(P_i,\overline{x})dR \qquad i=1,2,..LE$$

$$\frac{1}{2}u(P_i)=\sum_{j=1}^{LE}U_{ij}^*\cdot q(P_j)-\sum_{j=1}^{LE}Q_{ij}^*\cdot u(P_j)+f_i, \qquad i=1,2...LE\,.\; f_i=\int_{\Omega}f(\overline{x})u^*(P_i,\overline{x})d\Omega(\overline{x})$$

$$\frac{1}{2}\{u\}=\left[U^*\right]\{q\}-\left[Q^*\right]\{u\}+\{f\}.$$

LE linear equations with the unknows $u(P_j)$ (if the point $P_j\in\Gamma_q$) or $q(P_i)$ (if $P_i\in\Gamma_u$)

Finally: $[A]\{y\}=\{b\}$

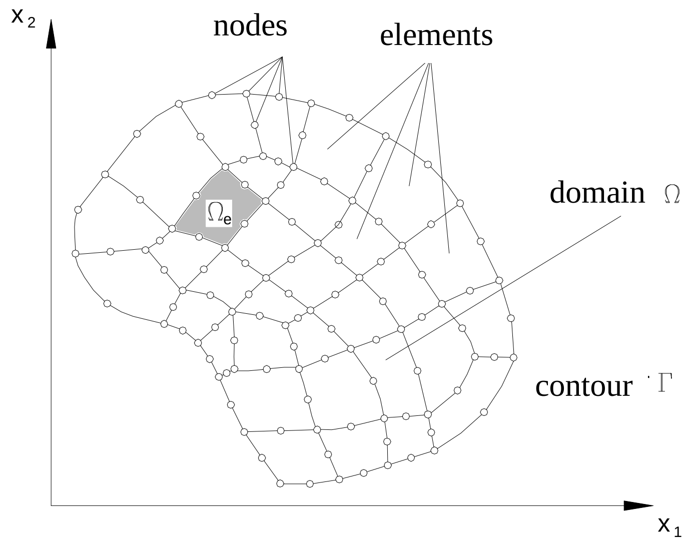
The solution {y} represents unknown boundary values of u and q.

The matrix A – full, unsymmetric

4. **Solution - provides complete information about the function** $u(\overline{x})$ **and its derivative** $q(\overline{x})$ **on the boundary**

Boundary Element Method reduces the number of unknown parameters (DOF of the discrete model) in comparison to FDM and FEM (domain methods).

Finite Element Method



Equivalent problem of minimising of the fuctional:

$$I(u)=\frac{1}{2}\int_{\Omega}\left[\left(\frac{\partial u}{\partial x_1}\right)^2+\left(\frac{\partial u}{\partial x_2}\right)^2-2f(x_1,x_2)u\right]d\Omega-\int_{\Gamma_q}q_0 u d\Gamma,$$

with the Dirchlet b. c.

$$u(\overline{x})=u_0, \qquad \overline{x}\in\Gamma_u$$

1. Discretization of the solution domain Ω into elements Ω_i, i= connected in the nodes

$$\Omega=\bigcup_{i=1}^{LE}\Omega_e \quad \text{i} \quad \Omega_i\cap\Omega_j=0 \qquad i\neq j,$$

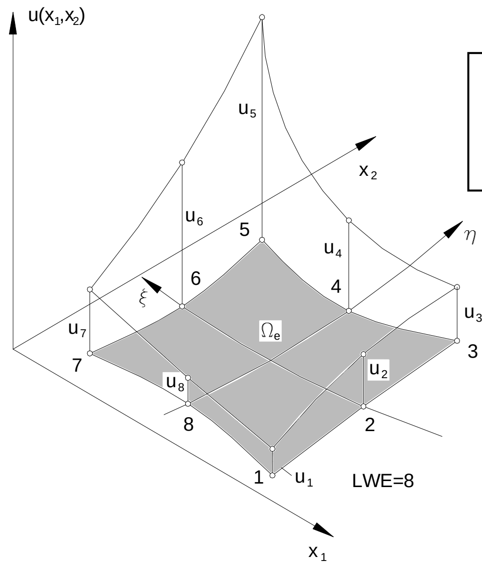
2.Approximation of function $u(\overline{x})$ **within the finite element** in the form of polynomials dependent on the unknown nodal values u_i

$$u(x_1,x_2)=\sum_{i=1}^{LWE}N_i(x_1,x_2)u_i$$

LWE – number of nodes of the element

u_i , $i = 1,...,LWE$ - nodal values of the approximated function,

$N_i(x_1,x_2)$ – shape functions



approximation of the function u(x,y) over the element Ω_e

3. Discrete form of the functional

$$I(u)\cong\sum_{i=1}^{LE}\frac{1}{2}\int_{\Omega_i}\left[\left(\frac{\partial u}{\partial x_1}\right)^2+\left(\frac{\partial u}{\partial x_2}\right)^2-2f(x_1,x_2)u\right]d\Omega_i-\sum_{j=1}^{LK}\int_{\Gamma_j}q_0 u d\Gamma_j$$

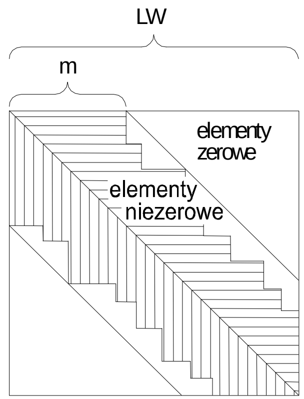
In each element

$$\frac{\partial u}{\partial x_1} = \sum_{i=1}^{LWE} \frac{\partial N_i}{\partial x_1} u_i,$$
$$\frac{\partial u}{\partial x_2} = \sum_{i=1}^{LWE} \frac{\partial N_i}{\partial x_2} u_i.$$

In this way the functional **I** is replaced by the function of several unknows u_i, $i = 1, 2, ..., LW$, where LW denotes the number of nodes of the finite element mesh. In the matrix form :

$$I(u) \approx \frac{1}{2} \lfloor u_1, u_2, u_3, ..., u_{LW} \rfloor \begin{bmatrix} k_{11} & k_{12} & k_{13} & \dots & k_{1LW} \\ k_{21} & k_{22} & k_{23} & & \\ k_{31} & k_{32} & & & \\ \dots & & & & \\ k_{LW1} & & & & k_{LW\,LW} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ \dots \\ u_{LW} \end{Bmatrix} - \lfloor u_1, u_2, u_3, \dots, u_{LW} \rfloor \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \\ \dots \\ b_{LW} \end{Bmatrix}$$

$$I \approx \frac{1}{2} \underset{1\times LW}{\lfloor u \rfloor} \underset{LW\times LW}{[K]} \underset{LW\times 1}{\{u\}} - \underset{1\times LW}{\lfloor u \rfloor} \underset{LW\times 1}{\{b\}}.$$



Necessary (and sufficient) condition of the minimum:

$$\frac{\partial I}{\partial u_i} = 0, \qquad i = 1, \dots, LW.$$

matrix: sparse, symmetrical, positive defined, banded

Hence

$$[K]\{u\} = \{b\},$$ (+ Dirichlet b.c.)

Set of the simultaneuous equations with unknown nodal values of the investigated function.