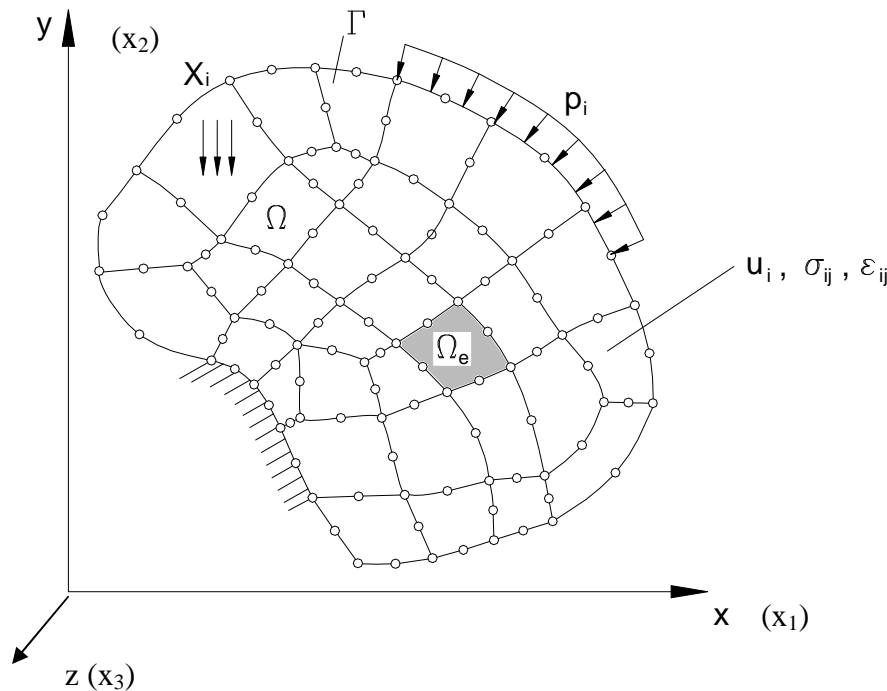


TWO AND THREE- DIMENSIONAL LINEAR ELASTOSTATICS

The finite elements of trusses and beams are, due to specific assumptions and simplifications, one –dimensional. All field problems of stress analysis are in fact three-dimensional. In some limited cases the mathematical description of the problem may be formally reduced to two dimensional models (plane stress state, plane strain state, axisymmetry) or ore even one dimensional as discussed bef.

Consider a linearly elastic body of volume Ω , which is bounded by surface Γ .



Data:

- Ω –the analysed volume (domain),
- Γ –the boundary,
- p_i –boundary tractions [N/m²].,
- X_i –body forces [N/m³].
- prescribed displacements u_i on on the part of the boundary Γ

Unknown internal fields:

- u_i –displacement field,
- ϵ_{ij} – strain state tensor,
- σ_{ij} – strss state tensor,

The body is referred to a three (or two) dimensional, rectangular, right-handed Cartesian coordinate system $x_i, i=1,3$ (or x,y,z). The body is in static equilibrium under the action of body forces X_i in Ω , *prescribed surface tractions* p_i and *prescribed displacements* u_i on on the boundary Γ The three unknown internal fields are *displacements* u_i , *strains* ϵ_{ij} and *stresses* σ_{ij} . All of them are defined in Ω .

Component notation (Einstein indicial notation) for Cartesian tensors

The notation is used in rectangular Cartesian coordinates. In this notation, writing u_i is equivalent to writing the three components u_1, u_2, u_3 of the displacement field \mathbf{u} .

The Einstein summation convention is a tensor notation, which is commonly used to implicitly define a sum. The convention states that when an index is repeated in a term that implies a sum over all possible values for that index.

Three examples:

$$\frac{\partial u_i}{\partial x_i} = \sum_i \frac{\partial u_i}{\partial x_i} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}$$

$$\frac{\partial u_i}{\partial x_j} n_j = \sum_j \frac{\partial u_i}{\partial x_j} n_j = \frac{\partial u_i}{\partial x_1} n_1 + \frac{\partial u_i}{\partial x_2} n_2 + \frac{\partial u_i}{\partial x_3} n_3$$

$a_{ij} x_j = b_i \quad i, j=1, n$ denotes the set of n linear equations

The indication of derivatives of tensors is simply illustrated in indicial notation by a comma.

$$f_{,i} = \frac{\partial f}{\partial x_i}$$

The comma in the above indicial notation indicates to take the derivative of f with respect to the coordinate x_i .

Examples: $u_{,i,i} = \frac{\partial u_i}{\partial x_i} = \sum_i \frac{\partial u_i}{\partial x_i} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}$

$$u_{,i,j} n_j = \frac{\partial u_i}{\partial x_j} n_j = \sum_j \frac{\partial u_i}{\partial x_j} n_j = \frac{\partial u_i}{\partial x_1} n_1 + \frac{\partial u_i}{\partial x_2} n_2 + \frac{\partial u_i}{\partial x_3} n_3$$

The Kronecker delta is a convenient way of expressing the identity in indicial notation: $\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

The Kronecker delta follows the rules of index notation: $A_{ik} = \delta_{ij} A_{jk}$

3 extensional strains

$$\epsilon_x = \frac{\partial u_x}{\partial x}$$

$$\epsilon_y = \frac{\partial u_y}{\partial y}$$

$$\epsilon_z = \frac{\partial u_z}{\partial z}$$

3 shearing strains

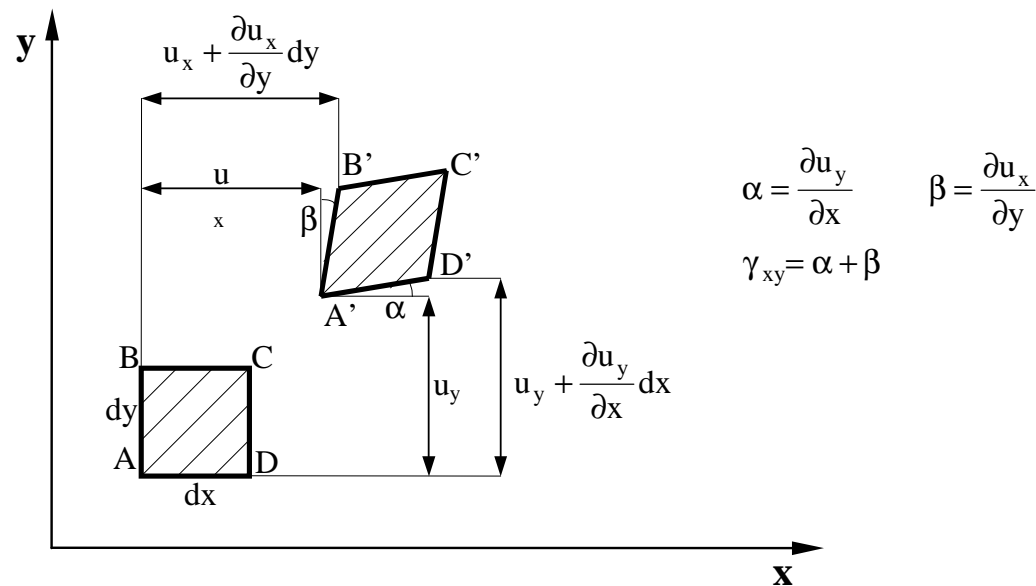
$$\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}$$

$$\gamma_{xz} = \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}$$

$$\gamma_{yz} = \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}$$

γ_{xy} , γ_{yz} , γ_{zx} - engineering shearing strains

Strain state:



The strains may be written in the form of symmetric matrix assuming that

$$\epsilon_{xy} = \gamma_{xy}/2, \quad \epsilon_{yz} = \gamma_{yz}/2, \quad \epsilon_{zx} = \gamma_{zx}/2. \text{ In this case the strains components form the symmetrical strain tensor.}$$

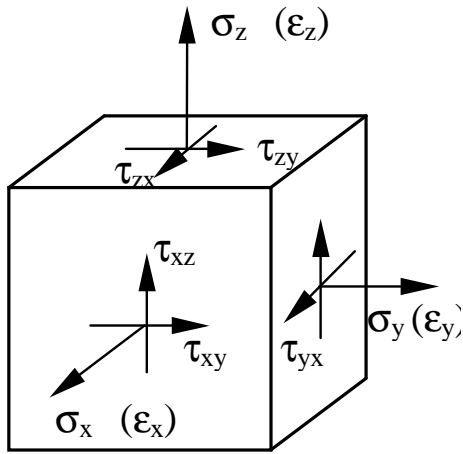
The components of the strain tensor ϵ_{ij} are often written in the form of symmetric matrix.

$$\epsilon = \begin{Bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{Bmatrix}$$

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad (\epsilon_{ij} = \epsilon_{ji}) \quad \text{- kinematic equations}$$

Stress state : stress tensor σ_{ij}

Constitutive equations (3D Hook's law)



$$\sigma_x = \sigma_{xx} \quad \sigma_y = \sigma_{yy} \quad \sigma_z = \sigma_{zz}$$

$$\left. \begin{aligned} \epsilon_x &= \frac{1}{E} [\sigma_x - \nu(\sigma_y + \sigma_z)] \\ \epsilon_y &= \frac{1}{E} [\sigma_y - \nu(\sigma_x + \sigma_z)] \\ \epsilon_z &= \frac{1}{E} [\sigma_z - \nu(\sigma_x + \sigma_y)] \\ \gamma_{xy} &= \frac{1}{G} \tau_{xy} \\ \gamma_{yz} &= \frac{1}{G} \tau_{yz} \\ \gamma_{xz} &= \frac{1}{G} \tau_{xz} \end{aligned} \right\} \Rightarrow \begin{aligned} \sigma_x &= \frac{E}{1+\nu} \left[\epsilon_x + \frac{\nu}{1-2\nu} (\epsilon_x + \epsilon_y + \epsilon_z) \right] \\ \sigma_y &= \frac{E}{1+\nu} \left[\epsilon_y + \frac{\nu}{1-2\nu} (\epsilon_x + \epsilon_y + \epsilon_z) \right] \\ \sigma_z &= \frac{E}{1+\nu} \left[\epsilon_z + \frac{\nu}{1-2\nu} (\epsilon_x + \epsilon_y + \epsilon_z) \right] \\ \tau_{xy} &= G \cdot \gamma_{xy} \\ \tau_{yz} &= G \cdot \gamma_{yz} \\ \tau_{xz} &= G \cdot \gamma_{xz} \end{aligned}$$

E-Young's modulus, $\mathbf{G} = \frac{\mathbf{E}}{2(1+\nu)}$ - shear modulus, ν - Poisson's ratio

$$\sigma_{ij} = 2G \left[\epsilon_{ij} + \frac{\nu}{1-2\nu} \delta_{ij} (\epsilon_{kk}) \right] \quad \epsilon_{ij} = \frac{1}{2G} \left(\sigma_{ij} - \frac{\nu}{1+\nu} \delta_{ij} \sigma_{kk} \right) \quad (\epsilon_{kk} = \epsilon_{11} + \epsilon_{22} + \epsilon_{33})$$

Strain energy density:

$$U' = \frac{1}{2} [\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \sigma_z \varepsilon_z + \tau_{xy} \gamma_{xy} + \tau_{yz} \gamma_{yz} + \tau_{zx} \gamma_{zx}]$$

$$U' = \frac{1}{2} \varepsilon_{ij} \sigma_{ij}$$

Principle of the total potential energy:

$$V = U - W_z = \frac{1}{2} \int_{\Omega} \sigma_{ij} \varepsilon_{ij} d\Omega - \int_{\Omega} X_i u_i d\Omega - \int_{\Gamma} p_i u_i d\Gamma = \min,$$

Matrix notation

Matrix notation is a modification of direct tensor notation in which everything is placed in matrix form, with some trickery used if need be. The main advantages of the matrix notation are historical compatibility with finite element formulations, and ready computer implementation in symbolic or numeric form.

The representation of scalars, which may be viewed as 1×1 matrices, does not change. Neither does the representation of vectors because vectors are column (or row) matrices. Two-dimensional *symmetric* tensors are converted to one-dimensional arrays that list only the independent components (six in three dimensions, three in two dimensions). Component order is a matter of convention, but usually the diagonal components are listed first followed by the off-diagonal components.

For the strain and stress tensors this “vectorization” process produces the vectors $\sigma = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix}$, $\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}$,

The relation between the strains and the displacement components in matrix notation:

$$\{ \varepsilon(x, y, z) \} = [R] \{ u(x, y, z) \},$$

$[R]$ is called symmetric gradient matrix in the continuum mechanics literature.

For 3 dimensional case :

$$\sigma = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix}, \quad \{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}, \quad [R] = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \end{bmatrix}, \quad \{u\} = \begin{Bmatrix} u_x \\ u_y \\ u_z \end{Bmatrix} = \begin{Bmatrix} u \\ v \\ w \end{Bmatrix}$$

In 2D case

$$\sigma = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}, \quad \{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}, \quad [R] = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}, \quad \{u\} = \begin{Bmatrix} u \\ v \end{Bmatrix}.$$

Hook's law:

$$\{\sigma\} = [D]\{\epsilon\},$$

$$[D] = \frac{E}{(1+\nu)(1-2\nu)}$$

1-ν	ν	ν	0	0	0
ν	1-ν	ν	0	0	0
ν	ν	1-ν	0	0	0
0	0	0	$\frac{1-2\nu}{2}$	0	0
0	0	0	0	$\frac{1-2\nu}{2}$	0
0	0	0	0	0	$\frac{1-2\nu}{2}$

Plane stress state ($\sigma_z = 0, \tau_{yz} = 0, \tau_{zx} = 0$)

$$[D] = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

Plane strain state ($\epsilon_z = 0, \gamma_{yz} = 0, \gamma_{zx} = 0$)

$$[D] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

Strain energy density

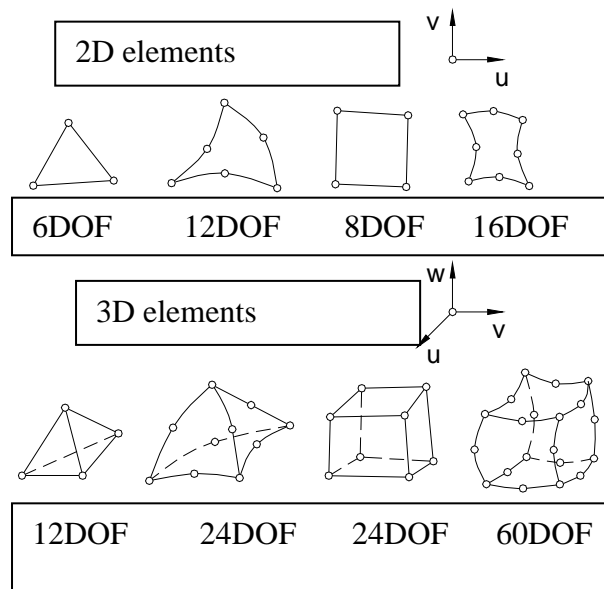
$$U' = \frac{1}{2} [\epsilon] \{\sigma\}$$

Total potential energy :

$$V = U - W_z = \frac{1}{2} \int_{\Omega} [\epsilon] \{\sigma\} d\Omega - \int_{\Omega} [X] \{u\} d\Omega - \int_{\Gamma} [p] \{u\} d\Gamma$$

Finite element method for 2D and 3D problems of theory of elasticity:

The domain Ω is divided into the subdomains (finite elements) Ω_i : $\Omega = \bigcup_{i=1}^{LE} \Omega_i$ $\Omega_i \cap \Omega_j = 0$ $i \neq j$.



2D and 3D finite elements

Displacement field over the element is interpolated from the nodal displacements:

$$\{u\} = [N(x, y, z)] \{q\}_e,$$

where $\{q\}_e$ - nodal displacements vector , $[N]$ - shape functions matrix.

For example for the simplest triangular element with 3 nodes and 6 DOF the relation is

$$\begin{Bmatrix} u(x, y) \\ v(x, y) \end{Bmatrix} = \begin{bmatrix} N_1(x, y) & 0 & N_2(x, y) & 0 & N_3(x, y) & 0 \\ 0 & N_1(x, y) & 0 & N_2(x, y) & 0 & N_3(x, y) \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} \quad \text{where } N_i \text{ are the linear functions}$$

Shape functions N_{ij} are usually polynomials defined in local (element) coordinate systems.

Displacements, strains and stresses within each element are defined as the functions of the coordinates of the considered point and the nodal displacements

$$\begin{aligned} \{u\} &= [N] \{q\}_e, \\ \{\varepsilon\} &= [R] \{u\} = [R][N] \{q\}_e = [B] \{q\}_e, \quad [B] - \text{strain-displacement matrix} \\ \{\sigma\} &= [D] \{\varepsilon\} = [D][B] \{q\}_e. \end{aligned}$$

The strain energy of the element Ω_e is:

$$\begin{aligned} U_e &= \frac{1}{2} \int_{\Omega_e} \{\varepsilon\}^T \{\sigma\} d\Omega_e, \\ U_e &= \frac{1}{2} \int_{\Omega_e} [q]_e^T [B]^T [D] [B] \{q\}_e d\Omega_e, \quad U_e = \frac{1}{2} [q]_e^T [k]_e \{q\}_e. \end{aligned}$$

Where

$$[k]_e = \int_{\Omega_e} [B]^T [D] [B] d\Omega_e = \int_{\Omega_e} [B^*] d\Omega_e,$$

is called **the stiffness matrix of the element** (symmetrical, singular, semi-positive defined) with the range equal to the number of DOF of the element. Matrix [B] depends on the position within the element so the integration requires the special numerical techniques.

Total strain energy of the structure is the sum of the finite elements energy:

$$U = \sum_{e=1}^{LE} U_e . \quad (\text{LE- number of finite elements in the model})$$

Using the global nodal displacement vector {q}

$$U = \frac{1}{2} \underset{1 \times n}{[q]} \underset{n \times n}{[K]} \underset{n \times 1}{\{q\}} ,$$

where n is total number of DOF of the model and [K] is the stiffness matrix of the model.

The next step in FEM algorithms is finding the equivalent nodal forces {F} corresponding to the distributed loads {p} and {X}.

The total potential energy of the model is:

$$V = U - W_z = \frac{1}{2} \underset{1 \times n}{[q]} \underset{n \times n}{[K]} \underset{n \times 1}{\{q\}} - \underset{1 \times n}{[q]} \underset{n \times 1}{\{F\}} ,$$

The minimum is determined by the conditions

$$\frac{\partial V}{\partial q_i} = 0 ,$$

$$[K] \{q\} = \{F\} . \quad (\text{to be solved using necessary displacement boundary conditions})$$

The strain and stress components in each finite element are found using the relations

$$\{\varepsilon\} = [B] \{q\}_e , \quad \{\sigma\} = [D] \{\varepsilon\} = [D][B] \{q\}_e$$