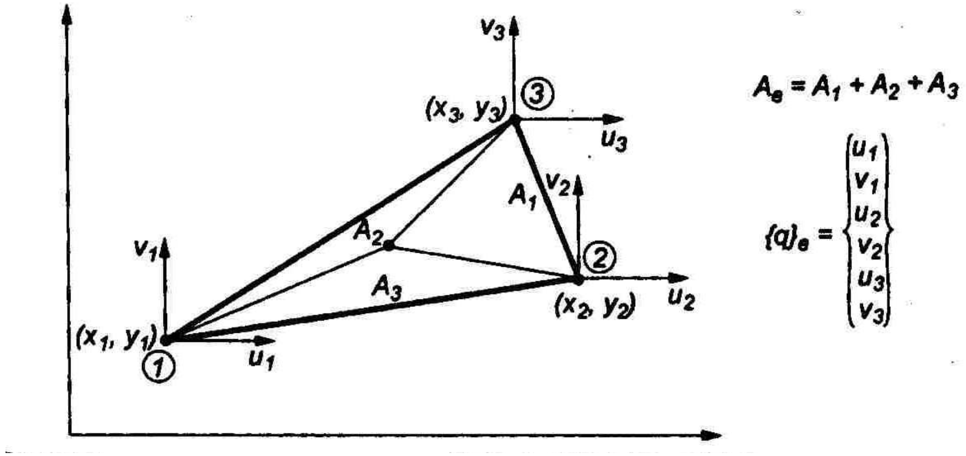
Constant Strain Triangle (CST)



$$u(x,y)=\sum_{i=1}^{3}N_i(x,y)\cdot u_i$$

$$v(x,y)=\sum_{i=1}^{3}N_i(x,y)\cdot v_i$$

$N_i(x_i, y_i) = 1, \qquad N_i(x_j, y_j) = 0 \quad \text{for } i \neq j$

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1(x,y) & 0 & N_2(x,y) & 0 & N_3(x,y) & 0 \\ 0 & N_1(x,y) & 0 & N_2(x,y) & 0 & N_3(x,y) \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}_e \qquad \{u\} = [N]\{q\}_e$$

$$N_i(x,y)=\frac{A_i(x,y)}{A_e}$$

$$A_e=\frac{1}{2}\begin{vmatrix}1 & 1 & 1\\ x_1 & x_2 & x_3\\ y_1 & y_2 & y_3\end{vmatrix}$$

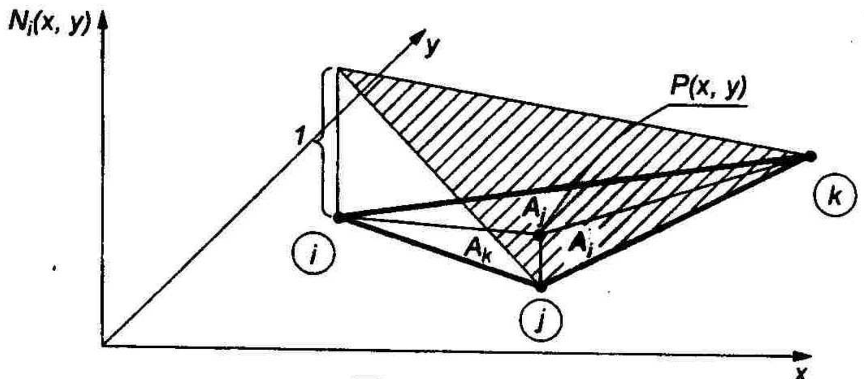
$$A_2=\frac{1}{2}\begin{vmatrix}1 & 1 & 1\\ x_1 & x & x_3\\ y_1 & y & y_3\end{vmatrix}$$

$$N_i(x,y)=\frac{1}{2A_e}\left(a_i+b_i x+c_i y\right)$$

$$a_i=x_j y_k-x_k y_j$$
$$b_i=y_j-y_k$$
$$c_i=x_k-x_j$$

$$\{\varepsilon\}=\begin{Bmatrix}\varepsilon_x(x,y)\\ \varepsilon_y(x,y)\\ \gamma_{xy}(x,y)\end{Bmatrix}=\underset{3\times2}{[R]}\underset{2\times1}{\begin{Bmatrix}u_{(x,y)}\\ v_{(x,y)}\end{Bmatrix}}=\underset{3\times2}{[R]}\underset{2\times6}{\left[N_{(x,y)}\right]}\underset{6\times1}{\{q\}_e}$$

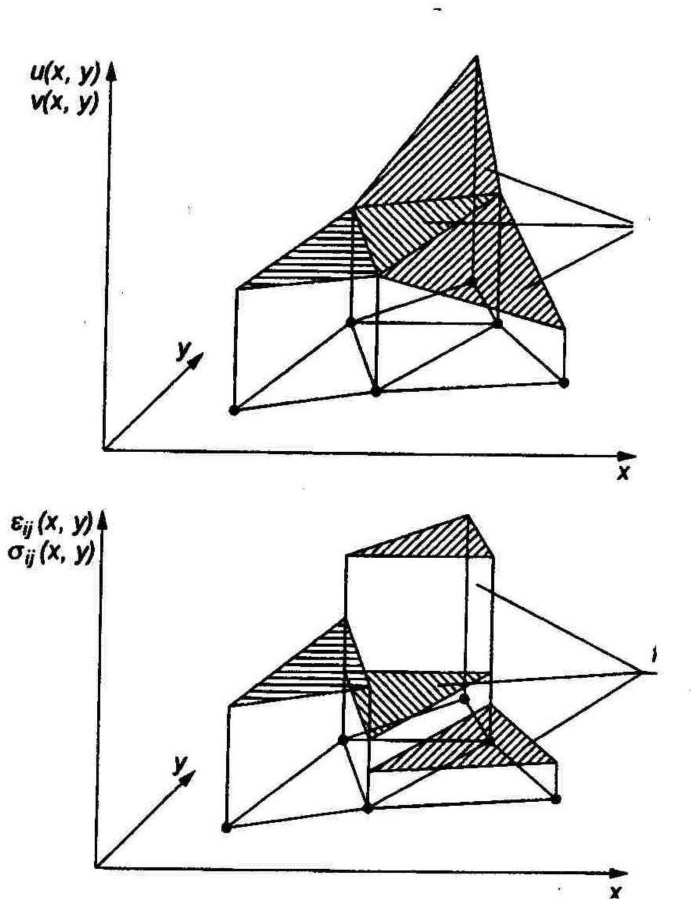
$$\{\varepsilon\}=\underset{3\times6}{[B]}\underset{6\times1}{\{q\}_e}$$



Strain- displacement matrix $[B]$:

$$[B]=[R][N]=\begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} \end{bmatrix}$$

$$[B]=\frac{1}{2A_e}\begin{bmatrix} b_1 & 0 & b_2 & 0 & b_3 & 0 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 \\ c_1 & b_1 & c_2 & b_2 & c_3 & b_3 \end{bmatrix}$$



With constant coefficients for each finite element.

CST – constant strain triangle! - linear displacement field within elements and constant strains and stresses

$$\{\sigma\}=[D]\{\varepsilon\}$$

$$\{\sigma\}=[D][B]\{q\}_e$$

STRAIN ENERGY OF THE ELEMENT

$$U_e = h_e \int_{A_e} \frac{1}{2} \lfloor \varepsilon \rfloor \{\sigma\} dA_e = A_e h_e \frac{1}{2} \lfloor q \rfloor_e [B]^T [D][B]\{q\}_e$$

$$U_e = \frac{1}{2} \lfloor q \rfloor_e [k]_e \{q\} e$$

The stiffines matrix of the CST element $[k]_e$

$$[k]_e = \frac{1}{2} A_e h_e \underset{6\times3}{[B]^T} \underset{3\times3}{[D]} \underset{3\times6}{[B]}$$

The strain energy of the entire model (N degrees of freedom)

$$U = \frac{1}{2} \lfloor q \rfloor [K] \{q\}$$

where $\{q\}$ is the total nodal displacement vector. $[K]$ matrix – symmetrical, semi-positive defined , singular

$$V = U - W_z = \frac{1}{2} \underset{1\times n}{\lfloor q \rfloor} \underset{n\times n}{[K]} \underset{n\times 1}{\{q\}} - \underset{1\times n}{\lfloor q \rfloor} \underset{n\times 1}{\{F\}} = \min!$$

<u>Global nodal forces vector $\{F\}$ is assembled from the equivalent nodal forces of all elements</u>

Minimum of V with respect to {q} → $[K]\{q\} = \{F\}$

Nodal forces of the Ω_e element equivalent to <u>the body load</u> $\lfloor X \rfloor$:

$$W_z^x = \int_{\Omega_e} \lfloor X \rfloor \{u\} d\Omega_e = \int_{\Omega_e} \lfloor X \rfloor [N] \{q\}_e \, d\Omega_e = \lfloor F^x \rfloor_e \{q\}_e ,$$

$$\lfloor F^x \rfloor_e = \int_{\Omega_e} \lfloor X \rfloor [N] d\Omega_e \qquad \text{(e.g. } F_1^X = \int_{\Omega_e} X_1(x,y) N_1(x,y) d\Omega_e \text{)}$$

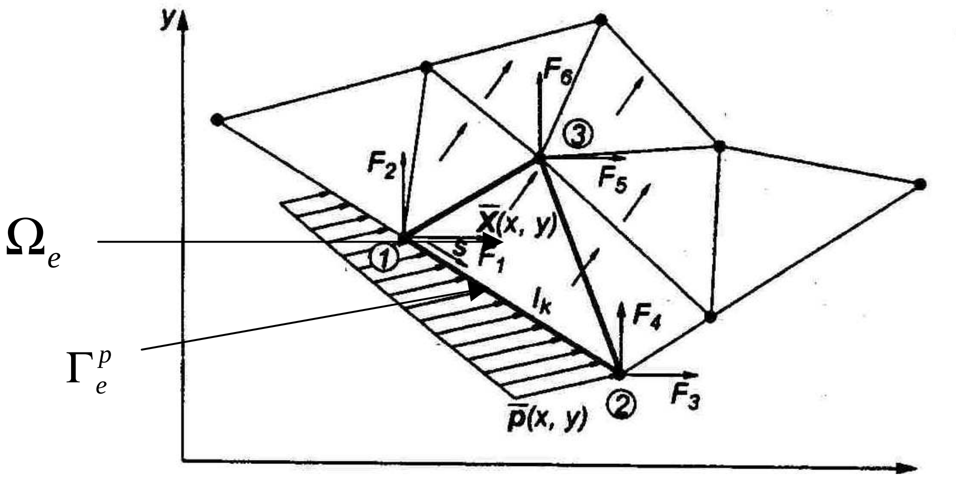
Nodal forces equivalent to the <u>surface traction p</u> acting on the edge Γ_e^p of the element Ω_e

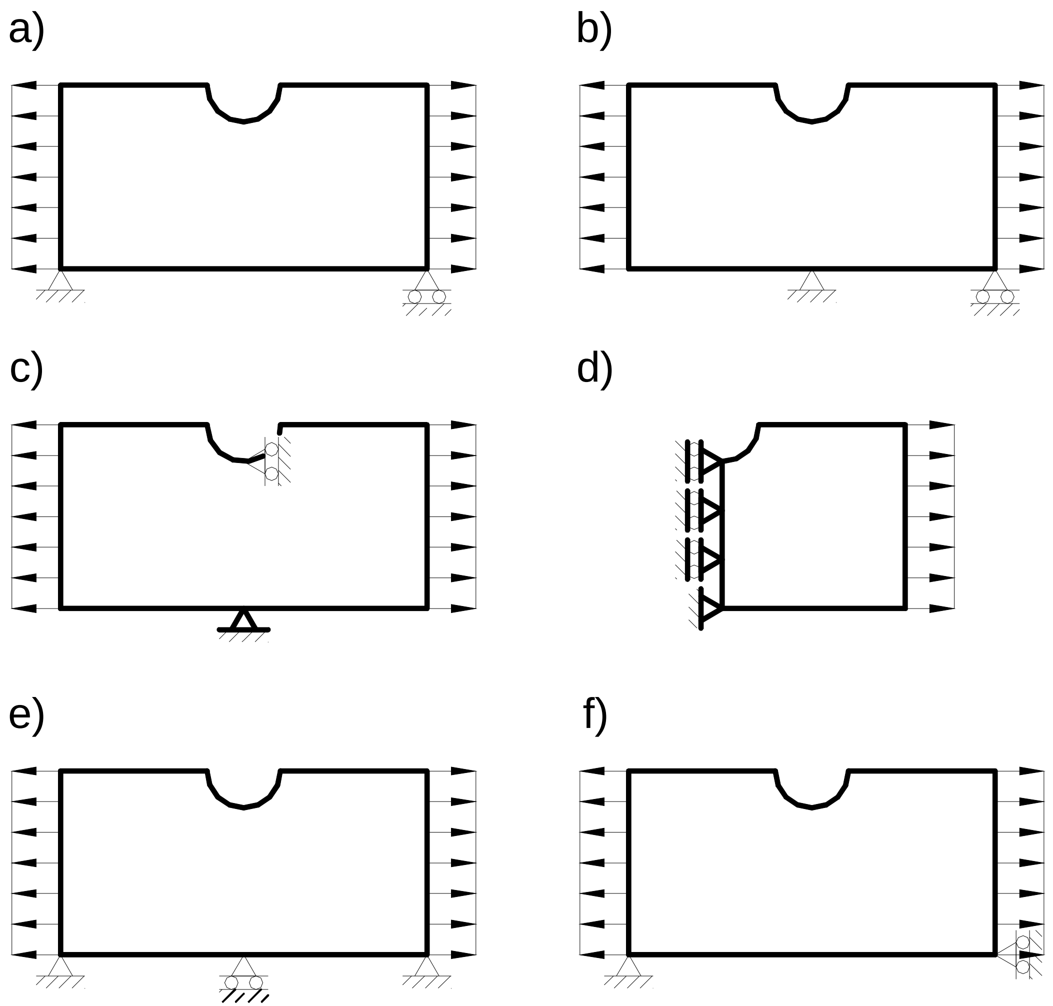
$$W_z^p = \int_{\Gamma_e^p} \lfloor p \rfloor \{u\} \, d\Gamma_e^p = \int_{\Gamma_e^p} \lfloor p \rfloor [N] \{q\}_e \, d\Gamma_e^p = \lfloor F^p \rfloor_e \{q\}_e ,$$

$$\lfloor F^p \rfloor_e = \int_{\Gamma_e^p} \lfloor p \rfloor [N] \, d\Gamma_e^p .$$

The total stiffnes matrix **K** is singular – the system of linear equations is modified by taking into account the current displacement boundary conditions.

$$\lfloor F \rfloor_e = \lfloor F_1, F_2, F_3, F_4, F_5, F_6 \rfloor = \lfloor F^X \rfloor_e + \lfloor F^p \rfloor_e$$





The 2D model of a tensioned plate (under external loads being in equilibruim). The correct and incorrect constraints (constrained rigid body motion,unconstrained deformation)

Finite element program:

Preprocessor

Information describing

- the geometry,
- the material properties ,
- the loads, the displacement boundary conditions.

Discretization of the model using the chosen type of finite elemets (e.g. CST)

Processor

Assembling the stiffness matrix using the stiffness matrices of all finite elements

- Building the set of simultaneous equations with included boundary conditions (displacement b.c. and equivalent nodal forces)
- Solution of the set of equations – calculation of all nodal displacements

Calculation of strain and stress components within all finite elements

Postprocessor

Graphical presentation of the results (contour maps, isolines , isosurfaces, graphs, animations)

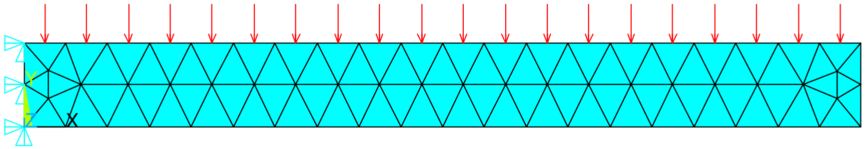
Listings, tables

User defined operations on the received results

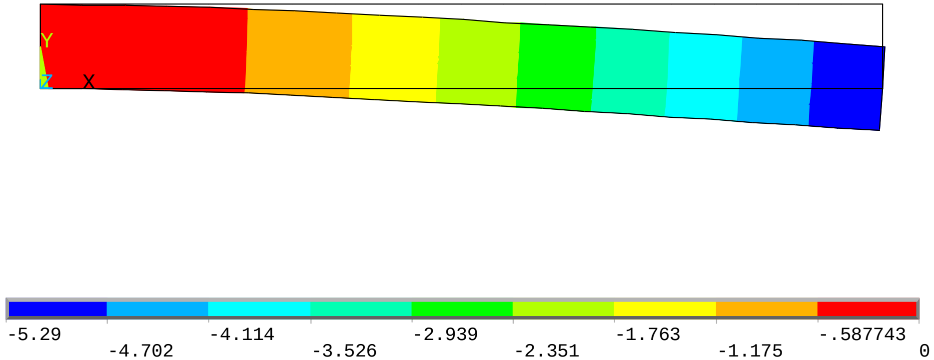
RESULTS OBTAINED USING CST ELEMENTS - AVERAGING

Example –2D FE model of the cantilever beam

Finite element mesh

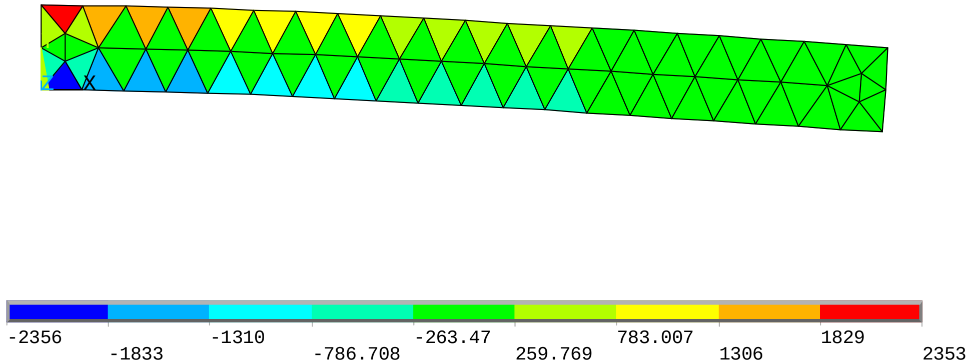


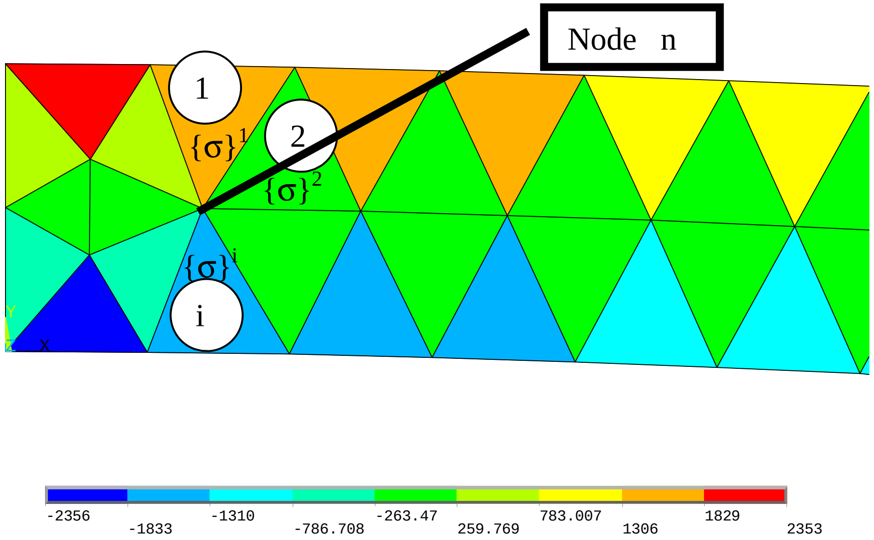
Vertical displacement distribution



Bending stress (σ_x) distribution

(**element solution)**





stress vectors in the CST elements

$$\{\sigma\}^1 \neq \{\sigma\}^2 \neq \{\sigma\}^3 \neq \ldots$$

Averaged presentation (named **nodal solution)**