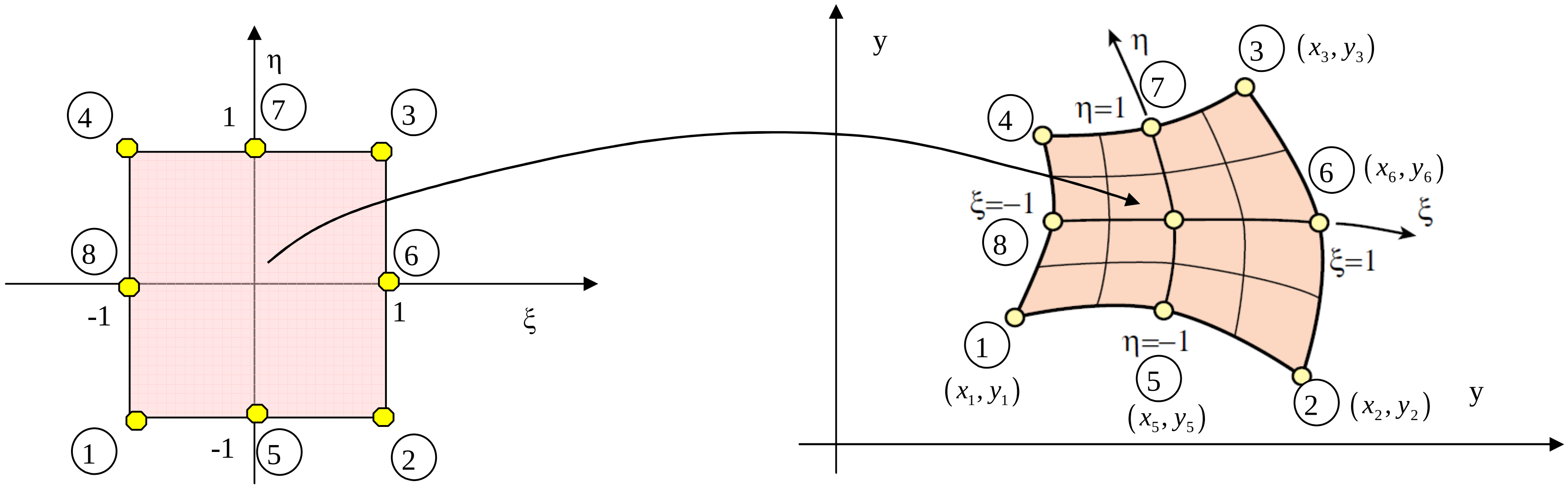
8-node quadrilateral element. Numerical integration

The technique used for the formulation of the linear triangle can be formally extended to construct quadrilateral elements as well as higher order triangles. But it is connected with some difficulties:

1. The construction of shape functions satisfying consistency requirements for higher order elements with curved boundaries becomes increasingly difficult.
2. Computations of shape function derivatives to evaluate the strain-displacement matrix.
3. Integrals that appear in the expressions of the element stiffness matrix and consistent nodal force vector can no longer be carried out in closed form.

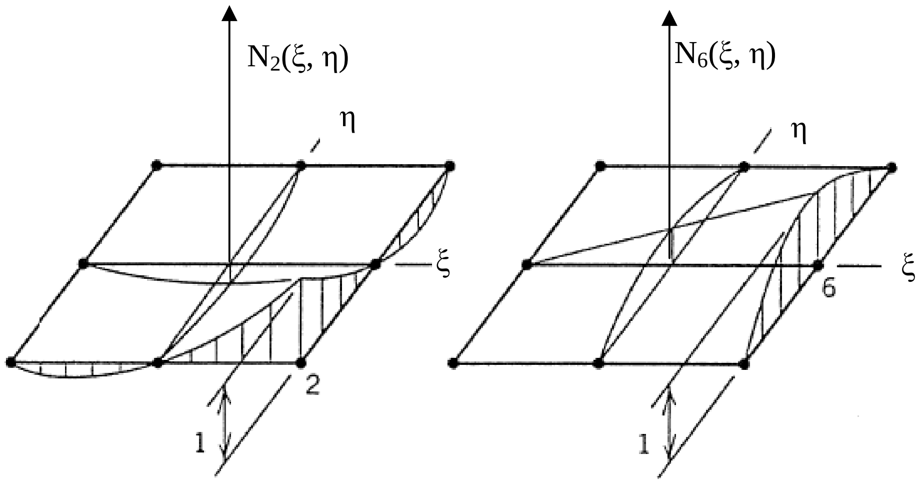
The 8-node element is defined by eight nodes having two degrees of freedom at each node: translations in the nodal x $(\boldsymbol{u})$ and y directions $(\boldsymbol{v})$. It provides more accurate results and can tolerate irregular shapes without much loss of accuracy. The 8-node are well suited to model curved boundaries.



$$(\xi,\eta) \rightarrow (x,y)$$

$(-1,-1) \rightarrow (x_1, y_1)$ $(1,-1) \rightarrow (x_2, y_2)$ $(1,1) \rightarrow (x_3, y_3)$ $(-1,1) \rightarrow (x_4, y_4)$

$(0,-1) \rightarrow (x_5, y_5)$ $(1,0) \rightarrow (x_6, y_6)$ $(0,1) \rightarrow (x_7, y_7)$ $(-1,0) \rightarrow (x_8, y_8)$



$$x(\xi,\eta)=\sum_{i=1}^{8} N_i(\xi,\eta)\,x_i$$

$$y(\xi,\eta)=\sum_{i=1}^{8} N_i(\xi,\eta)\,y_i$$

$$N_1(\xi,\eta)=-\frac{1}{4}(1-\xi)(1-\eta)(1+\xi+\eta) \qquad N_5(\xi,\eta)=\frac{1}{2}(1-\xi^2)(1-\eta)$$

$$N_2(\xi,\eta)=-\frac{1}{4}(1+\xi)(1-\eta)(1-\xi+\eta) \qquad N_6(\xi,\eta)=\frac{1}{2}(1+\xi)(1-\eta^2)$$

$$N_3(\xi,\eta)=-\frac{1}{4}(1+\xi)(1+\eta)(1-\xi-\eta) \qquad N_7(\xi,\eta)=\frac{1}{2}(1-\xi^2)(1+\eta)$$

$$N_4(\xi,\eta)=-\frac{1}{4}(1-\xi)(1+\eta)(1+\xi-\eta) \qquad N_8(\xi,\eta)=\frac{1}{2}(1-\xi)(1-\eta^2)$$

Shape functions N_2 and N_6

$$\begin{Bmatrix} x \\ y \end{Bmatrix}=\begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 & N_5 & 0 & \dots & N_8 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 & N_5 & \dots & 0 & N_8 \end{bmatrix}\begin{Bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \\ \vdots \\ x_8 \\ y_8 \end{Bmatrix}_e \qquad \begin{Bmatrix} x \\ y \end{Bmatrix}=[N]\begin{Bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \\ \vdots \\ x_8 \\ y_8 \end{Bmatrix}=[N]\{xy\}$$

$$u(\xi,\eta)=\sum_{i=1}^{8}N_i(\xi,\eta)u_i$$

$$v(\xi,\eta)=\sum_{i=1}^{8}N_i(\xi,\eta)v_i$$

$$\begin{Bmatrix}u\\v\end{Bmatrix}=\begin{bmatrix}N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 & N_5 & 0 & \dots & N_8 & 0\\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 & N_5 & \dots & 0 & N_8\end{bmatrix}\begin{Bmatrix}u_1\\v_1\\u_2\\v_2\\u_3\\v_3\\\vdots\\u_8\\v_8\end{Bmatrix}_e \qquad \begin{Bmatrix}u\\v\end{Bmatrix}=[N]\{q\}_e$$

$$\underset{3\times1}{\{\varepsilon\}}=\underset{3\times2}{[R]}\underset{2\times1}{\{u\}}=\underset{3\times2}{\begin{bmatrix}\frac{\partial}{\partial x} & 0\\ 0 & \frac{\partial}{\partial y}\\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x}\end{bmatrix}}\underset{2\times16}{\left[N(\xi,\eta)\right]}\underset{16\times1}{\{q\}_e}=\underset{3\times16}{[B]}\underset{16\times1}{\{q\}}$$

$$[B]=\begin{bmatrix}\frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 & \dots & \frac{\partial N_8}{\partial x} & 0\\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} & \dots & 0 & \frac{\partial N_8}{\partial y}\\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} & \dots & \frac{\partial N_8}{\partial y} & \frac{\partial N_8}{\partial x}\end{bmatrix}$$

Partial derivatives of shape functions with respect to the Cartesian coordinates *x* and *y* are required for the strain and stress calculations. Since the shape functions are not directly functions of *x* and *y* but of the natural (local) coordinates ξ and η, the determination of Cartesian partial derivatives is not trivial.

We need the Jacobian of two-dimensional transformations that connect the differentials of $\{x, y\}$ to those of $\{\xi, \eta\}$ and vice-versa

$$[J] = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{8} \frac{\partial N_i}{\partial \xi} \cdot x_i & \sum_{i=1}^{8} \frac{\partial N_i}{\delta \xi} \cdot y_i \\ \sum_{i=1}^{8} \frac{\partial N_i}{\partial \eta} \cdot x_i & \sum_{i=1}^{8} \frac{\partial N_i}{\eta} \cdot y_i \end{bmatrix} = [J(\xi, \eta)]$$

Matrix **J** is called the *Jacobian matrix* of (x, y) with respect to (ξ, η), whereas $\mathbf{J}^{-1}$ is the Jacobian matrix of (ξ, η) with respect to (x, y). **J** and $\mathbf{J}^{-1}$ are often called the *Jacobian* and *inverse Jacobian*, respectively. The scalar symbol *J* means the determinant of **J**: $J = |\mathbf{J}| = \det \mathbf{J}$. Jacobians play a crucial role in differential geometry.

$$\frac{\partial N_i}{\partial x} = \frac{\partial N_i}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} + \frac{\partial N_i}{\partial \eta} \frac{\partial \eta}{\partial x}$$
$$\frac{\partial N_i}{\partial y} = \frac{\partial N_i}{\partial \xi} \cdot \frac{\partial \xi}{\partial y} + \frac{\partial N_i}{\partial \eta} \frac{\partial \eta}{\partial y}$$

$$\begin{Bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{Bmatrix} = \begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} \end{bmatrix} \begin{Bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{Bmatrix}$$

$$\begin{Bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{Bmatrix} = [J] \begin{Bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{Bmatrix}$$

$$\{\varepsilon\} = \left[B(\xi,\eta)\right]\{q\}_e$$

$$U_e = \int_{\Omega_e(x,y))} \frac{1}{2}\lfloor \varepsilon \rfloor \{\sigma\}\, dxdy = \frac{1}{2}\int_{\Omega_e(x,y)} \lfloor q \rfloor_e [B]^T [D][B]\{q\}_e\, dxdy$$

$$\int_{A(x,y)} f(x,y)\, dxdy = \int_{A(\xi,\eta)} f(\xi,\eta) \det[J]\, d\xi d\eta \qquad dxdy = \det[J]\, d\xi d\eta$$

$$U_e = 1/2 \lfloor q \rfloor_e \int_{\Omega_e(\xi,\eta)} \underset{16\times3}{\left[B(\xi,\eta)\right]^T} \underset{3\times3}{[D]} \underset{3\times16}{\left[B(\xi,\eta)\right]} \det\left[J(\xi,\eta)\right] d\xi d\eta\ \{q\}_e$$

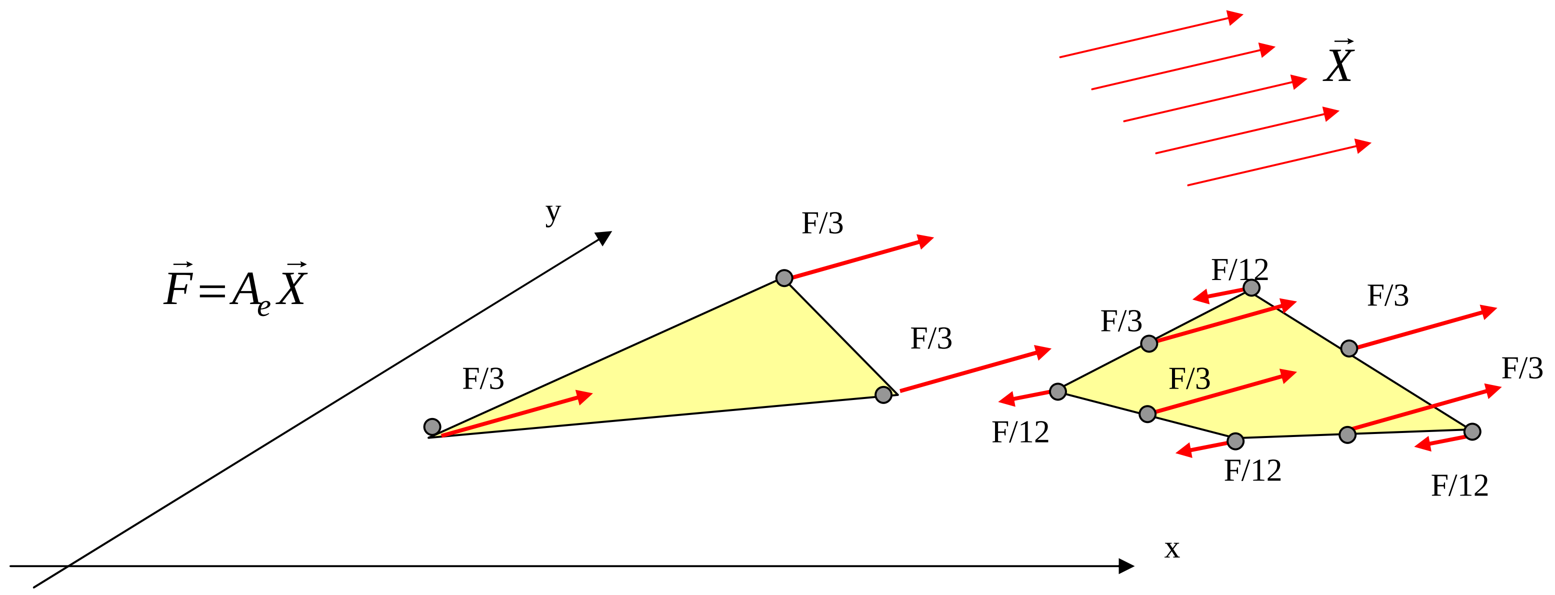
$$U_e = \frac{1}{2}\lfloor q \rfloor_e [k]_e \{q\}_e$$

$$[k]_e = \int_{\Omega_e(x,y)} [B]^T [D][B]\, dxdy = \int_{-1}^{1}\int_{-1}^{1} \underset{16x3}{\left[B(\xi,\eta)\right]^T} \underset{3\times3}{[D]} \underset{3\times16}{\left[B(\xi,\eta)\right]} \det\left[J(\xi,\eta)\right] d\xi d\eta \int \underset{16\times3}{\left[B(\xi,\eta)\right]}$$

Nodal forces of the Ω_e element equivalent to <u>the body load</u>:

$$W_z^x = \int_{\Omega_e} \lfloor X \rfloor \{u\} d\Omega_e = \int_{\Omega_e} \lfloor X \rfloor [N] \{q\}_e \, d\Omega_e = \lfloor F^x \rfloor_e \{q\}_e \,,$$

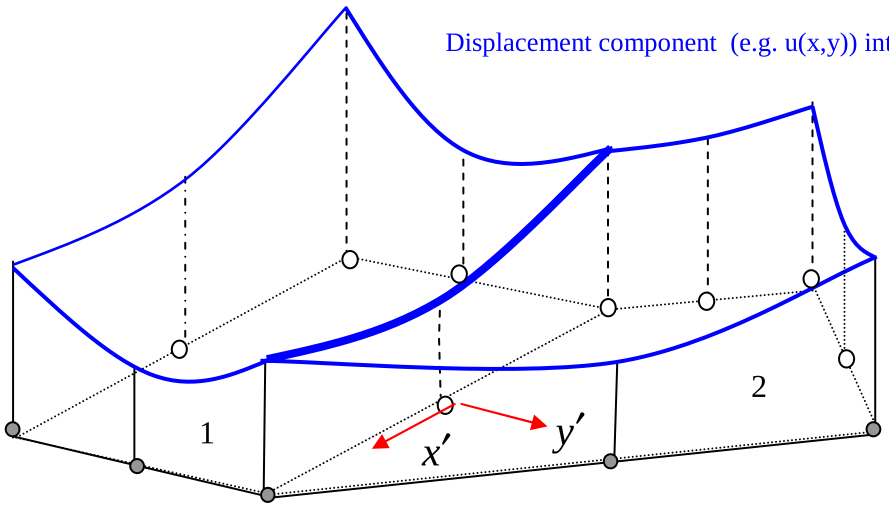
$$\lfloor F^x \rfloor_e = \int_{\Omega_e} \lfloor X \rfloor [N] \, d\Omega_e \,.$$



Work-equivalent nodal forces for uniform constant body load in the case of CST element and 8-node quadrilateral element

<u>Finite element method results: continuous displacement field and discontinous stress field</u>

Displacement component (e.g. u(x,y)) interpolation over two finite elements



$$\left.\frac{\partial u}{\partial x'}\right|_1 = \left.\frac{\partial u}{\partial x'}\right|_2 \qquad \left.\frac{\partial u}{\partial y'}\right|_1 \neq \left.\frac{\partial u}{\partial y'}\right|_2 \quad \Longrightarrow \quad \left(\varepsilon_{x'}\right)_1 = \left(\varepsilon_{x'}\right)_2 \qquad \left(\varepsilon_{y'}\right)_1 \neq \left(\varepsilon_{y'}\right)_2 \quad \Longrightarrow \left(\sigma_{ij}\right)_1 \neq \left(\sigma_{ij}\right)_2$$

Numerical Gauss integration in FE algorihms

The use of numerical integration is essential for evaluating element integrals of isoparametric elements. The standard practice has been to use *Gauss integration*[1] because such rules use a minimal number of sample points to achieve a desired level of accuracy. This property is important for efficient element calculations because we shall see that at each sample point we must evaluate a matrix product.

$$[k]_e = \int_{\Omega_e(\xi,\eta)} [B]^T [D][B]\, dxdy = \int_{\Omega_e(x,y)} \Big[\underset{16\times3}{B(\xi,\eta)}\Big]^T \underset{3\times3}{[D]} \Big[\underset{3\times16}{B(\xi,\eta)}\Big] \det\big[J(\xi,\eta)\big]\, d\xi d\eta$$

The numerical intergration have to be also performed for finding the equivalent nodal forces.

One dimensional integration

In general:

$$\int_a^b F(x)\, dx = \sum_{i=1}^{n} \alpha_i F_i(x_i) + R_n$$

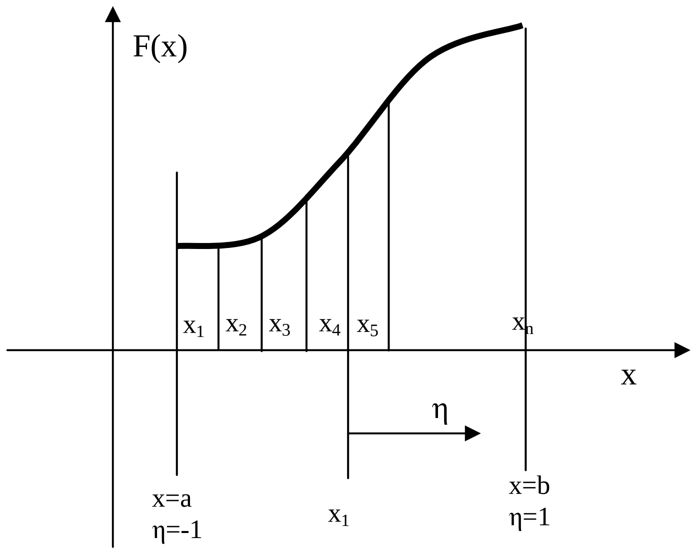
Introducing the new variable -1 ≤η≤ 1

$$x = \frac{(a+b)}{2} + \frac{b-a}{2}\cdot\eta \qquad dx = \frac{b-a}{2}\eta$$

$$\int_a^b F(x)\, dx = \int_{-1}^{1} f(\eta)\frac{b-a}{2}\, d\eta = \frac{b-a}{2}\int_{-1}^{1} f(\eta)\, d\eta$$

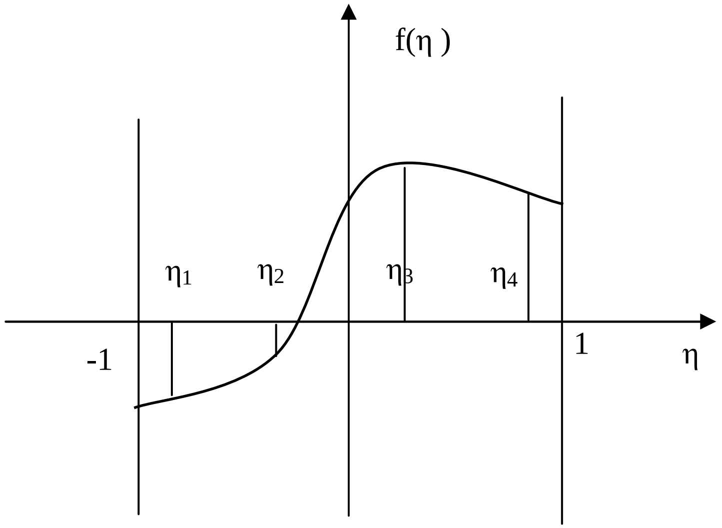
The Gauss integration

$$\int_{-1}^{1} f(\eta)\, d\eta = \sum_{i=1}^{n} w_i f(\eta_i) + R_n \qquad R_n = 0\left(\frac{d^{2n} f}{d\eta^{2n}}\right)$$



Here $n \geq 1$ is the number of especially defined Gauss integration points, w_i are the integration weights, and ηi are sample-point abcissae in the interval [−1,1]. The use of the interval [−1,1] is no restriction, because an integral over another range, from *a* to *b* can be transformed to the standard interval via a simple linear transformation of the independent variable, as shown above. The values η_i and w_i are defined in such a way to aim for best accuracy. Indeed, if we assume a polynomial expression, it is easy to check that for *n* sampling a polynomial of degree $2n - 1$ can be exactly integrated .

Table below shows the positions and weighting coefficients for gaussian integration.



Abscissae and weight coefficients of the gaussian Quadrature

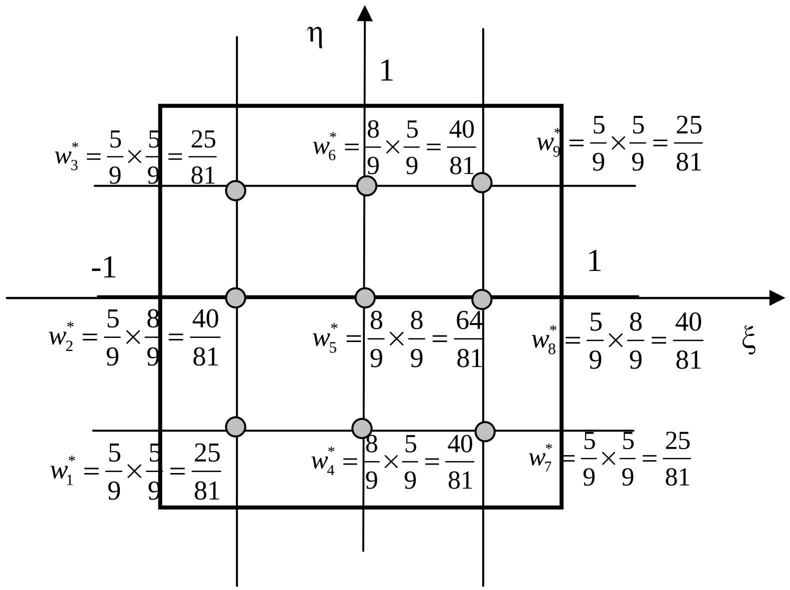
n	$ç_i$ (i=1,n)	W_i (i=1,n)
1	0	2
2	$-1/\sqrt{3}$ $+1/\sqrt{3}$	1 1
3	$-\sqrt{0.6}$ 0 $+\sqrt{0.6}$	5/9 8/9 5/9
4	-0.861136311594953 -0.339981043584856 +0.339981043584856 +0.861136311594953	0.347854845137454 0.652145154862546 0.652145154862546 0.347854845137454
5		

Remarks: The sum of weighing coefficients is always 2
The integration gives the exact solution for polynomials of 2n-1 degree.

Numerical integration – rectangular region:

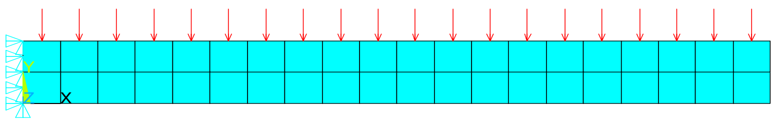
$$\int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)\, d\xi d\eta \approx \int_{-1}^{1}\left(\sum_{i=1}^{n} f(\xi_i,\eta)\, w_i\right) d\eta \approx \sum_{j=1}^{n} w_j \sum_{i=1}^{n} w_i\, f(\xi_i,\eta_j) =$$

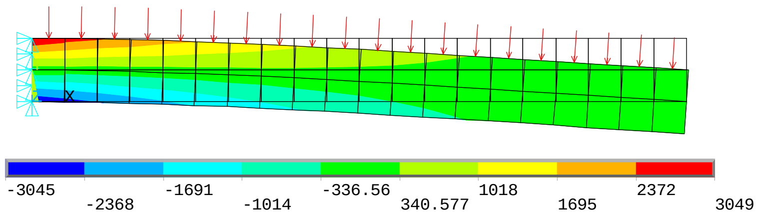
$$= \sum_{i=1}^{n}\sum_{i=1}^{n} w_i w_j\, f(\xi_i,\eta_j) = \sum_{k=1}^{m} w_k^* \; f(\xi_k,\eta_k)$$



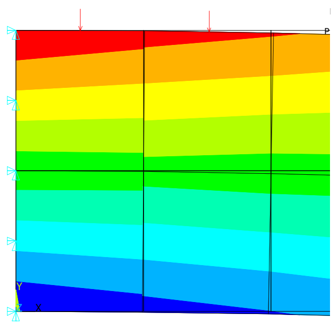
RESULTS OBTAINED USING 8-NODE ELEMENTS - AVERAGING

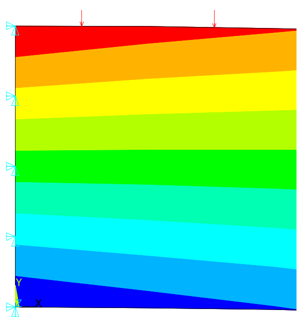
Example –2D FE model of the cantilever beam

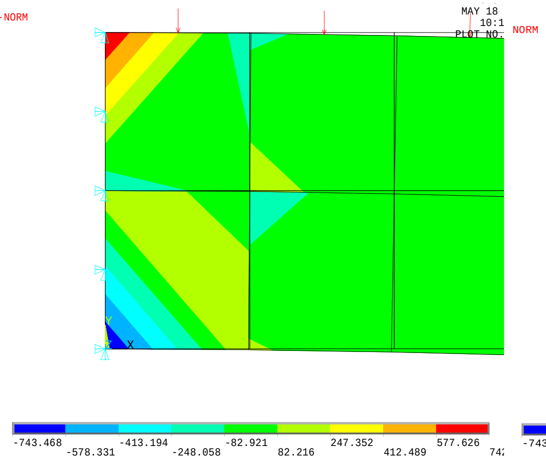


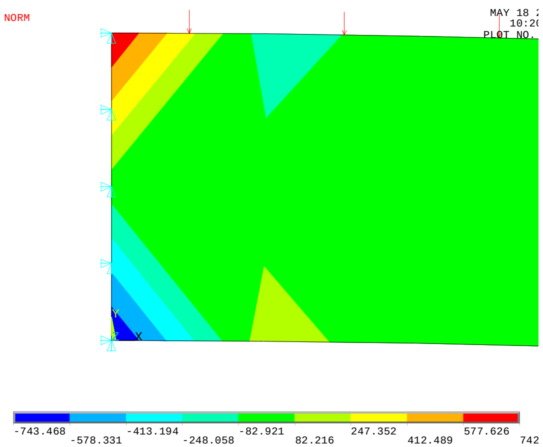


Bending stress σ_x distribution (element solution)









σ_x element and nodal solution σ_y element and nodal solution