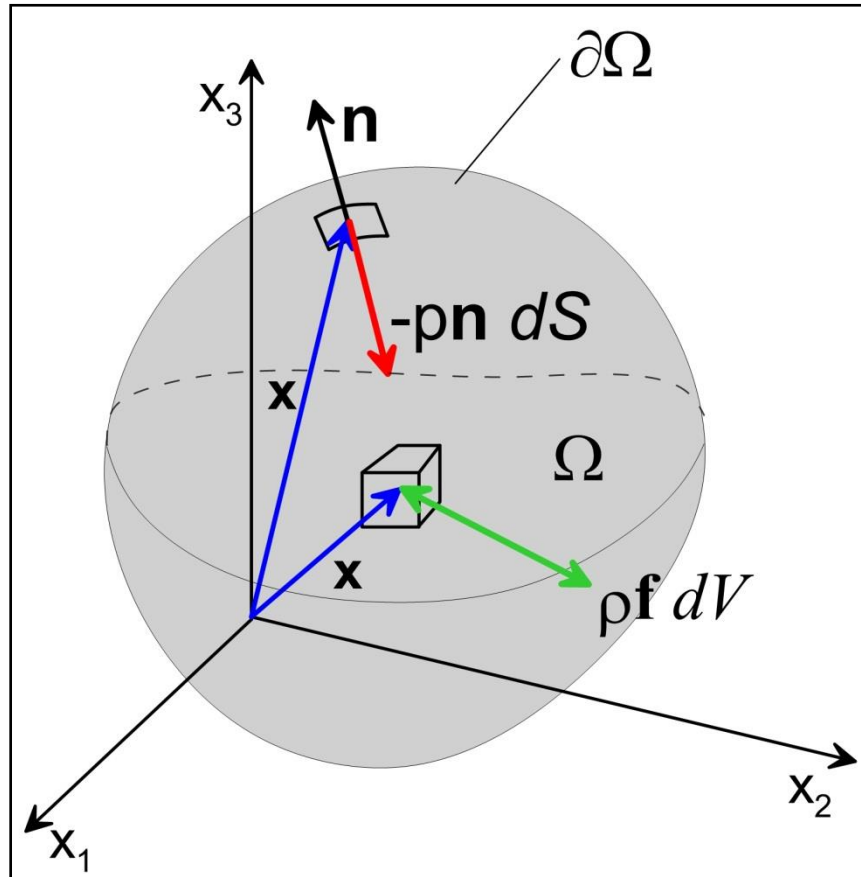


# LECTURE 1

# ELEMENTS OF STATICS OF FLUIDS



## BALANCE OF EXTERNAL FORCES



Total surface force  $\mathbf{F}_S = - \int_{\partial\Omega} \mathbf{p}\mathbf{n} dS$

Total volumetric force  $\mathbf{F}_V = \int_{\Omega} \rho \mathbf{f} dV$

Balance of forces  $\mathbf{F}_S + \mathbf{F}_V = \mathbf{0}$

We obtain the integral form of the balance equation

$$- \int_{\partial\Omega} \mathbf{p}\mathbf{n} dS + \int_{\Omega} \rho \mathbf{f} dV = \mathbf{0}$$

We need a differential form of the balance equation. To this aim, we will transform the surface integral into the volume one.

$$\int_{\partial\Omega} p\mathbf{n} dS = \mathbf{e}_1 \int_{\partial\Omega} pn_1 dS + \mathbf{e}_2 \int_{\partial\Omega} pn_2 dS + \mathbf{e}_3 \int_{\partial\Omega} pn_3 dS$$

Consider one of the above scalar integrals ...

$$\int_{\partial\Omega} pn_1 dS = \int_{\partial\Omega} [p, 0, 0] \cdot \mathbf{n} dS = \int_{\Omega} \text{div}[p, 0, 0] dV = \int_{\Omega} \frac{\partial p}{\partial x_1} dV$$

*GGO theorem*

We conclude that

$$\int_{\partial\Omega} pn_k dS = \int_{\Omega} \frac{\partial p}{\partial x_k} dV, \quad k = 1, 2, 3 \Rightarrow \int_{\partial\Omega} p\mathbf{n} dS = \int_{\Omega} \nabla p dV$$

Thus, the balance equation takes the form of  $\int_{\Omega} (-\nabla p + \rho \mathbf{f}) dV = \mathbf{0}$

Since the volume  $\Omega$  can be chosen arbitrary, the above equality implies that the integrand vanishes identically in the whole volume of fluid. We get the following differential equation

$$-\nabla p + \rho \mathbf{f} = \mathbf{0}$$

**Do we need to worry about the balance of moments of the forces?**

These moments are equal

$$\mathbf{M}_S = - \int_{\partial\Omega} \mathbf{x} \times p \mathbf{n} dS \quad , \quad \mathbf{M}_V = \int_{\Omega} \mathbf{x} \times \rho \mathbf{f} dV$$

The fluid is motionless if and only if  $\mathbf{M}_S + \mathbf{M}_V = \mathbf{0}$

We will show that this condition is satisfied automatically! Again, consider the surface integral

$$\int_{\partial\Omega} \mathbf{x} \times p \mathbf{n} dS = \mathbf{e}_1 \int_{\partial\Omega} p (\mathbf{x} \times \mathbf{n})_1 dS + \mathbf{e}_2 \int_{\partial\Omega} p (\mathbf{x} \times \mathbf{n})_2 dS + \mathbf{e}_3 \int_{\partial\Omega} p (\mathbf{x} \times \mathbf{n})_3 dS$$

We will transform this surface integral to the volumetric one ...

Consider the first integral ...

$$\begin{aligned} \int_{\partial\Omega} p(\mathbf{x} \times \mathbf{n})_1 dS &= \int_{\partial\Omega} p(x_2 n_3 - x_3 n_2) dS = \\ &= \int_{\partial\Omega} [0, 0, px_2] \cdot \mathbf{n} dS - \int_{\partial\Omega} [0, px_3, 0] \cdot \mathbf{n} dS = \int_{\Omega} \left[ \frac{\partial}{\partial x_3} (px_2) - \frac{\partial}{\partial x_2} (px_3) \right] dV = \\ &= \int_{\Omega} \left[ x_2 \frac{\partial}{\partial x_3} p - x_3 \frac{\partial}{\partial x_2} p \right] dV = \int_{\Omega} (\mathbf{x} \times \nabla p)_1 dV \end{aligned}$$

In general, we have

$$\int_{\partial\Omega} p(\mathbf{x} \times \mathbf{n}) dS = \int_{\Omega} (\mathbf{x} \times \nabla p) dV$$

The condition for the balance of the moments can be written as

$$\int_{\Omega} \mathbf{x} \times (-\nabla p + \rho \mathbf{f}) dV = \mathbf{0}$$

The above equation is actually the identity as the integrand vanishes identically at any point of the fluid volume. Thus, there is only one **equation of fluid statics**, namely

$$\nabla p = \rho \mathbf{f}$$

**Is this equation always solvable? The answer is NO!**

The solution exists only if the right-hand side of this equation (i.e.,  $\rho \mathbf{f}$ ) is the **potential field**. This means that the **rotation of this field should be identically equal to zero**

$$\text{rot}(\rho \mathbf{f}) \equiv \nabla \times (\rho \mathbf{f}) = \mathbf{0}$$

Let us remind that the *rot* operator is defined as follows (in Cartesian coordinates)

$$\text{rot } \mathbf{w} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ w_1 & w_2 & w_3 \end{vmatrix} = \left( \frac{\partial}{\partial x_2} w_3 - \frac{\partial}{\partial x_3} w_2 \right) \mathbf{e}_1 + \left( \frac{\partial}{\partial x_3} w_1 - \frac{\partial}{\partial x_1} w_3 \right) \mathbf{e}_2 + \left( \frac{\partial}{\partial x_1} w_2 - \frac{\partial}{\partial x_2} w_1 \right) \mathbf{e}_3$$

The Reader is recommended to verify the following identity (exercise!)

$$\nabla \times (\rho \mathbf{f}) = \rho \nabla \times \mathbf{f} + \nabla \rho \times \mathbf{f} = \mathbf{0}$$

The above equality is the necessary condition for the fluid to remain in rest in the external field  $\mathbf{f}$ . It does not tell us much, unless we assume something more about properties of the field  $\mathbf{f}$ .

Assume that the  $\mathbf{f}$  is the **potential force field**. It means that there exists such scalar field  $\Phi$ , that

$$\mathbf{f} = \text{grad} \Phi \equiv \nabla \Phi$$

Then

$$\nabla \times \mathbf{f} \equiv \text{rot} \mathbf{f} = \text{rot} \text{grad} \Phi \equiv \nabla \times \nabla \Phi = \mathbf{0}$$

and

$$\nabla \times (\rho \mathbf{f}) = \nabla \rho \times \mathbf{f} = \mathbf{0}$$

We conclude that the **gradient of density in a motionless fluid is everywhere aligned with the force field**, i.e.

$$\nabla \rho \parallel \mathbf{f}$$

If we re-write the balance equation as  $\frac{1}{\rho} \nabla p = f$

then for the potential force field  $\mathbf{f}$  we also have

$$0 = \nabla \times \left( \frac{1}{\rho} \nabla p \right) = \nabla \left( \frac{1}{\rho} \right) \times \nabla p + \frac{1}{\rho} \nabla \times \nabla p = -\frac{1}{\rho^2} \nabla \rho \times \nabla p$$

We conclude that  $\nabla \rho \parallel \nabla p$

Thus, the **surfaces of constant density and surfaces of constant pressure (isobaric) coincide**. This implies that there exists a global relation between density and pressure

$$\rho = \rho(p)$$

We say that the **fluid is in barotropic conditions** (or – shortly – the **fluid is barotropic**). This is a very special situation because – in general – we need both pressure and temperature to determine density (generally, the fluids are **baroclinic**). Note that **if the barotropic fluid remains in rest then the external force field must be a potential one**. In other words: the **barotropic fluid cannot stay in rest in the force field which is not potential**.



If the fluid is barotropic then the differential equation of fluid statics can be solved (integrated). To this aim, we define the function (sometimes called the pressure potential)  $P$  such that

$$P'(p) = \frac{1}{\rho(p)}$$

Thus, the function  $P$  can be defined as the indefinite integral

$$P = P(p) = \int \frac{dp}{\rho(p)}$$

Next, we define the composite function as follows  $\tilde{P}(\mathbf{x}) = P[p(\mathbf{x})]$ . The derivatives with respect to spatial variables are

$$\frac{\partial}{\partial x_k} \tilde{P}(\mathbf{x}) = P'[p(\mathbf{x})] \frac{\partial}{\partial x_k} p(\mathbf{x}) , \quad k = 1, 2, 3$$

Equivalently, we have the following relation

$$\nabla \tilde{P} = \frac{1}{\rho} \nabla p$$

Since the force field  $\mathbf{f}$  is assumed potential, there exist the scalar field  $\Phi$  such that

$$\mathbf{f} = \nabla \Phi.$$

Then, the equation of statics can be written in the following form

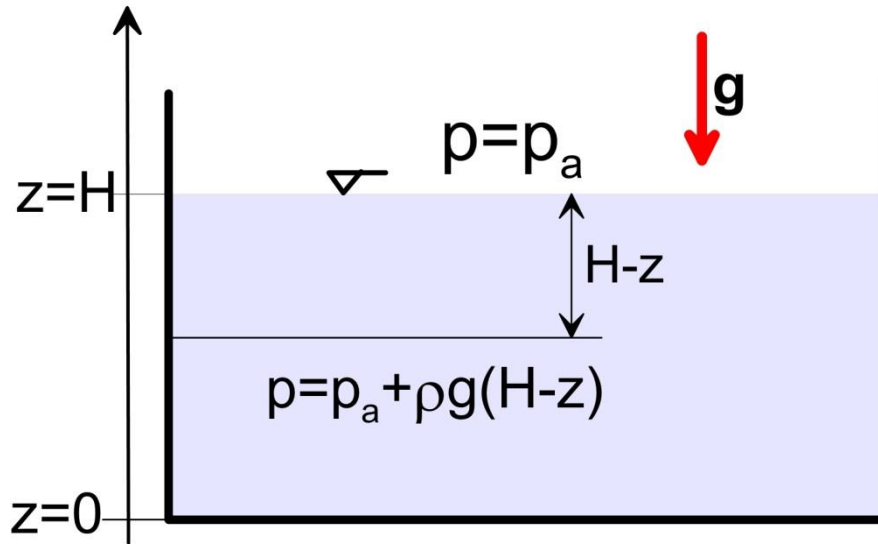
$$\nabla \tilde{P} = \nabla \Phi$$

from which we conclude that the pressure and force field potential differ by the additive constant

$$\tilde{P} = \Phi + \text{const}$$

## EXAMPLES

### 1. Constant-density fluid (liquid) in the uniform, unidirectional gravity field



Assume that  $\mathbf{f} = [0, 0, -g]$

The potential of  $\mathbf{f}$  is  $\Phi = -gz + const$

Since density is fixed we have  $P = p / \rho$

Thus  $p = \rho\Phi + C = -\rho gz + C$

The value of the pressure constant should be determined. Usually, the pressure at certain point (or points) of the fluid volume is known. For instance, the pressure at the free surface is equal to the ambient pressure. Then, we can write

$$p(H) = p_a = C - \rho gH \quad \Rightarrow \quad C = p_a + \rho gH$$

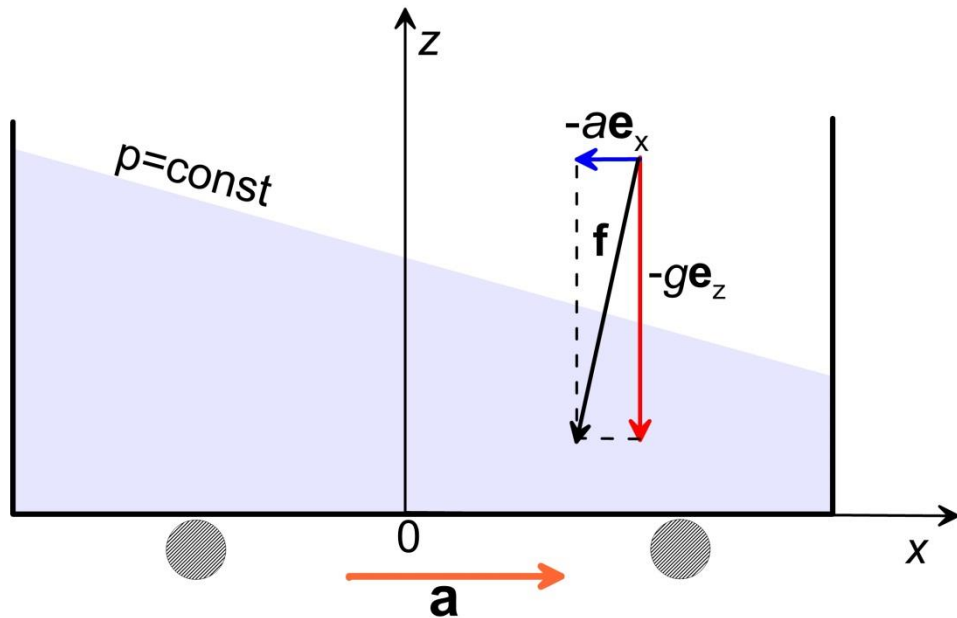
Thus, the final pressure distribution is

$$p(z) = p_a + \rho g(H - z)$$

Note that the **isobaric surfaces are horizontal**

$$p = cont \quad \Rightarrow \quad z = const$$

2. Fluid in the combined force field in the noninertial reference frame: uniform gravity plus inertial force caused by linear motion with steady acceleration.



Determination of the pressure field goes as follows

$$\mathbf{f} = [-a, 0, -g] \Rightarrow \Phi = -ax - gz + const$$

⇓

$$p = -\rho ax - \rho gz + C$$

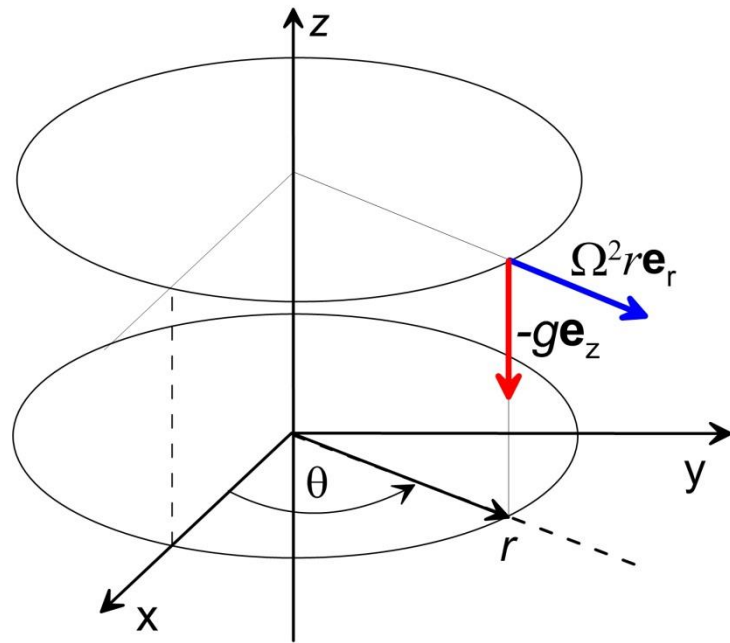
As before, the pressure constant has to be determined from additional condition.

Isobaric surfaces are described as follows

$$p = const \Rightarrow ax + gz = const \Rightarrow z = -\frac{a}{g}x + const$$

Note that  $\mathbf{f}$  is always **perpendicular** to such surface (the **general rule!**)

**3. Fluid in the combined force field in the noninertial reference frame: uniform gravity plus centrifugal force field caused by steady rotation around vertical axis.**



It is natural to use the cylindrical coordinate system. Then, the force field is given by the formula

$$\mathbf{f} = [f_r, f_\theta, f_z,] = [\Omega^2 r, 0, -g]$$

where  $\Omega$  denotes the angular velocity of rotation. This field is axisymmetric (no circumferential component, no dependence in the angle  $\theta$ ). The potential function is related to the  $\mathbf{f}$  components as follows

$$f_r = \frac{\partial}{\partial r} \Phi(r, z) \quad , \quad f_z = \frac{\partial}{\partial z} \Phi(r, z)$$

Clearly, the potential is

$$\Phi(r, z) = \frac{1}{2} \Omega^2 r^2 - gz + const$$

The corresponding pressure field is

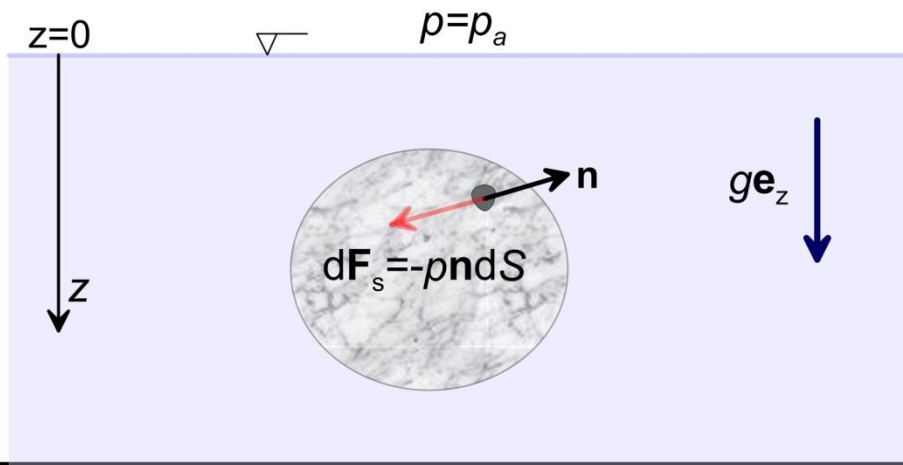
$$p(r, z) = \frac{1}{2} \rho \Omega^2 r^2 - \rho gz + const$$

Note that the isobaric surfaces are the axisymmetric and parabolic

$$p = const \quad \Rightarrow \quad z(r) = \frac{\Omega^2}{2g} r^2 + const$$

**Note:** more examples will be demonstrated during the tutorial meetings.

# THE LAW OF ARCHIMEDES



We will show the formal proof of the most classical results of the hydrostatics known as the Law of Archimedes.

We will calculate the total hydrostatic force acting on the body immersed in the motionless liquid

$$\begin{aligned}
 \mathbf{F}_s &= - \int_{\partial\Omega} p \mathbf{n} dS = - \int_{\partial\Omega} (p_a + \rho g z) \mathbf{n} dS = - \int_{\Omega} \nabla (p_a + \rho g x_3) dV = \\
 &= - \mathbf{e}_z \rho g \int_{\Omega} dV = - \rho g \underbrace{|\Omega|}_{\substack{\text{volume of} \\ \text{displaced} \\ \text{liquid}}} \mathbf{e}_z = -\mathbf{G}
 \end{aligned}$$

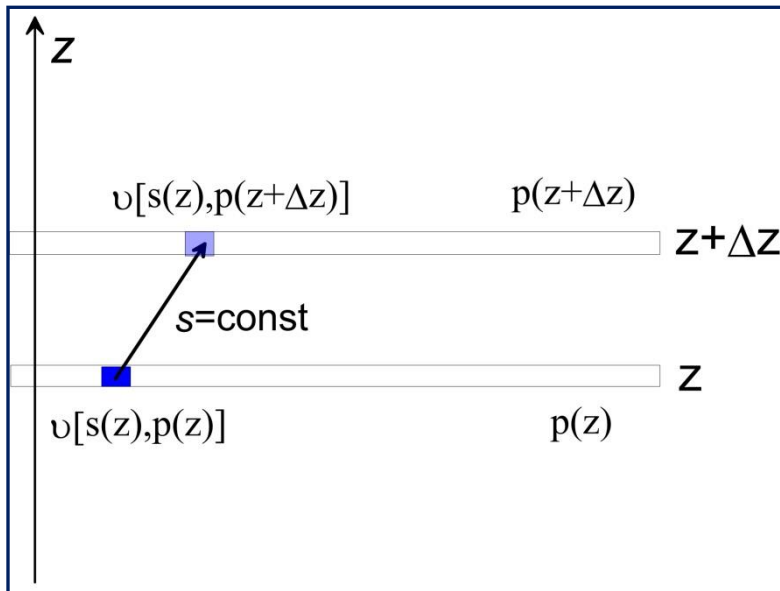
We get the well-known result: the **reaction is directed against the gravity force**, i.e., towards the free surface (this is why we call this reaction a **displacement force**) and its value is **equal to the weight of displaced liquid**.

We will show that the displacement force vector is applied at the geometric center of the immersed body. To this aim we calculate the hydrostatic moment with respect to the origin of the coordinate system. We have the following

$$\begin{aligned}
 \mathbf{M}_0 &= \int_{\partial\Omega} \mathbf{x} \times (-p\mathbf{n}) dS = - \int_{\partial\Omega} (p_a + \rho g z) \mathbf{x} \times \mathbf{n} dS = p_a \int_{\partial\Omega} \mathbf{n} \times \mathbf{x} dS + \\
 &+ \rho g \int_{\partial\Omega} \mathbf{n} \times (z\mathbf{x}) dS = p_a \int_{\Omega} \underbrace{\nabla \times \mathbf{x}}_0 dV + \rho g \int_{\Omega} \nabla \times (z\mathbf{x}) dV = \rho g \int_{\Omega} \nabla z \times \mathbf{x} dV + \\
 &\rho g \int_{\Omega} z \underbrace{\nabla \times \mathbf{x}}_0 dV = \rho g \mathbf{e}_z \times \underbrace{\int_{\Omega} \mathbf{x} dV}_{=\mathbf{x}_C |\Omega|} = \mathbf{x}_C \times \mathbf{F}_S
 \end{aligned}$$

The obtained equality means that the vector  $\mathbf{F}_S$  is applied  $\mathbf{x}_C$  as stated.

## STABILITY OF A LAYER OF GAS IN THE UNIFORM GRAVITY FIELD



Consider motionless layer of gas under action of the uniform vertical gravity field. We assume that all thermodynamic parameters are the functions of the vertical coordinate  $z$ .

Consider the “virtual” adiabatic displacement of the fluid element from its original position  $z$  to the slightly higher level  $z + \Delta z$  ( $\Delta z > 0$ ). After this displacement the pressure in the element will adjust to the pressure  $p(z + \Delta z)$  while its entropy remains the same (virtual displacement is adiabatic and reversible). Thus, its specific volume  $v$

will become different that the value corresponding to static configuration – the motion must appear. The motionless **layer of gas is stable if the fluid element shifted upwards sinks back to its original position**, i.e., when the following inequality holds

$$v[p(z + \Delta z), s(z + \Delta z)] - v[p(z + \Delta z), s(z)] > 0$$

Since  $\Delta z$  is arbitrarily small, the above condition is equivalent to  $\left(\frac{\partial v}{\partial s}\right)_p \frac{ds}{dz} > 0$ .



We will show that this condition can be re-formulated in terms of the vertical temperature gradient. To this aim, we have to play a bit with thermodynamic relations. First, we will show that

$$\left(\frac{\partial v}{\partial s}\right)_p = \frac{T}{c_p} \left(\frac{\partial v}{\partial T}\right)_p$$

Indeed, recall that  $dQ = Tds$ ,  $c_p = \left(\frac{dQ}{dT}\right)_p = T \left(\frac{ds}{dT}\right)_p$ ,  $c_v = T \left(\frac{ds}{dT}\right)_v$

Then

$$\left(\frac{\partial v}{\partial T}\right)_p = \left(\frac{\partial v}{\partial s}\right)_p \left(\frac{\partial s}{\partial T}\right)_p = \frac{c_p}{T} \left(\frac{\partial v}{\partial s}\right)_p \Rightarrow \left(\frac{\partial v}{\partial s}\right)_p = \frac{T}{c_p} \left(\frac{\partial v}{\partial T}\right)_p$$

Since  $T, c_p > 0$  the condition of stability can be re-written as  $\left(\frac{\partial v}{\partial T}\right)_p \frac{ds}{dz} > 0$

Next, most substances expand while being heated (at constant pressure), thus  $\left(\frac{\partial v}{\partial T}\right)_p > 0$

We conclude that the **condition of stability of the gas layer** is:  $\frac{ds}{dz} > 0$

If we consider the specific entropy  $s$  to be a function of pressure and temperature  $s = s(p, T)$ , then the vertical gradient of entropy can be expressed as

$$\frac{ds}{dz} = \left( \frac{\partial s}{\partial p} \right)_T \frac{dp}{dz} + \left( \frac{\partial s}{\partial T} \right)_p \frac{dT}{dz}$$

We have already seen that  $\left( \frac{\partial s}{\partial T} \right)_p = \frac{c_p}{T}$ . We will also show that  $\left( \frac{\partial s}{\partial p} \right)_T = - \left( \frac{\partial v}{\partial T} \right)_p$ .

To this aim, consider the free enthalpy (Gibbs function)  $g$  defined by the formula ( $u$  denotes the specific internal energy of gas)

$$g = u + pv - Ts$$

Let's calculate the full differential of this function. Using the 1<sup>st</sup> Principle of Thermodynamics we get

$$dg = du + d(pv) - d(Ts) = \underbrace{du + pdv + vdp}_{dQ = Tds} - Tds - sdT = vdp - sdT$$

From the above we conclude that

$$\left( \frac{\partial g}{\partial p} \right)_T = v, \quad \left( \frac{\partial g}{\partial T} \right)_p = -s \quad \Rightarrow \quad \left( \frac{\partial v}{\partial T} \right)_p = - \left( \frac{\partial s}{\partial p} \right)_T$$

Using the above formulae, we can write the stability condition as follows

$$\frac{ds}{dz} = \frac{c_p}{T} \frac{dT}{dz} - \left( \frac{\partial v}{\partial T} \right)_p \frac{dp}{dz} > 0$$

The last step is to use the stability equation to calculate the pressure gradient

$$\nabla p = \rho f \quad \Rightarrow \quad \frac{dp}{dz} = -\rho g = -\frac{g}{v}$$

We get the inequality

$$\frac{c_p}{T} \frac{dT}{dz} + \left( \frac{\partial v}{\partial T} \right)_p \frac{g}{v} > 0$$

or, equivalently

$$-\frac{dT}{dz} < \frac{\beta g T}{c_p}$$

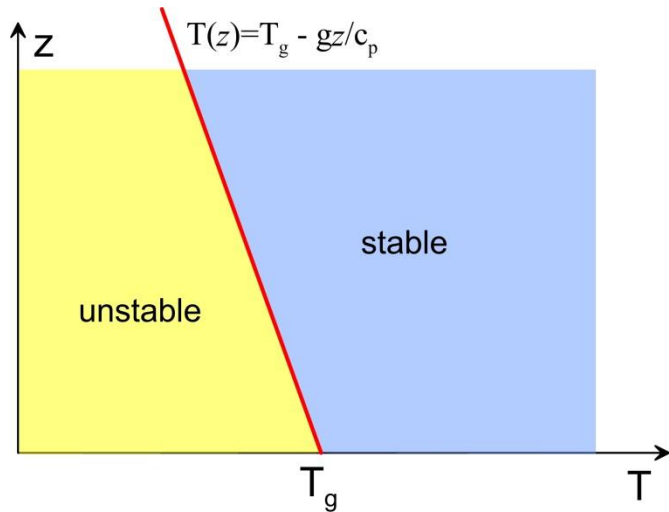
where  $\beta = \frac{1}{v} \left( \frac{\partial v}{\partial T} \right)_p$  is the thermal expansion coefficient.

For the Clapeyron gas, we can calculate  $\beta$  using the equation of state, namely

$$pv = RT, \quad \left( \frac{\partial v}{\partial T} \right)_p = \frac{R}{p} \quad \Rightarrow \quad \beta = \frac{R}{pv} = \frac{1}{T}$$

Thus, the final form of the stability condition is  $-\frac{dT}{dz} < \frac{g}{c_p}$ .

For the atmospheric layer near the Earth's surface we have  $g = 9.81 \frac{m}{s^2}$ ,  $c_p = 1005 \frac{J}{kg \cdot K}$ .



Thus, the stability condition is  $-\frac{dT}{dz} < 0.0098 \frac{K}{m}$ .

Summarizing, we can write

$$\frac{dT}{dz} = \begin{cases} -\frac{g}{c_p} \quad (s = const) - \text{neutral balance} \\ > -\frac{g}{c_p} \quad (\frac{ds}{dz} > 0) - \text{stable} \\ < -\frac{g}{c_p} \quad (\frac{ds}{dz} < 0) - \text{unstable} \end{cases}$$

Note that the neutral gradient of temperature in the (ideal) atmospheric layer near the Earth's surface is approximately equal to the drop by 1 degree per each 100 meters of altitude. **If the layer is unstable then the motion will appear spontaneously.** Such motion of gas caused by thermal/gravitational instability is called **convection**.