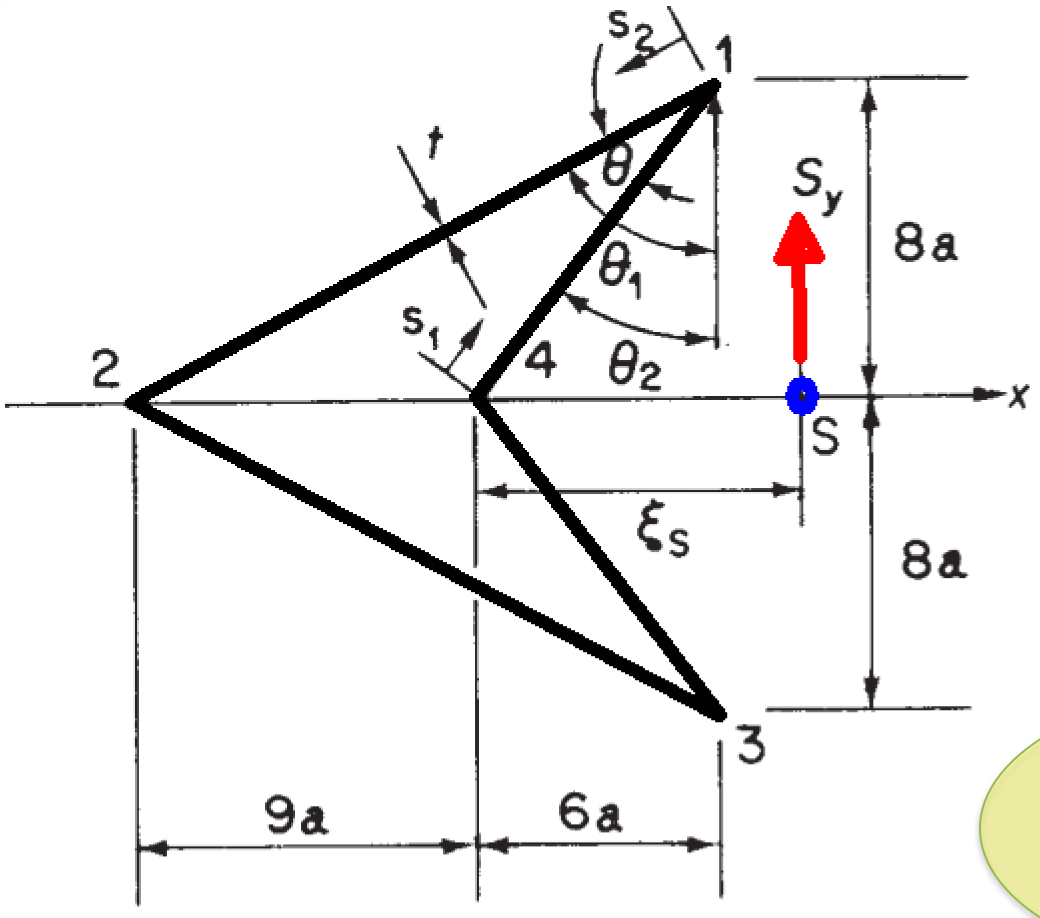
Shear of closed section thin-walled beams - Example



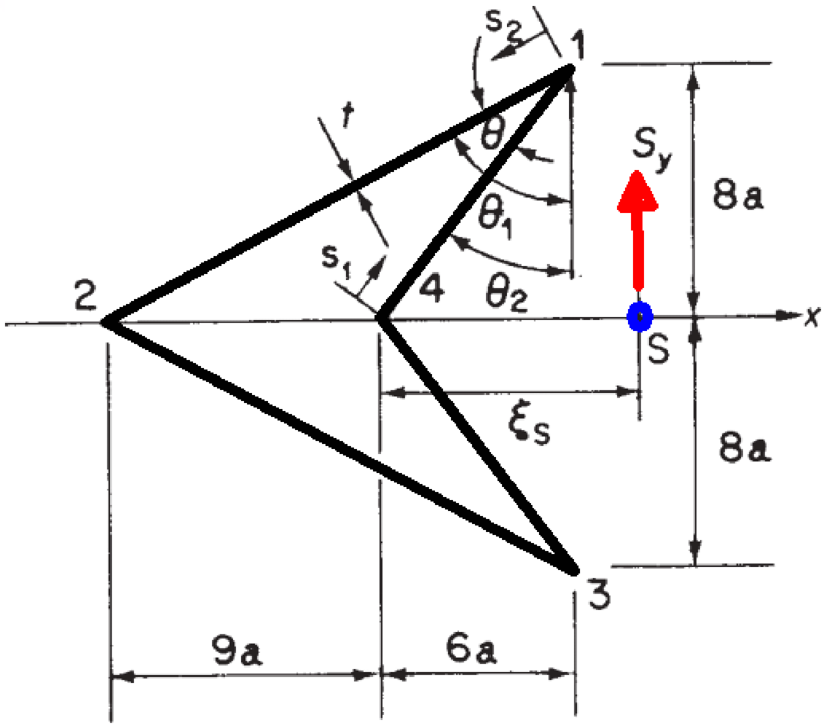
$$q_s = -\frac{S_y}{I_{xx}} \int_0^s t y \, \mathrm{d}s + q_{s,0}$$

$$I_{xx} = 2\left[\int_0^{10a} t\left(\frac{8}{10}s_1\right)^2 \mathrm{d}s_1 + \int_0^{17a} t\left(\frac{8}{17}s_2\right)^2 \mathrm{d}s_2\right]$$

$$I_{xx} = \frac{a^3 t \sin^2 \beta}{12}$$

$$I_{xx} = 1152 a^3 t.$$

Shear of closed section thin-walled beams - Example

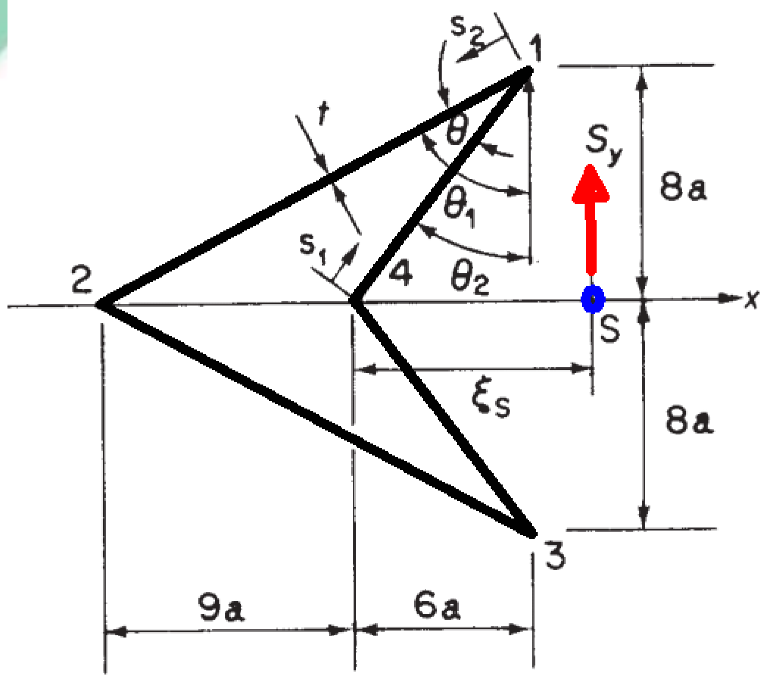


$$q_{b,41} = \frac{-S_y}{1152a^3 t}\int_0^{s_1} t\left(\frac{8}{10}s_1\right)\mathrm{d}s_1 = \frac{-S_y}{1152a^3}\left(\frac{2}{5}s_1^2\right)$$

$$q_{b,12} = \frac{-S_y}{1152a^3}\left[\int_0^{s_2}(17a - s_2)\frac{8}{17}\mathrm{d}s_2 + 40a^2\right]$$

$$q_{b,12} = \frac{-S_y}{1152a^3}\left(-\frac{4}{17}s_2^2 + 8as_2 + 40a^2\right)$$

Shear of closed section thin-walled beams - Example



$$q_{s,0} = \frac{2S_y}{54a \times 1152a^3}\left[\int_0^{10a} \frac{2}{5}s_1^2\,ds_1 + \int_0^{17a}\left(-\frac{4}{17}s_2^2 + 8as_2 + 40a^2\right)ds_2\right]$$

$$q_{s,0} = \frac{S_y}{1152a^3}(58.7a^2)$$

Taking moment w.r.t. to point 2:

$$S_y(\xi_S + 9a) = 2\int_0^{10a} q_{41}\,17a\sin\theta\,ds_1$$

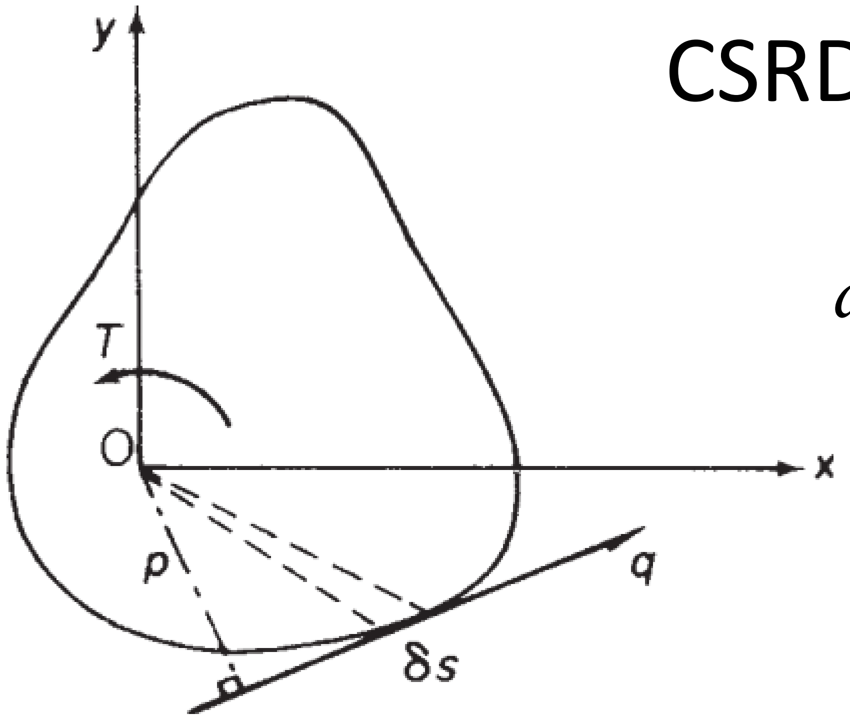
$$S_y(\xi_S + 9a) = \frac{S_y 34a\sin\theta}{1152a^3}\int_0^{10a}\left(-\frac{2}{5}s_1^2 + 58.7a^2\right)ds_1$$

$$\xi_S = -3.35a$$

Thin-walled beams – torsion

open section vs closed section

Torsion of closed section thin-walled beams



Assumptions:

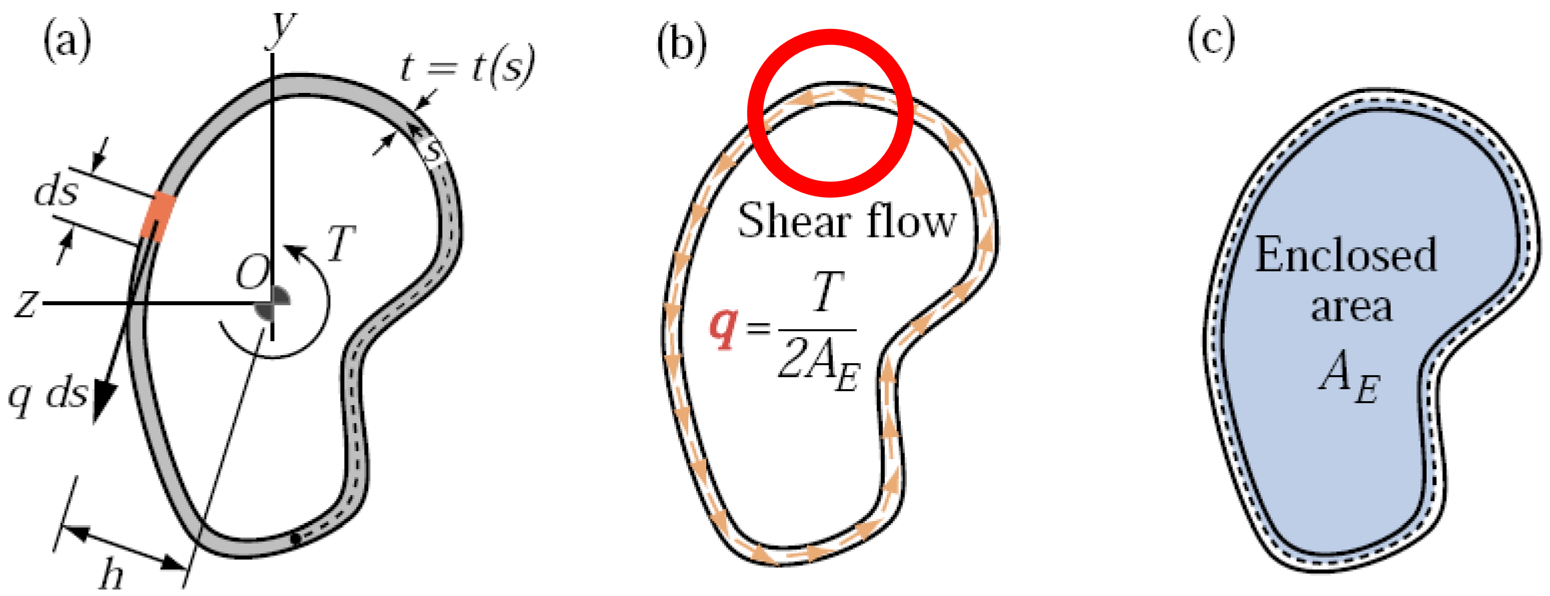
CSRD

$dT = q\,ds \,.\, p$ ⟹ $T = \oint pq \, \mathrm{d}s$

$T = 2Aq$

Bredt-Batho formula

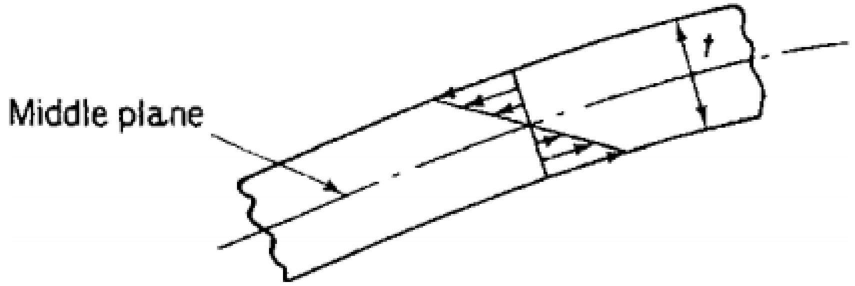
Torsion of closed section thin-walled beams

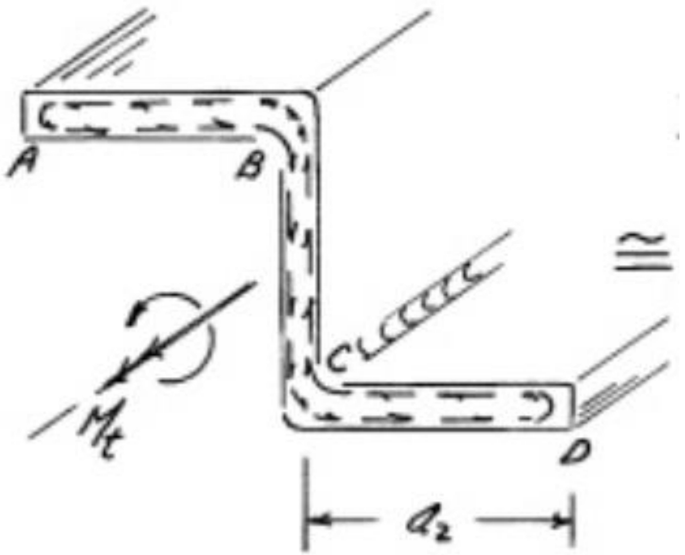


Torsion of single-cell closed TW section.

$$\frac{d\theta}{dz} = \frac{T}{4A^2} \oint \frac{ds}{Gt}$$

Torsion of open section thin-walled beams





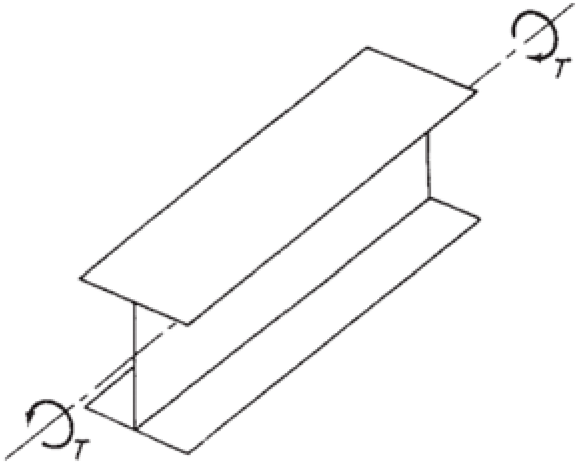
$$\cong \quad T = GJ\frac{\mathrm{d}\theta}{\mathrm{d}z}$$

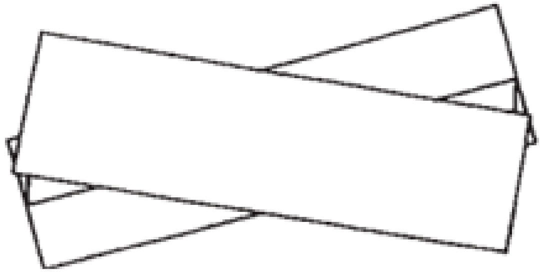
$$J = \sum \frac{st^3}{3} \quad \text{or} \quad J = \frac{1}{3}\int_{\text{sect}} t^3\,\mathrm{d}s$$

Thin-walled beams – constrained torsion

St. Venant vs Wagner torsion-bending

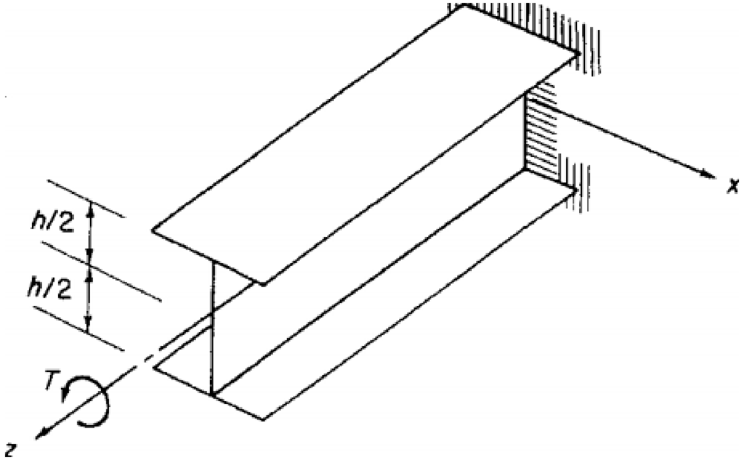
Constrained torsion of thin-walled beams

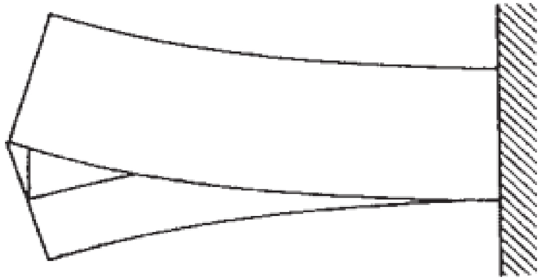




- **I-beam with constant torque**
- **Positive moment, but <u>negative</u> top flange displacement *u***

(a)

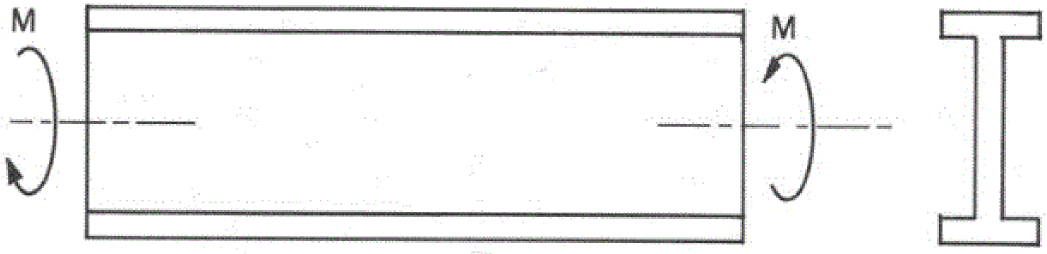


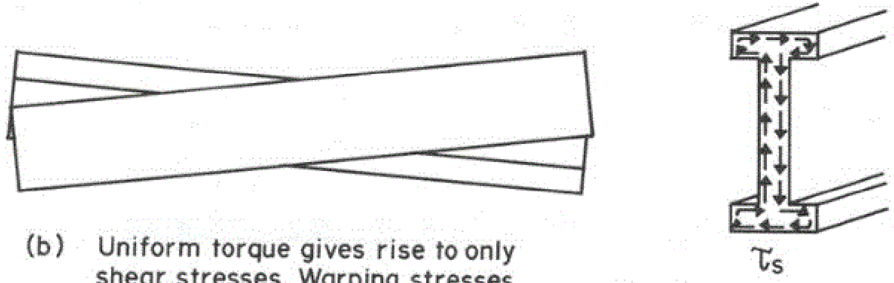


angle of twist → θ

rate of twist → $d\theta/dz = \theta'$

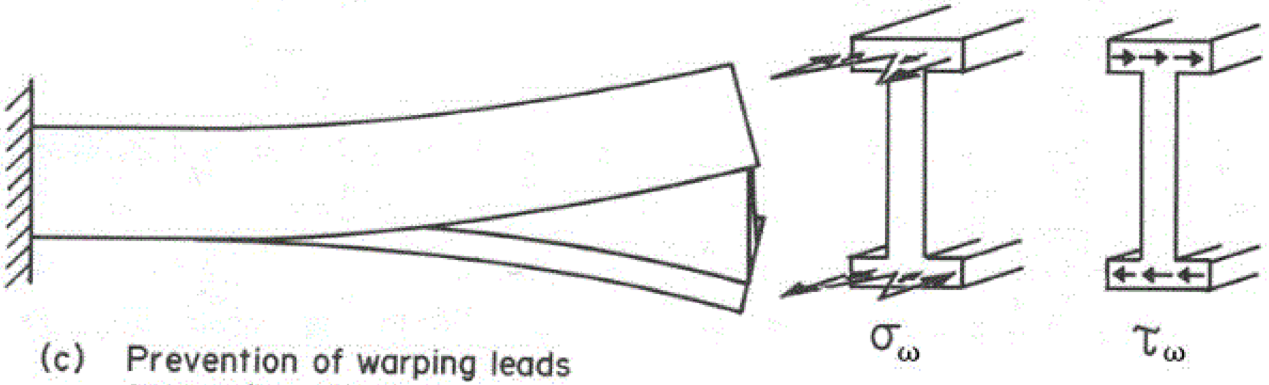
Constrained torsion of thin-walled beams





(b) Uniform torque gives rise to only shear stresses. Warping stresses are zero and every cross-section warps by the same amount. This is Saint Venant torsion.

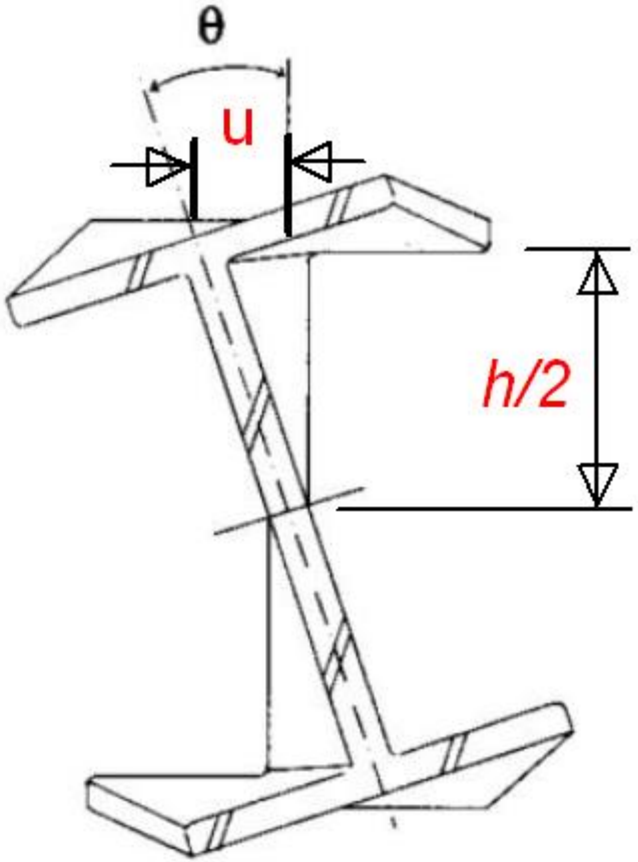
Saint Venant shear stresses on left hand end

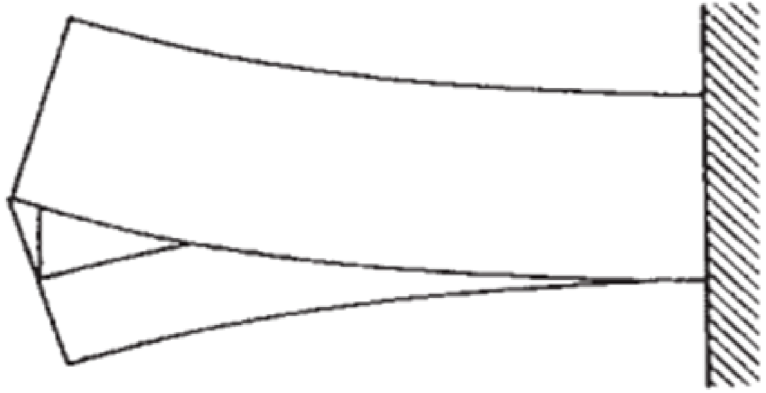


(c) Prevention of warping leads to warping stresses

Warping stresses on left hand end.

Constrained torsion of thin-walled beams





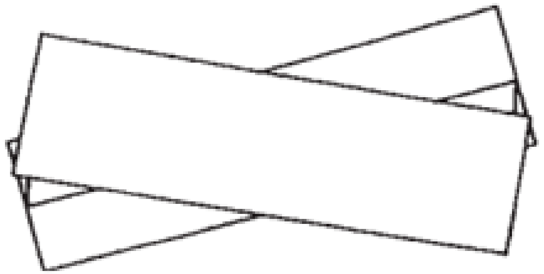
Flange BENDING u=u(z)

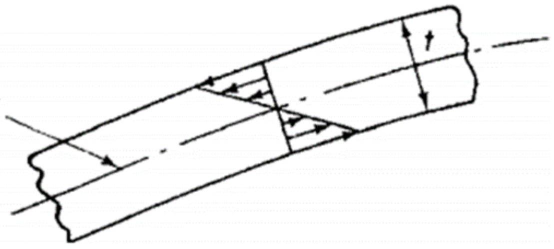
Rate of twist $d\theta/dz = \theta'$ *is* NOT constant

twist is nonlinear

Constrained torsion of thin-walled beams

FREE S-V torsion stiffness





$$T_J = GJ_s \frac{d\theta}{dz} = C_T \frac{d\theta}{dz} = C_T \theta'$$

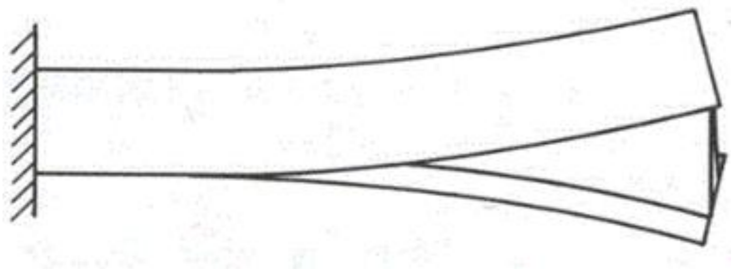
$$J_s = \frac{1}{3} \sum s_i t_i^3$$

$$GJ_s = C_T \Rightarrow \quad S-V\ torsional\ stiffness$$

$GJ_s = C_T \Rightarrow \quad S-V \; torsional \; stifness$

Constrained torsion of thin-walled beams

Stiffness of bending of flanges



Flange BENDING   Vlasov theory (Wagner effect)

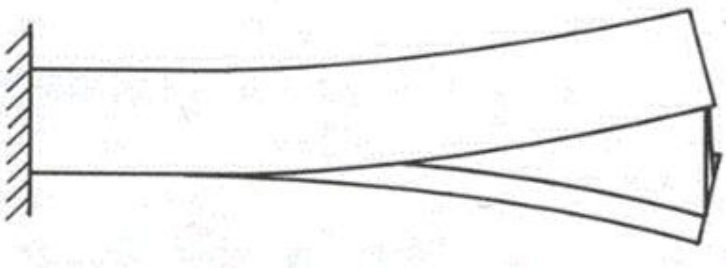
Transverse bending (shear force $\boldsymbol{S_F}$ exists), bending moment is variable (is a function)

$$M_y = -EI_{yy}u''$$

$$M_F = -EI_F \frac{d^2u}{dz^2}$$

Constrained torsion of thin-walled beams

Stiffness of bending of flanges



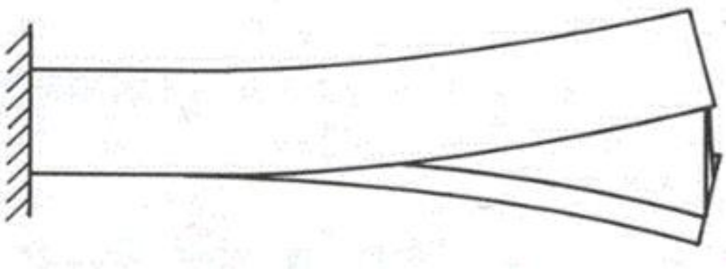
Transverse bending (shear force $\boldsymbol{S_F}$ exists), bending moment is variable (is a function of z)

$$u = \frac{h}{2}\theta$$

$$M_F = -EI_F \frac{d^2u}{dz^2} \qquad S_F = \frac{dM_F}{dz} = -EI_F \frac{d^3u}{dz^3}$$

Constrained torsion of thin-walled beams

Stiffness of bending of flanges



Transverse bending (shear force $\boldsymbol{S_F}$ exists), bending moment is variable (is a function of z)

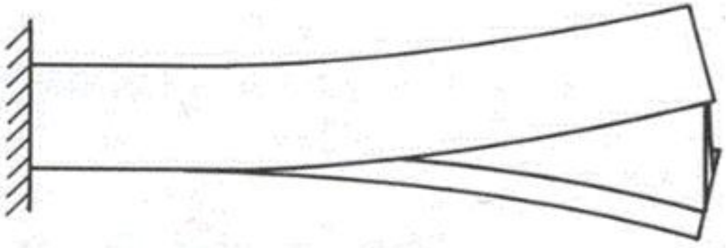
$$u = \frac{h}{2}\theta \quad so \quad \frac{d^3u}{dz^3} = \frac{h}{2}\frac{d^3\theta}{dz^3}$$

$$T_\Gamma = S_F h = -EI_F \frac{h}{2}\frac{d^3\theta}{dz^3} \cdot h \qquad I_F = \frac{t_f b^3}{12}$$

Constrained torsion of thin-walled beams

Combining two mechanisms …

Total torque is transmitted thru sum of both:



$$T_J = GJ_s \frac{d\theta}{dz} = C_T \frac{d\theta}{dz}$$

$$T_\Gamma = -EI_F \frac{h^2}{2} \frac{d\theta^3}{dz^3}$$

$$T = T_J + T_\Gamma$$

Constrained torsion of thin-walled beams

Combining two mechanisms …

$$T = T_J + T_\Gamma = GJ_s \frac{d\theta}{dz} - EI_F \frac{h^2}{2} \frac{d\theta^3}{dz^3}$$

$$T = T_J + T_\Gamma = C_T \frac{d\theta}{dz} - C_\omega \frac{d\theta^3}{dz^3}$$

$$C_\omega = EI_\omega \quad where \quad I_\omega = \frac{b^3 h^2 t_F}{24} \quad \left[m^6\right]$$

$$or: \quad C_\omega = E\Gamma_R \quad where \quad \Gamma_R = \int_C 4A_R^2 t ds \quad torsion - bending\ constant$$

Constrained torsion of thin-walled beams

$$T = T_J + T_\Gamma = C_T \frac{d\theta}{dz} - C_\omega \frac{d\theta^3}{dz^3}$$

ODE to be solved + necessary BC

In general, T=T(z) may vary along TWB

For T=const the solution is:

$$\frac{d\theta}{dz} = \frac{T}{GJ} + A\cosh\mu z + B\sinh\mu z \qquad \mu^2 = GJ/E\Gamma_R$$

Constrained torsion of thin-walled beams

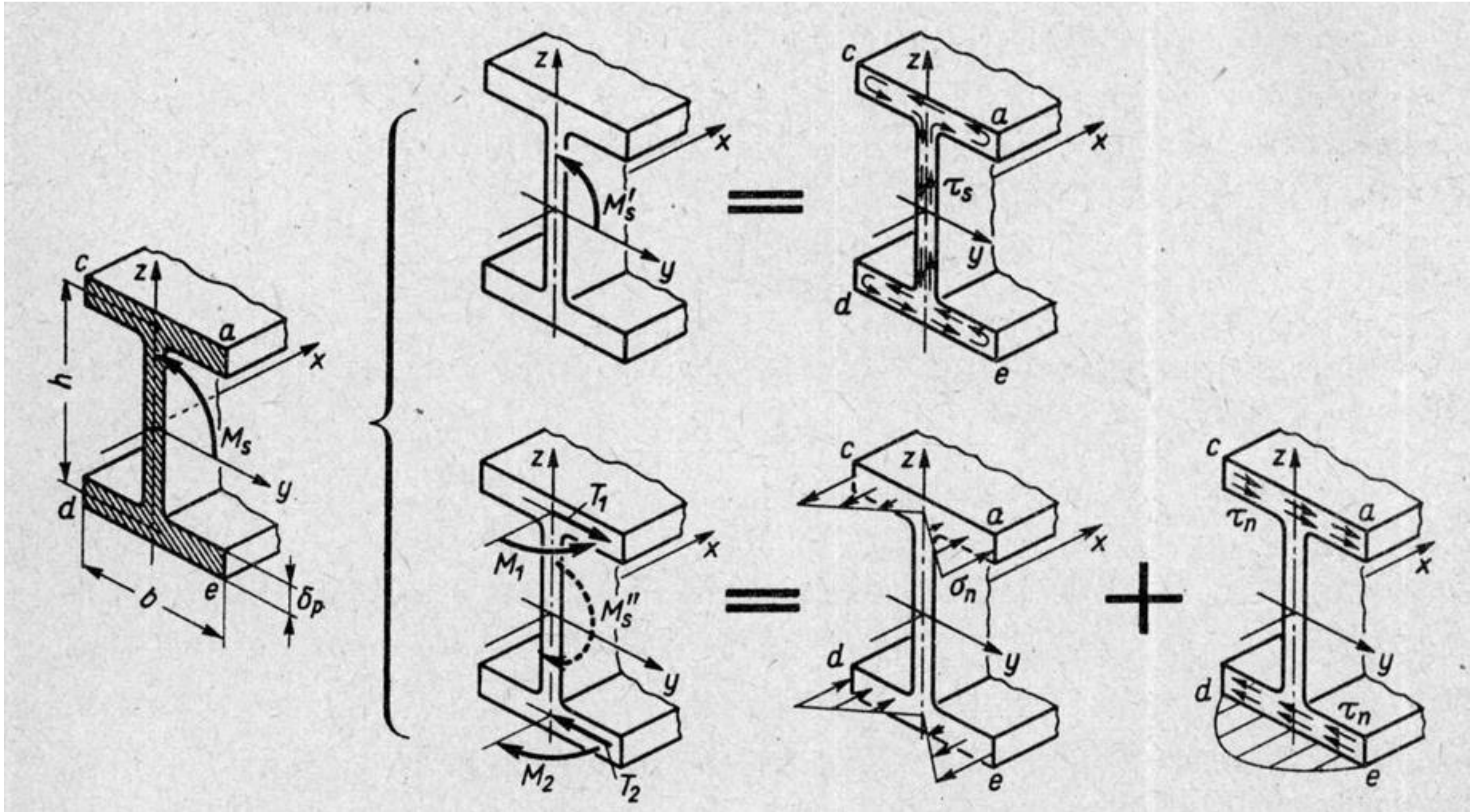
The BC are:

- at built-in end (z=0) $\mathrm{d}\theta/\mathrm{d}z = 0$
- at free end (z=L) $\mathrm{d}^2\theta/\mathrm{d}z^2 = 0$

$$\frac{\mathrm{d}\theta}{\mathrm{d}z} = \frac{T}{GJ}\left[1 - \frac{\cosh \mu(L - z)}{\cosh \mu L}\right]$$

$$\mu^2 = GJ/E\Gamma_R$$

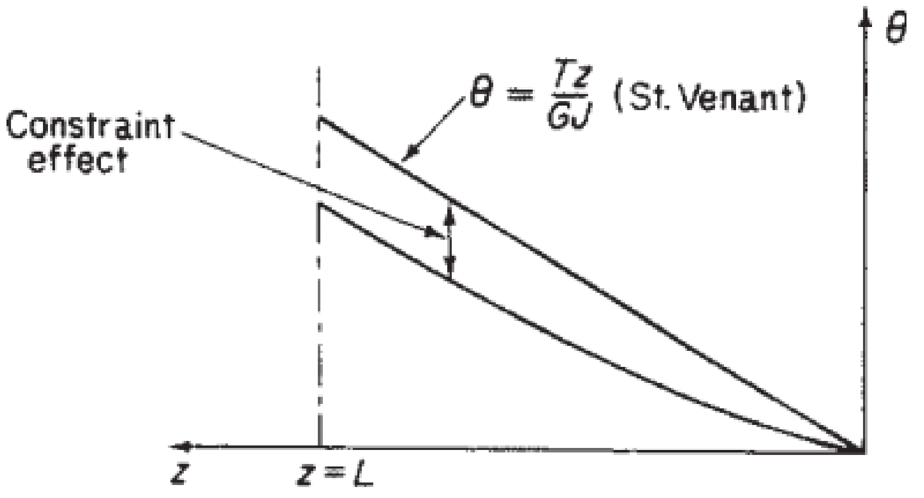
Constrained torsion of thin-walled beams



Constrained torsion of thin-walled beams

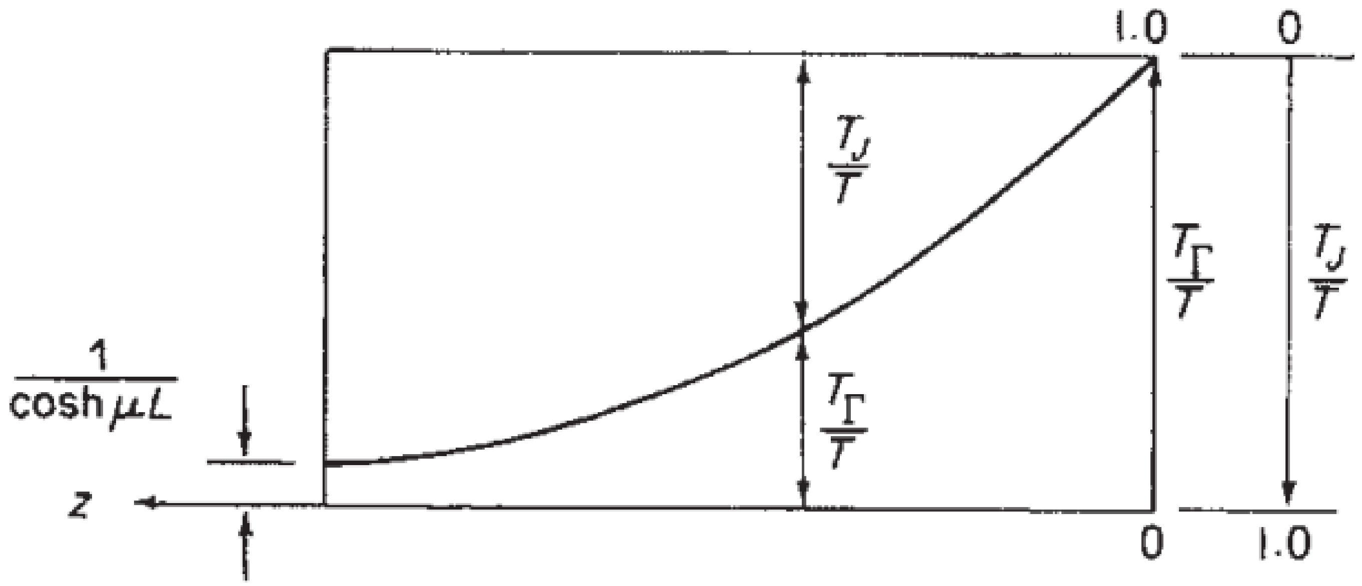
$$\mu^2 = GJ/E\Gamma_R$$

for $\mu L < 0.5$ short TWB
for $\mu L > 5$ long TWB



Constrained torsion of thin-walled beams

$\mu^2 = GJ/E\Gamma_R$

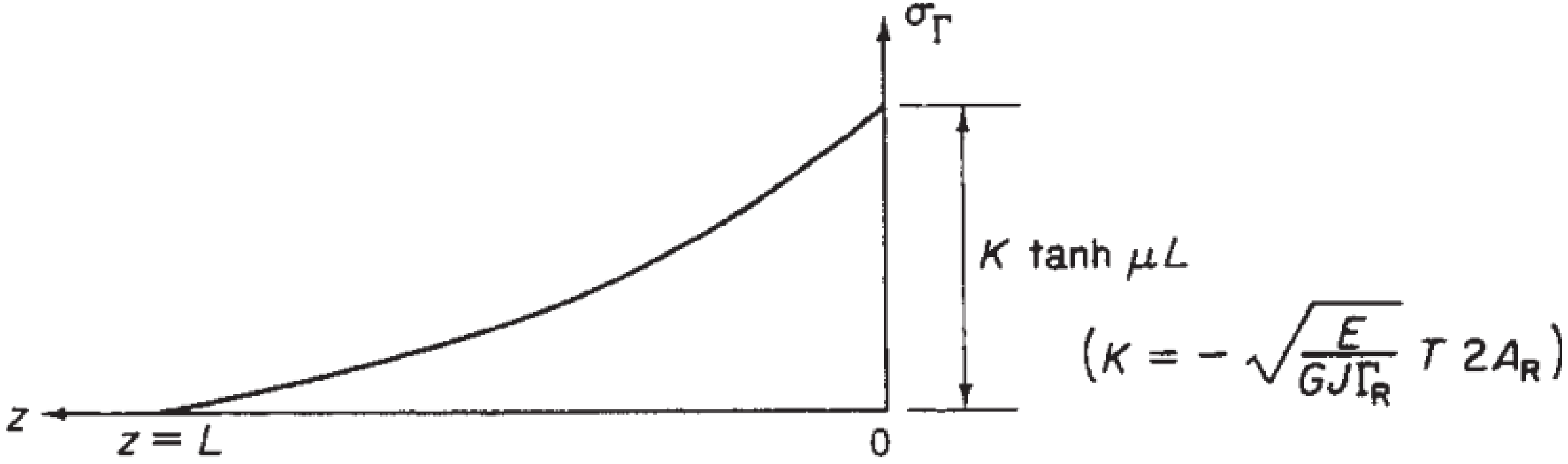


for $\mu L < 0.5$ short TWB

for $\mu L > 5$ long TWB

Constrained torsion of thin-walled beams

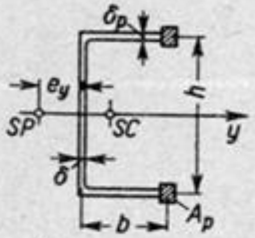
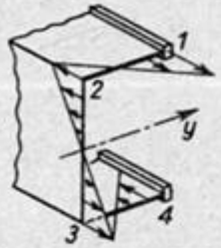
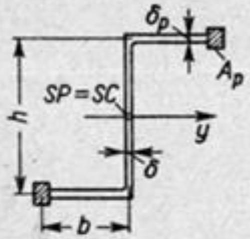
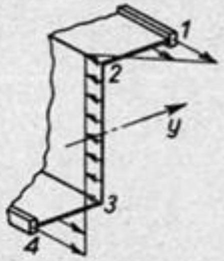
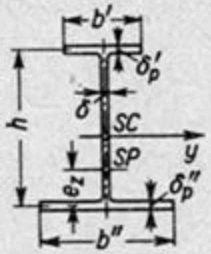
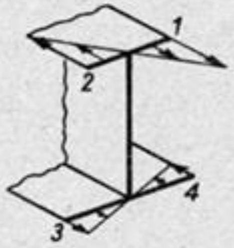
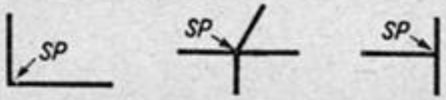
$$\mu^2 = GJ/E\Gamma_R$$



for $\mu L < 0.5$ short TWB

for $\mu L > 5$ long TWB

Constrained torsion of thin-walled beams

Nr	Profile	Shear centre	Stress values and location	Stiffness
1		$e_y = \frac{b(1+2\psi)}{2(1+\psi+\eta)}$ $\psi = \frac{A_p}{b\delta_p}$ $\eta = \frac{h\delta}{6b\delta_p}$	 $\varkappa_1 = \frac{1+2\eta}{4(1+\psi+\eta)}$ $\varkappa_2 = -\frac{1+2\psi}{4(1+\psi+\eta)}$ $\varkappa_3 = -\varkappa_2,\ \varkappa_4 = -\varkappa_1,\ F_\omega = bh$	$C_s = \frac{2Eb^3h^2\delta_p}{3} \times$ $\times [\varkappa_1^2(1+3\psi)+\varkappa_1\varkappa_2+\varkappa_2^2(1+3\eta)]$
2		$SP \equiv SC$ $\psi = \frac{A_p}{b\delta_p}$ $\eta = \frac{h\delta}{6b\delta_p}$	 $\varkappa_1 = \frac{1+6\eta}{4(1+\psi+3\eta)}$ $\varkappa_2 = -\frac{1+2\psi}{4(1+\psi+3\eta)}$ $\varkappa_3 = \varkappa_2,\ \varkappa_4 = \varkappa_1,\ F_\omega = bh$	$C_s = \frac{2Eb^3h^2\delta_p}{3} \times$ $\times [\varkappa_1^2(1+3\psi)+\varkappa_1\varkappa_2+\varkappa_2^2(1+9\eta)]$
3		$e_z = \frac{h}{1+\psi}$ $\psi = \left(\frac{b''}{b'}\right)^3 \left(\frac{\delta_p''}{\delta_p'}\right)$	 $\varkappa_1 = -\varkappa_2 = \frac{\psi}{2(1+\psi)}$ $\varkappa_3 = -\varkappa_4 = \frac{1}{2(1+\psi)} \frac{b''}{b'}$ $F_\omega = b'h$	$C_s = \frac{Eb'^3h^2\delta_p'\psi}{12(1+\psi)}$
4		Przy dowolnych wymiarach, grubości, liczbie i ustawieniu ścianek płaskich zawsze $\sigma_\omega \equiv 0,\ C_s \equiv 0$		

Constrained torsion of thin-walled beams